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ON A QUEUEING SYSTEM WITH SEQUENTIAL SERVICE

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Abstract. We consider a queueing system with sequential service such that the amount of unfinished work in the system satisfies the Langevin equation. We study the total time spent by a customer in the system under busy traffic.

1. Introduction

We study a one channel queueing system with sequential service. The system contains n servers; every customer should be served sequentially by each of the servers. The input for the first server is a generalized Poisson process with parameter λ; the speed of service at a server k is proportional to the amount of unfinished work at this server. Any customer arriving at the system should sequentially be served by every server; the total service time for the i-th customer is denoted by η^i.

Let $x_k(t)$, $k = 1, \dots, n$, be the unfinished work at the k-th server at the moment t, and let $\mu_k x_k(t)$, $\mu_k > 0$, be the speed of service at server $k = 1, \dots, n$. Assume that the total unfinished work in the system

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$$

satisfies the Langevin equation

$$dx(t) = Ax(t)\,dt + dz(t), \tag{1}$$

where $z(t) = (z_1(t), 0, \dots, 0)^T \in \mathbb{R}^n$ is a generalized Poisson process with parameter λ and jumps $\eta^1 = \eta, \eta^2, \dots, \eta^i, \dots$, and

$$A = \left\| \begin{matrix} -\mu_1 & 0 & 0 & \dots & 0 & 0 \\ \mu_1 & -\mu_2 & 0 & \dots & 0 & 0 \\ 0 & \mu_2 & -\mu_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mu_{n-1} & -\mu_n \end{matrix} \right\| \tag{2}$$

for some $\mu_k > 0$, $k = 1, \dots, n$.

2. Setting of the problem

A one channel queueing system containing a single server whose unfinished work satisfies the differential equation

$$dx(t) = -\mu x(t)\,dt + dz(t)$$

is considered in [1] for the case where $z(t)$ is a generalized Poisson process and $\mu > 0$. For such systems, the waiting time until the start of the service and the time spent by a

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customer in the system are studied in [1] under the condition of busy traffic (that is, as $\lambda \to \infty$).

It is of interest to study the behavior of the time spent by a customer in a one channel queueing system with sequential service, and this is the aim of this paper.

The time spent by a customer in a queueing system depends on the discipline in the system; we consider FIFO systems ("First In First Out"). In what follows, such a system is called a filter system.

3. Conditions for the existence of a stationary regime in a filter system

It is shown in [1] that the process $x(t)$ has the limit distribution as $t \to \infty$; moreover, the limit distribution does not depend on the initial value $x_0 = x(0)$ if and only if

a) eigenvalues of A belong to the left half-plane, and
b) $\mathsf{E}(\ln|\eta|; |\eta| > 1) < \infty$.

It is also proved in [1] that the limit distribution is a unique stationary distribution of the process $x(t)$ if assumptions a) and b) hold.

Note that condition a) is equivalent to

$$\mu_1 > 0, \qquad \mu_2 > 0, \qquad \dots, \qquad \mu_n > 0,$$

as is easily seen from (2).

4. Auxiliary results

Let the stationary distribution of the process $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ coincide with the distribution of $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$. By $F(y_1, y_2, \dots, y_n)$ we denote the distribution function of $\eta = (\eta_1, 0, \dots, 0)^T$ and by $\mathsf{E}\,\eta = (\mathsf{E}\,\eta_1, 0, \dots, 0)^T$ we denote the expectation of η where $\mathsf{E}\,\eta_1 = \int_0^\infty \dots \int_0^\infty y_1\, dF(y_1, y_2, \dots, y_n) = m_1$. Let $\varphi(s) = \mathsf{E}\{\exp i(s, \eta)\}$ be the characteristic function of η, where $s = (s_1, \dots, s_n)$.

Lemma 4.1. *If the stationary distribution of the process $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ coincides with the distribution of $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$ and $\mathsf{E}\,\eta_1 = m_1 < \infty$, then the vector $\lambda^{-1}(\xi_1, \xi_2, \dots, \xi_n)^T$ converges in probability to the vector $m_1(\mu_1^{-1}, \mu_2^{-1}, \dots, \mu_n^{-1})^T$ as $\lambda \to \infty$.*

Proof. It is shown in [1] that the characteristic function of ξ is of the form

$$\Xi_\xi(s) = \exp\left\{-\lambda \int_0^\infty \left(1 - \varphi\left(\exp\left\{uA^T\right\} s\right)\right) du\right\},$$

whence

$$\ln \Xi_{\lambda^{-1}\xi}(s) = \ln \Xi_\xi\left(\lambda^{-1} s\right) = \int_0^\infty \frac{\varphi\left(\exp\left\{uA^T\right\}\lambda^{-1}s\right) - 1}{\lambda^{-1}}\, du$$
$$= \left|\lambda^{-1} = l\right| = \int_0^\infty \frac{\varphi\left(\exp\left\{uA^T\right\} ls\right) - 1}{l}\, du.$$

Put

$$\exp\left\{uA^T\right\} ls = (\theta_1, \theta_2, \dots, \theta_n) = (lk_1(u), lk_2(u), \dots, lk_n(u)),$$

where

$$k_i(u) = \sum_{l=1}^{n} a_{il} \exp\left\{-\mu_l u\right\} \sum_{j=1}^{r_l} c_{ij} u^{j-1}, \qquad i = 1, \dots, n,$$

r_l is the multiplicity of the eigenvalue $\lambda_l = -\mu_l$ of the matrix A, and a_{il}, c_{ij} are some constants. Then

$$\varphi\left(\theta_1, \theta_2, \dots, \theta_n\right) = 1 + (im_1, 0, \dots, 0)\left(\theta_1, \theta_2, \dots, \theta_n\right) + o_u\left(\rho\right)$$
$$\text{for} \quad \rho = \sqrt{\theta_1^2 + \theta_2^2 + \dots + \theta_n^2} = |\,l\,| \sqrt{k_1^2(u) + k_2^2(u) + \dots + k_n^2(u)}.$$

Thus

$$\begin{aligned}\ln \Xi_{\lambda^{-1}\xi}(s) &= \int_0^\infty \left((im_1, 0, \dots, 0) \exp\left\{uA^T\right\} s + \frac{o_u(\rho)}{l} \right) du \\ &= (im_1, 0, \dots, 0) \int_0^\infty \exp\left\{uA^T\right\} du \,(s_1, s_2, \dots, s_n)^T + \int_0^\infty \frac{o_u(\rho)}{l}\, du.\end{aligned}$$

Put

$$\varlimsup_{l\to 0} \left| \int_0^\infty \frac{o_u(\rho)}{l}\, du \right| = \varepsilon.$$

For an arbitrary $\delta > 0$, we choose κ such that $|o_u(\rho)| < \delta\rho$ if $\rho < \kappa$. Then

$$\begin{aligned}\varepsilon &< \varlimsup_{l\to 0} \int_0^\infty \frac{\delta\rho}{l}\, du = \varlimsup_{l\to 0} \int_0^\infty \frac{\delta |l| \sqrt{k_1^2(u) + k_2^2(u) + \dots + k_n^2(u)}}{l}\, du \\ &\le \delta \int_0^\infty \left(k_1^2(u) + k_2^2(u) + \dots + k_n^2(u)\right)^{1/2} du \le \delta \int_0^\infty \left(n \max_i k_i^2(u)\right)^{1/2} du \\ &\le \delta n^{1/2} \int_0^\infty \max_i |k_i(u)|\, du.\end{aligned}$$

Since $k_i(u) \le n^2 \max_l |a_{il}| \max_j |c_{ij}| \exp\{-\mu_l u\}\, u^{j-1}$, $i = 1, \dots, n$, we have

$$\varepsilon < \delta n \max_{i,l} |a_{il}| \max_{i,j} |c_{ij}| \int_0^\infty u^{j-1} \exp\{-\mu_l u\}\, du = \delta n \max_{i,l} |a_{il}| \max_{i,j} |c_{ij}|\, (j-1)!\, \mu_l^{-j},$$

whence $\varepsilon = 0$.

Since A is nonsingular, the eigenvalues of the matrix A belong to the left half-plane. Hence

$$\int_0^\infty \exp\left\{uA^T\right\} du = -\left(A^T\right)^{-1}$$

(see [2]) and

$$\lim_{\lambda\to\infty} \ln \Xi_{\lambda^{-1}\xi}(s) = (im_1, 0, \dots, 0) \left(-A^T\right)^{-1} (s_1, s_2, \dots, s_n)^T.$$

It follows from (2) that the matrix $\left(-A^T\right)^{-1}$ is of the form

$$\left(-A^T\right)^{-1} = \left\| \begin{matrix} \mu_1^{-1} & \mu_2^{-1} & \mu_3^{-1} & \dots & \mu_{n-1}^{-1} & \mu_n^{-1} \\ 0 & \mu_2^{-1} & \mu_3^{-1} & \dots & \mu_{n-1}^{-1} & \mu_n^{-1} \\ 0 & 0 & \mu_3^{-1} & \dots & \mu_{n-1}^{-1} & \mu_n^{-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mu_{n-1}^{-1} & \mu_n^{-1} \\ 0 & 0 & 0 & \dots & 0 & \mu_n^{-1} \end{matrix} \right\|.$$

Thus

$$\lim_{\lambda\to\infty} \Xi_{\lambda^{-1}\xi}(s) = \exp\left\{ i \left(s_1 m_1 \mu_1^{-1} + s_2 m_1 \mu_2^{-1} + \dots + s_n m_1 \mu_n^{-1} \right) \right\}$$

and

$$\lambda^{-1} (\xi_1, \xi_2, \dots, \xi_n)^T \xrightarrow[\lambda\to\infty]{\mathsf{P}} m_1 \left(\mu_1^{-1}, \mu_2^{-1}, \dots, \mu_n^{-1}\right)^T. \qquad \square$$

Lemma 4.2. *If $z(t) = (z_1(t), 0, \dots, 0)^T$ is a generalized Poisson process and*

$$\mathsf{E}\, \eta_1 = m_1 < \infty,$$

then the vector $\lambda^{-1} (z_1(t), 0, \dots, 0)^T$ converges in probability to the vector $(m_1 t, 0, \dots, 0)^T$ as $\lambda \to \infty$.

Proof. The characteristic function of $\lambda^{-1}z(t)$ is given by

$$\Psi_{\lambda^{-1}z(t)}(s) = \mathsf{E}\exp\left\{is\lambda^{-1}z(t)\right\} = \exp\left\{\lambda t\left(\varphi\left(\lambda^{-1}s\right)-1\right)\right\}.$$

Therefore,

$$\lim_{\lambda\to\infty}\ln\Psi_{\lambda^{-1}z(t)}(s) = \lim_{\lambda\to\infty}\lambda t\left(\varphi(\lambda^{-1}s)-1\right) = \left|\lambda^{-1}=l, \lambda^{-1}s_i=\theta_i\right|$$
$$= \lim_{l\to 0}\frac{t\left(\varphi(ls)-1\right)}{l} = \lim_{l\to 0} t\frac{d}{dl}\varphi(ls) = \lim_{l\to 0} t\sum_{i=1}^{n}\frac{\partial\varphi}{\partial\theta_i}\frac{d\theta_i}{dl}.$$

Since $d\theta_i/dl = s_i$, we get

$$\lim_{\lambda\to\infty}\Psi_{\lambda^{-1}z(t)}(s) = \exp\{im_1 t s_1\},$$

which completes the proof of the lemma. □

5. The time spent by a customer in a filter system under busy traffic

Assume that a customer arriving at the system at a moment t_0 needs y units of service time,

$$x\left(t_0\right) = \left(x_1\left(t_0\right), x_2\left(t_0\right), \ldots, x_n\left(t_0\right)\right) = \left(x_1^* + y, x_2^*, \ldots, x_n^*\right).$$

Let S be the total time spent by the customer in the system and let W_i be the waiting time at the i-th server, $i = 1, \ldots, n$. To determine the distributions of these characteristics we consider the amount $\beta_i(t)$ of work finished by the server i until the moment $t \geq t_0$. Without loss of generality let $t_0 = 0$. Then

$$\{S > t\} = \left\{x_1^* + x_2^* + \cdots + x_n^* + y > \beta_n(t)\right\}, \tag{3}$$
$$\{W_i > t\} = \left\{x_1^* + x_2^* + \cdots + x_i^* > \beta_i(t)\right\} \tag{4}$$

by the definition of the process $\beta_i(t)$, where $d\beta_i(t) = \mu_i x_i(t)\,dt$, $\beta_i(0) = 0$. Rewriting system (1) as follows:

$$\begin{cases} dx_1(t) = dz_1(t) - \mu_1 x_1(t)\,dt, \\ dx_2(t) = \mu_1 x_1(t)\,dt - \mu_2 x_2(t)\,dt, \\ \cdots \\ dx_n(t) = \mu_{n-1}x_{n-1}(t)\,dt - \mu_n x_n(t)\,dt \end{cases} \tag{5}$$

and summing up all the equations of system (5), we obtain

$$dx_1(t) + dx_2(t) + \cdots + dx_n(t) = dz_1(t) - \mu_n x_n(t)\,dt,$$
$$\mu_n x_n(t)\,dt = dz_1(t) - dx_1(t) - dx_2(t) - \cdots - dx_n(t),$$
$$d\beta_n(t) = dz_1(t) - dx_1(t) - dx_2(t) - \cdots - dx_n(t),$$
$$\int_0^t d\beta_n(u) = \int_0^t dz_1(u) - \int_0^t dx_1(u) - \int_0^t dx_2(u) - \cdots - \int_0^t dx_n(u),$$
$$\beta_n(t) = z_1(t) - x_1(t) + x_1(0) - x_2(t) + x_2(0) - \cdots - x_n(t) + x_n(0).$$

Thus

$$\beta_n(t) = z_1(t) - x_1(t) - x_2(t) - \cdots - x_n(t) + x_1^* + x_2^* + \cdots + x_n^* + y.$$

This equality together with (3) implies that

$$\{S > t\} = \left\{x_1^* + \cdots + x_n^* + y > z_1(t) - x_1(t) - \cdots - x_n(t) + x_1^* + \cdots + x_n^* + y\right\}$$
$$= \left\{x_1(t) + x_2(t) + \cdots + x_n(t) > z_1(t)\right\},$$

whence

$$\mathsf{P}\{S > t\} = \mathsf{P}\left\{x_1(t) + x_2(t) + \cdots + x_n(t) > z_1(t)\right\}. \tag{6}$$

Proceeding in a similar way we get

$$\mathsf{P}\{W_i > t\} = \mathsf{P}\{x_1(t) + x_2(t) + \cdots + x_i(t) > z_1(t) + y\}. \tag{7}$$

Now we consider the total time T spent by a customer in the system and the waiting time W_i^s at the i-th server. In what follows we assume that the system remains in the stationary regime. The distributions of these characteristics are given by equalities (6) and (7), respectively. Moreover, the stationary distribution of

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$$

coincides with the distribution of $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ (see [3]).

Theorem 5.1. *If* $\mathsf{E}\,\eta_1 = m_1 < \infty$*, then*

$$\left(\mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_n^{-1}\right)^{-1} T \overset{\mathrm{w}}{\Rightarrow} 1 \qquad \text{as} \quad \lambda \to \infty.$$

Proof. Equality (6) implies that

$$\mathsf{P}\{T > t\} = \mathsf{P}\{\xi_1 + \xi_2 + \cdots + \xi_n > z_1(t)\} = \mathsf{P}\left\{\lambda^{-1}(\xi_1 + \xi_2 + \cdots + \xi_n) > \lambda^{-1} z_1(t)\right\}.$$

We obtain from Lemmas 4.1 and 4.2 that

$$\begin{aligned}\mathsf{P}\{T > t\} &\overset{\mathrm{w}}{\Rightarrow} \mathsf{P}\left\{m_1\mu_1^{-1} + m_1\mu_2^{-1} + \cdots + m_1\mu_n^{-1} > m_1 t\right\} \\ &= \mathsf{P}\left\{\mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_n^{-1} > t\right\} = \begin{cases} 1, & t < t_{\mathrm{cr}}^1, \\ 0, & t \geq t_{\mathrm{cr}}^1, \end{cases}\end{aligned}$$

as $\lambda \to \infty$, where $t_{\mathrm{cr}}^1 = \mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_n^{-1}$. This proves Theorem 5.1. □

Theorem 5.2. *If* $\mathsf{E}\,\eta_1 = m_1 < \infty$*, then*

$$\left(\mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_i^{-1}\right)^{-1} W_i^s \overset{\mathrm{w}}{\Rightarrow} 1 \qquad \text{as} \quad \lambda \to \infty.$$

Proof. Equality (7) implies that

$$\begin{aligned}\mathsf{P}\{W_i^s > t\} &= \mathsf{P}\{\xi_1 + \xi_2 + \cdots + \xi_i > z_1(t) + y\} \\ &= \mathsf{P}\left\{\lambda^{-1}(\xi_1 + \xi_2 + \cdots + \xi_i) > \lambda^{-1} z_1(t) + \lambda^{-1} y\right\}.\end{aligned}$$

Lemmas 4.1 and 4.2 imply that

$$\begin{aligned}\mathsf{P}\{W_i^s > t\} &\overset{\mathrm{w}}{\Rightarrow} \mathsf{P}\left\{m_1\mu_1^{-1} + m_1\mu_2^{-1} + \cdots + m_1\mu_i^{-1} > m_1 t\right\} \\ &= \mathsf{P}\left\{\mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_i^{-1} > t\right\} = \begin{cases} 1, & t < t_{\mathrm{cr}}^2, \\ 0, & t \geq t_{\mathrm{cr}}^2, \end{cases}\end{aligned}$$

as $\lambda \to \infty$, since $\lambda^{-1} y \overset{\mathsf{P}}{\to} 0$ as $\lambda \to \infty$, where $t_{\mathrm{cr}}^2 = \mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_i^{-1}$. This proves Theorem 5.2. □

6. Corollaries

Using Theorems 5.1 and 5.2 one can study the following characteristics of queueing systems under busy traffic:

- the time T_i spent by a customer in the system until the service at the i-th server is terminated;
- the time T_i' spent by a customer in the system after arriving at the i-th server;
- the total time V_i spent by a customer in the i-th server;
- the waiting time K_i at the i-th server;
- the service time S_i at the i-th server

provided that the system remains in the stationary regime.

Theorem 6.1. *If* $\mathsf{E}\,\eta_1 = m_1 < \infty$, *then* $\left(\mu_i^{-1} + \dots + \mu_n^{-1}\right)^{-1} T_i' \stackrel{\mathrm{w}}{\Rightarrow} 1$ *as* $\lambda \to \infty$.

Proof. Since one starts to count the time T_i' at the moment when the $(i-1)$-th server begins the service, we have

$$\mathsf{P}\left\{T > t\right\} = \mathsf{P}\left\{W_{i-1}^s + T_i' > t\right\}.$$

Assuming that all random variables and stochastic processes are defined on the common probability space we use Theorems 5.1 and 5.2 and complete the proof of Theorem 6.1. □

Theorem 6.2. *If* $\mathsf{E}\,\eta_1 = m_1 < \infty$, *then* $\mu_i K_i \stackrel{\mathrm{w}}{\Rightarrow} 1$ *as* $\lambda \to \infty$.

Proof. The proof follows from Theorems 5.1, 5.2, and 6.1, since

$$\mathsf{P}\{T > t\} = \mathsf{P}\left\{W_{i-1}^s + K_i + T_{i+1}' > t\right\}. \quad \square$$

Theorem 6.3. *If* $\mathsf{E}\,\eta_1 = m_1 < \infty$, *then* $\mu_i V_i \stackrel{\mathrm{w}}{\Rightarrow} 1$ *as* $\lambda \to \infty$.

Proof. Since T_i coincides with T if the system contains i servers, and

$$\mathsf{P}\left\{T_i > t\right\} = \mathsf{P}\left\{W_i^s + S_i > t\right\},$$

we have

$$S_i \stackrel{\mathrm{w}}{\Rightarrow} 0.$$

Now the proof follows from Theorem 6.2 in view of $\mathsf{P}\left\{V_i > t\right\} = \mathsf{P}\left\{K_i + S_i > t\right\}$. □

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