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THRESHOLD STRUCTURE OF OPTIMAL STOPPING STRATEGIES FOR AMERICAN TYPE OPTION. II

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Abstract. The paper presents results of theoretical studies of optimal stopping domains of American type options in discrete time. Sufficient conditions on the payoff functions and the price process for the optimal stopping domains to have one-threshold structure are given. We consider monotone, convex and inhomogeneous-in-time payoff functions. The underlying asset's price is modelled by an inhomogeneous discrete time Markov process.

1. Introduction

The present paper is a continuation of Jönsson, Kukush, and Silvestrov [5]. In the mentioned paper we presented sufficient local conditions on the payoff functions and the price process for American type put options with convex nonincreasing payoff functions such that the optimal stopping domains have a one-threshold structure.

Here we give examples of put type payoff functions and the corresponding concrete form of the sufficient condition introduced in [5].

Further, we present sufficient conditions equivalent to the conditions presented in [5] such that the optimal stopping domains for American type call options have a one-threshold structure.

All notation used and the list of related publications are introduced in [5].

2. Examples

In the following examples we will consider homogeneous-in-time payoff functions and constant interest rate, i.e., $g_n(x) = g(x)$ and $r_n = r > 0$ for all $n = 0, 1, \dots, N$.

Further we assume that the price of the underlying asset follows a multiplicative random walk, i.e., for all $n = 1, \dots, N$,

$$A_n(x, Y_n) = xY_n.$$

In particular, we have the cases:

(i) binomial random walk

$$Y_n = \begin{cases} U & \text{with prob. } p, \\ D & \text{with prob. } 1-p, \end{cases}$$

where $0 < D < e^r < U$, and $0 \le p \le 1$ is the probability for an up-movement;

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(ii) geometrical random walk with log-normal increments

$$Y_n = e^{\mu+\sigma\xi_n}, \qquad n = 1, 2, \dots,$$

where $\sigma > 0$ and ξ_n are independent standard normal random variables.

Example 1. For the standard American put option with payoff function

$$g(x) = \max\{0, K - x\},$$

condition **B2** takes the form

$$\mathsf{E}\, Y_{n+1} \le e^r,$$

for $0 \le x \le K$ and each $n = 0, 1, \dots, N-1$.

We see that condition **B2** covers the risk-neutral case $\mathsf{E}\, Y_{n+1} = e^r$ for both the binomial random walk and the geometrical random walk with log-normal increments.

Hence, the risk-neutrality is sufficient for the existence of a one-threshold structure in the case of the standard American put option. This result is known, e.g., from Merton [7], Kim [6], Jacka [2], and Broadie and Detemple [1].

Example 2. Let us consider a time-homogeneous piecewise linear payoff function with two intervals with different slopes given by

$$g(x) = \begin{cases} a_1(K_1 - x) + a_2(K_2 - K_1), & 0 \le x < K_1, \\ a_2(K_2 - x), & K_1 \le x < K_2, \\ 0, & x \ge K_2, \end{cases} \tag{1}$$

where $K_1 < K_2$ are the first and second strike prices and $0 < a_2 \le a_1$ are the scale pricing coefficients for price intervals $[0, K_1)$ and $[K_1, K_2)$, respectively. Note that for $a_2 = a_1 = 1$, we have the standard American put option with strike price K_2.

Inserting the payoff function (1) into condition **B2** gives

$$\begin{aligned} & e^r \left(-a_1 I_{[0,K_1]}(x) - a_2 I_{(K_1,K_2]}(x)\right) \\ & \quad \le \mathsf{E}\big\{-a_1 Y_{n+1} I_{[0,K_1]}(xY_{n+1})\big\} + E\big\{-a_2 Y_{n+1} I_{(K_1,K_2]}(xY_{n+1})\big\} \\ & \qquad - a_2\, \mathsf{E}\big\{Y_{n+1} I_{(K_2,\infty)}(xY_{n+1})\big\} \\ & \quad = \mathsf{E}\{-a_2 Y_{n+1}\} - (a_1 - a_2)\, \mathsf{E}\left\{Y_{n+1} I_{[0,K_1/x]}(Y_{n+1})\right\}. \end{aligned}$$

If $0 \le x \le K_1$, then

$$\mathsf{E}\{-a_2 Y_{n+1}\} - (a_1 - a_2)\, \mathsf{E}\big\{Y_{n+1} I_{[0,K_1/x]}(Y_{n+1})\big\} \ge -a_1\, \mathsf{E}\, Y_{n+1} \ge -a_1 e^r,$$

where the last inequality should hold if condition **B2** holds.

If $K_2 < x < \infty$, then

$$\begin{aligned} & \mathsf{E}\{-a_2 Y_{n+1}\} - (a_1 - a_2)\, \mathsf{E}\big\{Y_{n+1} I_{[0,K_1/x]}(Y_{n+1})\big\} \\ & \quad \ge \mathsf{E}\{-a_2 Y_{n+1}\} - (a_1 - a_2)\, \mathsf{E}\big\{Y_{n+1} I_{[0,K_1/K_2]}(Y_{n+1})\big\} \\ & \quad \ge -a_2 e^r, \end{aligned}$$

where the last inequality should hold if condition **B2** holds.

Thus the following condition should hold to fulfill condition **B2**:

F2: $\mathsf{E}\, Y_{n+1} + (a_1/a_2 - 1) E\{Y_{n+1} I_{[0,K_1/K_2]}(Y_{n+1})\} \le e^r$.

If $I_{[0,K_1/K_2]}(Y_{n+1}) \ne 0$, then combining **F2** and the fact that $a_2 \le a_1$ we get

$$a_2 \le a_1 \le a_2 \left[1 + \frac{e^r - \mathsf{E}\, Y_{n+1}}{\mathsf{E}\{Y_{n+1} I_{[0,K_1/K_2]}(Y_{n+1})\}}\right]. \tag{2}$$

There are three possible cases that can occur in (2).

First, if $\mathsf{E}\, Y_{n+1} = e^r$, then $a_1 = a_2$ is a sufficient condition for a one-threshold structure.

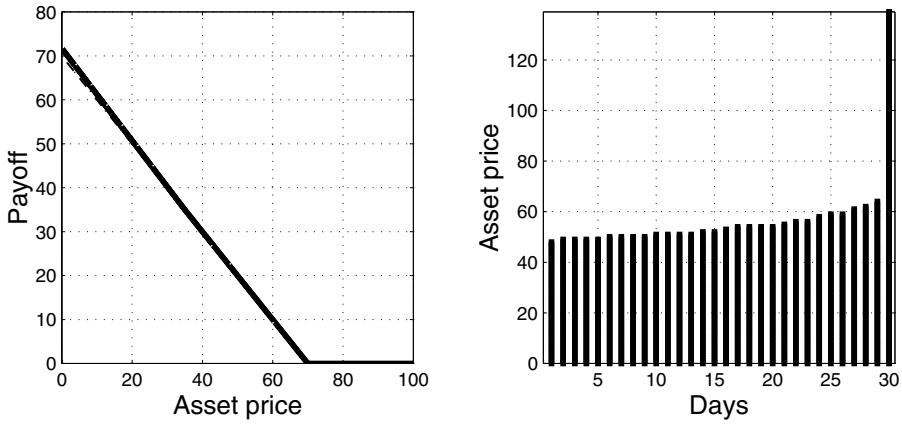

FIGURE 1. Piecewise linear payoff function with two slopes, $a_1 = 1.1$ and $a_2 = 1.0$, and strike prices $K_1 = 35$ and $K_2 = 70$, and corresponding optimal stopping domains for American type options. The values of the parameters are $\mu = 0.0$, $\sigma = 0.05$ and 4% yearly risk-free interest rate.

Second, if $\mathsf{E}\, Y_{n+1} < e^r$, then

$$c = \frac{e^r - \mathsf{E}\, Y_{n+1}}{\mathsf{E}\big\{Y_{n+1} I_{[0,K_1/K_2]}(Y_{n+1})\big\}} > 0$$

and $a_2 \le a_1 \le a_2 + c \cdot a_2$ is a sufficient condition.

Finally, if $\mathsf{E}\, Y_{n+1} > e^r$, we get $a_1 < a_2$, but this is not a function from the class $\mathbf{G}$, so we cannot say anything in this case.

For the binomial random walk, (2) takes the form

$$a_2 \le a_1 \le a_2 \frac{e^r - pU}{(1-p)D}$$

if $D \le K_1/K_2$. To assume that $U \le K_1/K_2 < 1$ does not make any sense since this implies that $e^r < 1$, i.e., $r < 0$.

If $D > K_1/K_2$ we get $\mathsf{E}\, Y_{n+1} \le e^r$ as a sufficient condition for a one-threshold structure, since $I_{[0,1]}(Y_{n+1}) = 0$ in **F2**.

If Y_{n+1} is log-normal distributed, (2) takes the form

$$a_2 \le a_1 \le a_2 + a_2 \frac{e^r - e^{\mu+\sigma^2/2}}{e^{\mu+\sigma^2/2}(\Phi(K_1/K_2 - \mu/\sigma - \sigma) - \phi(-\mu/\sigma - \sigma))}.$$

In Figure 1 and Figure 2, two examples of piecewise linear payoff functions and the corresponding optimal stopping domains are given.

An intuitive understanding of the structure of the optimal stopping domains is the following: We can assume that the change of the asset value is reasonably large, i.e., the probability of very large jumps in the asset value is small. When the asset value is approaching the first strike price K_1 from above, the probability of reaching below K_1 at moment $n+1$ will increase. This means that the expected discounted value of the future payoff will increase and finally at some level become greater than the payoff at moment n. Thus, there will exist an interval of values around the first strike price such that it is better to hold the option at least until moment $n+1$.

Note that the structures of the optimal stopping domains shown are not the only variants we can expect.

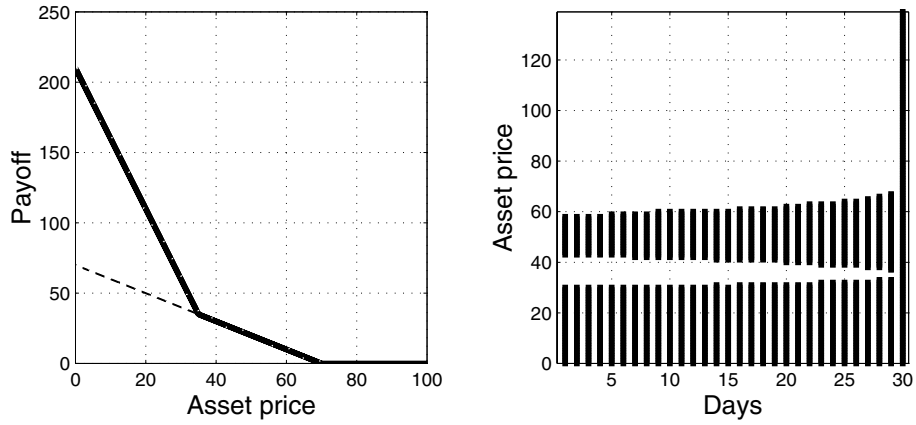

FIGURE 2. Piecewise linear payoff function with two slopes, $a_1 = 5.0$ and $a_2 = 1.0$, and strike prices $K_1 = 35$ and $K_2 = 70$, and corresponding optimal stopping domains for American type options. The values of the parameters are $\mu = 0.0$, $\sigma = 0.02$ and 8% yearly risk-free interest rate.

3. Threshold structure of optimal stopping domains for call options

In this section we consider the structure of the optimal stopping domains for American type call options.

In Jönsson, Kukush, and Silvestrov [3] the structure of the optimal stopping domains for American type call options with nonnegative, nondecreasing and convex payoff functions is investigated for a multiplicative price process, i.e., $S_n = S_{n-1}Y_n$.

Here we will present the sufficient local conditions presented in [3], for the same class of payoff functions, formulated however for the general Markov process

$$S_n = A_n(S_{n-1}, Y_n), \qquad n = 1, \dots, N, \tag{3}$$

where $A_n(x, y)$, $n = 1, 2, \dots, N$, is a measurable function acting on $\mathbb{R}^+ \times \mathbb{Y}$ to $\mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{Y}$ is a measurable space.

Instead of condition **A2** given in Jönsson, Kukush, and Silvestrov [5] we now impose the following condition on the price process:

$\tilde{\mathbf{A}}$2: *For each $y \in \mathbb{Y}$ and $n = 1, 2, \dots, N$, $A_n(x, y)$ is a nondecreasing and convex function in x.*

This assumption does not look as natural as **A2** in [5], though it holds, e.g., for the classical multiplicative model $A_n(x, Y_n) = xY_n$, $n = 1, 2, \dots, N$.

Denote by $\tilde{\mathbf{G}}$ the class of nondecreasing and convex functions acting from $\mathbb{R}^+$ to $\mathbb{R}^+$.

In the case of call options, we consider payoff functions with the following property

$\tilde{\mathbf{G}}$1: *For every $n = 0, 1, \dots, N$, the payoff function g_n belongs to the class $\tilde{\mathbf{G}}$.*

Hereafter $g'_{(+)}(x)$ denotes the right derivative for a function $g(x) \in \tilde{\mathbf{G}}$. Moreover, $g'_{(+)}(x)$ coincides with the ordinary derivative at all points where the function $g(x)$ is differentiable. Note that $g'_{(+)}(x)$ is a nonnegative and nondecreasing function.

We define the operator $\tilde{T}_n$, acting on nonnegative and nondecreasing functions $f(x)$ defined on $\mathbb{R}^+$ by the formula

$$\tilde{T}_n f(x) = \mathsf{E}\big\{ f(A_n(S_{n-1}, Y_n)) \mid S_{n-1} = x \big\} = \mathsf{E}\, f(A_n(x, Y_n)). \tag{4}$$

For convenience we repeat the definitions of the reward functions and the optimal stopping domains given in [5]. The reward functions are defined recursively starting with

$$w_0(x) = g_N(x), \tag{5}$$

and for the moments $n = N-1, \dots, 0$,

$$w_{N-n}(x) = \max\left\{g_n(x), e^{-r_{n+1}} T_{n+1} w_{N-n-1}(x)\right\}. \tag{6}$$

For $n = 0, \dots, N-1$, using equation (6), we can define the optimal stopping domains by

$$\Gamma_n = \left\{x \colon g_n(x) \geq e^{-r_{n+1}} T_{n+1} w_{N-n-1}(x)\right\}. \tag{7}$$

For the expiration date N,

$$\Gamma_N = [0, \infty).$$

The following three lemmas correspond to Lemmas 1–3 in Jönsson, Kukush, and Silvestrov [5].

Lemma 1. *If a function $g(x) \in \tilde{\mathbf{G}}$, $\tilde{T}_n g(x)$ is finite and condition $\tilde{\mathbf{A}}\mathbf{2}$ holds, then the functions $\tilde{T}_n g(x) \in \tilde{\mathbf{G}}$, $n = 1, 2, \dots, N$.*

Lemma 2. *Let $f \in \tilde{\mathbf{G}}$ and $\tilde{T}_n f$ be finite. Then, under condition $\tilde{\mathbf{A}}\mathbf{2}$, for all $x \geq 0$,*

$$(\tilde{T}_n f)'_{(+)}(x) = \mathsf{E}[f(A_n(x, Y_n))]'_{(+)} = \mathsf{E}\, f'_{(+)}(A_n(x, Y_n)) A'_{n,(+)}(x, Y_n).$$

Proof. The proof is similar to the proof of Lemma 2 in [5] and is not included here. The proof can be found in Jönsson, Kukush, and Silvestrov [4]. □

Lemma 3. *If conditions $\tilde{\mathbf{A}}\mathbf{2}$ and $\tilde{\mathbf{G}}\mathbf{1}$ hold, then for every $n = N, \dots, 0$, the reward functions $w_{N-n}(x) \in \tilde{\mathbf{G}}$.*

We would like the optimal stopping domains to have the following structure:

$$\Gamma_n = [d_n, \infty), \quad \text{with } d_n < \infty,\ n = 0, \dots, N-1; \tag{8}$$

that is, the optimal stopping domains should be closed semi-infinite intervals.

The results given in [4] and [3] show that, in general, under $\mathbf{A1}$ (see [5]), $\tilde{\mathbf{A}}\mathbf{2}$ and $\tilde{\mathbf{G}}\mathbf{1}$, an optimal stopping domain may possess more complicated structure, for example be a union of intervals.

From (7) and condition $\mathbf{A1}$, given in [5], we see that $0 \in \Gamma_n$ for all n. To avoid the situations where 0 is an isolated point in Γ_n we will redefine the optimal stopping domains as follows:

$$\tilde{\Gamma}_n = \mathrm{cl}\left(\Gamma_n \cap (0, \infty)\right), \tag{9}$$

where $\mathrm{cl}(A)$ is the closure of the set A.

Note that $\tilde{\Gamma}_n = \Gamma_n$ if 0 is a limit point of Γ_n and $\tilde{\Gamma}_n = \Gamma_n \setminus \{0\}$ if 0 is an isolated point.

We impose the following conditions:

$\tilde{\mathbf{E}}\mathbf{1}$: *For each $n = 0, 1, \dots, N-1$,*

$$\liminf_{x \to \infty} \frac{e^{-r_{n+1}} \tilde{T}_{n+1} g_{n+1}(x)}{g_n(x)} < 1.$$

$\tilde{\mathbf{E}}\mathbf{2}$: *For any $0 < d < \infty$ and for each $n = 0, 1, \dots, N-1$,*

$$\lim_{x \to \infty} \mathsf{P}\{A_{n+1}(x, Y_{n+1}) < d\} = 0.$$

We shall show that these two conditions guarantee that $\tilde{\Gamma}_n \neq \varnothing$.

The following condition plays a key role:

$\tilde{\mathbf{B}}$1: *For every $n = 0, 1, \dots, N-1$, and $x \in \mathbb{R}^+$,*

$$g'_{n,(+)}(x) \geq e^{-r_{n+1}} \mathsf{E}\, g'_{n+1,(+)}(A_{n+1}(x, Y_{n+1}))A'_{n+1,(+)}(x, Y_{n+1}).$$

Due to $\tilde{\mathbf{B}}$**1**, for each $x > 0$ and every $n = 0, 1, \dots, N-1$,

$$e^{-r_{n+1}} \mathsf{E}\, g'_{n+1,(+)}(A_{n+1}(x, Y_{n+1}))A'_{n+1,(+)}(x, Y_{n+1}) < \infty.$$

Theorem 1. *Under conditions* **A1**, $\tilde{\mathbf{A}}$**2**, $\tilde{\mathbf{G}}$**1**, $\tilde{\mathbf{B}}$**1**, $\tilde{\mathbf{E}}$**1** *and* $\tilde{\mathbf{E}}$**2**, *for each*

$$n = 0, 1, \dots, N-1,$$

there exists $0 \leq d_n < \infty$ such that the optimal stopping domain has the one-threshold structure

$$\tilde{\Gamma}_n = [d_n, \infty).$$

Proof. The structure of the optimal stopping domains given in (8) corresponds to the following two properties of the optimal stopping domain: **(a)** $\tilde{\Gamma}_n \neq \varnothing$ and closed; **(b)** if $x' \in \tilde{\Gamma}_n$, then $x'' \in \tilde{\Gamma}_n$ for any $x' \leq x'' < \infty$.

Consider moment $N-1$. From condition $\tilde{\mathbf{E}}$**1** we know that there exists an increasing sequence $\{x_k\}$ such that $x_k \to \infty$ and, for all $k \geq 1$,

$$g_{N-1}(x_k) \geq (1+\lambda)e^{-r_N}\tilde{T}_N g_N(x_k), \tag{10}$$

with $\lambda > 0$ independent of k.

Relation (10) implies that $x_k \in \tilde{\Gamma}_{N-1}$, for all $k \geq 1$, i.e., $\tilde{\Gamma}_{N-1} \neq \varnothing$.

Furthermore, $\tilde{\Gamma}_{N-1}$ is bounded below by 0. Thus there exists a point

$$0 \leq d_{N-1} := \inf\{x \colon \tilde{\Gamma}_{N-1}\} < \infty.$$

Moreover, the functions $g_{N-1}(x)$ and $e^{-r_N}\tilde{T}_N g_N(x)$ are continuous by condition $\tilde{\mathbf{G}}$**1** and Lemma 1, respectively. It follows from the continuity of these functions and the defining formula (7) that $\tilde{\Gamma}_{N-1}$ is closed, and, therefore, $d_{N-1} \in \tilde{\Gamma}_{N-1}$. Thus property **(a)** holds for $\tilde{\Gamma}_{N-1}$.

By condition $\tilde{\mathbf{B}}$**1** and Lemma 2, for any $x \geq 0$,

$$\begin{aligned} g'_{N-1,(+)}(x) &\geq e^{-r_N} \mathsf{E}\, g'_{N,(+)}(A_N(x, Y_N))A'_{N,(+)}(x, Y_N) \\ &= (e^{-r_N}\tilde{T}_N w_0)'_{(+)}(x). \end{aligned} \tag{11}$$

Assume that $x' \in \tilde{\Gamma}_{N-1}$. By this assumption and relation (11), for any $x' \leq x'' < \infty$,

$$\begin{aligned} g_{N-1}(x'') &= g_{N-1}(x') + \int_{x'}^{x''} g'_{N-1,(+)}(t)\,dt \\ &\geq e^{-r_N}\tilde{T}_N w_0(x') + \int_{x'}^{x''} e^{-r_N}(\tilde{T}_N w_0)'_{(+)}(t)\,dt \\ &= e^{-r_N}\tilde{T}_N w_0(x''). \end{aligned}$$

Thus, **(b)** holds for $\tilde{\Gamma}_{N-1}$.

Hence, we have proved that $\tilde{\Gamma}_{N-1}$ has the one-threshold structure described in (8).

We now continue with moment $N-2$. From condition $\tilde{\mathbf{E}}$**1** we know that there exists an increasing sequence $\{x_k\}$, such that $x_k \to \infty$ and, for all $k \geq 1$,

$$g_{N-2}(x_k) \geq (1+\lambda)e^{-r_{N-1}}\tilde{T}_{N-1} g_{N-1}(x_k), \tag{12}$$

with $\lambda > 0$ independent of k. By the definition of $\tilde{T}_{N-1}$ and w_1,

$$\begin{aligned}\tilde{T}_{N-1}w_1(x_k) &= \mathsf{E}\, g_{N-1}(A_{N-1}(x_k, Y_{N-1}))I(A_{N-1}(x_k, Y_{N-1}) \geq d_{N-1}) \\ &\quad + \mathsf{E}\, e^{-r_N}\tilde{T}_N w_0(A_{N-1}(x_k, Y_{N-1}))I(A_{N-1}(x_k, Y_{N-1}) < d_{N-1}) \\ &:= I_1(x_k) + I_2(x_k).\end{aligned}$$

Consider the inequality

$$(1+\lambda)e^{-r_{N-1}}\tilde{T}_{N-1}g_{N-1}(x_k) \geq e^{-r_{N-1}}(I_1(x_k) + I_2(x_k)). \tag{13}$$

From the definition of $\tilde{T}_{N-1}$,

$$\begin{aligned}&e^{-r_{N-1}}T_{N-1}g_{N-1}(x_k) \\ &\quad = e^{-r_{N-1}}\,\mathsf{E}\, g_{N-1}(A_{N-1}(x_k, Y_{N-1})) \\ &\quad \geq e^{-r_{N-1}}\,\mathsf{E}\, g_{N-1}(A_{N-1}(x_k, Y_{N-1}))I(A_{N-1}(x_k, Y_{N-1}) \geq d_{N-1}) \\ &\quad = e^{-r_{N-1}}I_1(x_k).\end{aligned} \tag{14}$$

Note that $A_{N-1}(x_k, Y_{N-1}) \xrightarrow{\mathsf{P}} \infty$ as $x_k \to \infty$, since $A_{N-1}(x, Y_{N-1})$ is nondecreasing and convex in x and $\mathsf{P}\{A_{N-1}(x, Y_{N-1}) < d\} \to 0$ as $x \to \infty$ for any $0 < d < \infty$ by $\tilde{\mathbf{E}}\mathbf{2}$. Furthermore, $g_{N-1}(x) \to \infty$ as $x \to \infty$, since $g_{N-1} \in \tilde{G}$. Thus,

$$I_1(x_k) = \mathsf{E}\, g_{N-1}(A_{N-1}(x_k, Y_{N-1}))I(A_{N-1}(x_k, Y_{N-1}) \geq d_{N-1}) \to \infty, \tag{15}$$

as $k \to \infty$. From (15) we get that there exists an integer $k_1 \geq 1$ such that $I_1(x_k) > 0$, for all $k \geq k_1$, since $I_1(x_k)$ is nondecreasing.

Thus (14) implies that for $k \geq k_1$,

$$\lambda e^{-r_{N-1}}\tilde{T}_{N-1}g_{N-1}(x_k) \geq \lambda e^{-r_{N-1}}I_1(x_k) \geq \lambda e^{-r_{N-1}}I_1(x_{k_1}) > 0. \tag{16}$$

Consider $I_2(x_k)$. Since $\tilde{T}_N w_0(x)$ and $A_{N-1}(x, Y_{N-1})$ are nondecreasing in x and by condition $\tilde{\mathbf{E}}\mathbf{2}$,

$$\begin{aligned}I_2(x_k) &= \mathsf{E}\, e^{-r_N}\tilde{T}_N w_0(A_{N-1}(x, Y_{N-1}))I(A_{N-1}(x, Y_{N-1}) < d_{N-1}) \\ &\leq e^{-r_N}\tilde{T}_N w_0(d_{N-1})\,\mathsf{P}\{A_{N-1}(x, Y_{N-1}) < d_{N-1}\} \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Thus there exists an integer $k_2 \geq 1$ such that for all $k \geq k_2$,

$$\lambda e^{-r_{N-1}}T_{N-1}g_{N-1}(x_k) > e^{-r_{N-1}}I_2(x_k). \tag{17}$$

Hence, it follows from (14) and (17) that there exists an integer $k_0 = \max\{k_1, k_2\}$ such that (13) holds for all $k \geq k_0$ and by (12), for all $k \geq k_0$,

$$g_{N-2}(x_k) \geq e^{-r_{N-1}}\tilde{T}_{N-1}w_1(x_k). \tag{18}$$

Relation (18) implies that, for all $k \geq k_0$, $x_k \in \tilde{\Gamma}_{N-2}$, i.e., $\tilde{\Gamma}_{N-2} \neq \varnothing$.

Since $\tilde{\Gamma}_{N-2} \neq \varnothing$ and bounded below, there exists a point

$$0 \leq d_{N-2} := \inf\{x \colon x \in \tilde{\Gamma}_{N-2}\} < \infty.$$

From condition $\tilde{\mathbf{G}}\mathbf{1}$ and Lemmas 1 and 3, respectively, the functions $g_{N-2}(x)$ and $\tilde{T}_{N-1}w_1$ are continuous. By the continuity of these functions and the defining formula (7), $\tilde{\Gamma}_{N-2}$ is closed and, therefore, $d_{N-2} \in \tilde{\Gamma}_{N-2}$. Thus property (a) holds for $\tilde{\Gamma}_{N-2}$.

From Lemmas 2 and 3, we have

$$\big(\tilde{T}_{N-1}w_1\big)'_{(+)}(x) = \mathsf{E}[w_1(A_{N-1}(x, Y_{N-1}))]'_{(+)}. \tag{19}$$

We now have to consider two cases concerning the expression on the right-hand side.

First, consider the case $A_{N-1}(x, Y_{N-1}) \geq d_{N-1}$. This is true if and only if $x \geq x_*(Y_{N-1})$, where $x_*(Y_{N-1}) = \inf\{x \geq 0\colon A_{N-1}(x, Y_{N-1}) \geq d_{N-1}\}$. Then, for $x \geq x_*(Y_{N-1})$,

$$w_1(A_{N-1}(x, Y_{N-1})) = g_{N-1}(A_{N-1}(x, Y_{N-1}))$$

and

$$\begin{aligned} \big[w_1(A_{N-1}(x, Y_{N-1}))\big]'_{(+)} &= \big(g_{N-1}(A_{N-1}(x, Y_{N-1}))\big)'_{(+)} \\ &= g'_{N-1,(+)}(A_{N-1}(x, Y_{N-1}))A'_{N-1,(+)}(x, Y_{N-1}). \end{aligned} \tag{20}$$

Second, consider the case $A_{N-1}(x, Y_{N-1}) < d_{N-1}$. This is true if and only if $x < x_*(Y_{N-1})$. Then

$$w_1(A_{N-1}(x, Y_{N-1})) = e^{-r_N}\tilde{T}_N g_N(z)\big|_{z=A_{N-1}(x,Y_{N-1})},$$

and by Lemma 2 and condition $\tilde{\mathbf{B}}\mathbf{1}$,

$$\begin{aligned} &\big[w_1(A_{N-1}(x, Y_{N-1}))\big]'_{(+)} \\ &\quad = e^{-r_N}\big(\tilde{T}_N g_N\big)'_{(+)}(z)\big|_{z=A_{N-1}(x,Y_{N-1})} \cdot A'_{N-1,(+)}(x, Y_{N-1}) \\ &\quad \leq g'_{N-1,(+)}(A_{N-1}(x, Y_{N-1}))A'_{N-1,(+)}(x, Y_{N-1}). \end{aligned} \tag{21}$$

Using relations (20) and (21) in (19) we get, for any $x \geq 0$,

$$\big(\tilde{T}_{N-1}w_1\big)'_{(+)}(x) \leq \mathsf{E}\, g'_{N-1,(+)}(A_{N-1}(x, Y_{N-1}))A'_{N-1,(+)}(x, Y_{N-1}),$$

and by condition $\tilde{\mathbf{B}}\mathbf{1}$, for any $x \geq 0$,

$$g'_{N-2,(+)}(x) \geq e^{-r_{N-1}}\big(\tilde{T}_{N-1}w_1\big)'_{(+)}(x). \tag{22}$$

Assume that $x' \in \tilde{\Gamma}_{N-2}$. By this assumption and relation (22), for any $x' \leq x'' < \infty$,

$$\begin{aligned} g_{N-2}(x'') &= g_{N-2}(x') + \int_{x'}^{x''} g'_{N-2,(+)}(t)\,dt \\ &\geq e^{-r_{N-1}}\tilde{T}_{N-1}w_1(x') + \int_{x'}^{x''} e^{-r_{N-1}}\big(\tilde{T}_{N-1}w_1\big)'_{(+)}(t)\,dt \\ &= e^{-r_{N-1}}\tilde{T}_{N-1}w_1(x''). \end{aligned}$$

Thus, **(b)** holds for $\tilde{\Gamma}_{N-2}$.

Hence, we have proved that $\tilde{\Gamma}_{N-2}$ has the structure described in (8).

Repeating the induction reasoning used for the case $N-2$, etc., it is possible to prove that the stopping domain $\tilde{\Gamma}_n$ has the structure given in (8) for all $n = N-3, \dots, 0$. □

Remark. Without $\tilde{\mathbf{E}}\mathbf{1}$ and $\tilde{\mathbf{E}}\mathbf{2}$, the structure of the stopping domains is still as in (8), but in this case, d_n can equal ∞, which means that $\tilde{\Gamma}_n = \varnothing$.

Remark. If the payoff functions $g_n(0) \equiv 0$ for all $n = 0, \dots, N$, then $d_n = 0$ for all $n = 0, 1, \dots, N$, by condition $\tilde{\mathbf{B}}\mathbf{1}$.

The following condition guarantees that $d_n > 0$ for all $n = 0, 1, \dots, N-1$:

$\tilde{\mathbf{D}}\mathbf{1}$: *For every $n = 0, \dots, N-1$,*

$$g_n(0) < \min_{n+1 \leq k \leq N} e^{-R_{n+1,k}} g_k(0),$$

where $R_{n+1,k} = r_{n+1} + \dots + r_k, 1 \leq n+1 \leq k \leq N$.

Indeed, by $\tilde{\mathbf{D}}\mathbf{1}$, condition $\mathbf{A1}$, the fact that $g_{n+1}(x)$ is nondecreasing and that $g_{n+1}(x) \le w_{N-(n+1)}(x)$ for all $x \ge 0$, we have

$$\begin{aligned} g_n(0) &< \min_{n+1\le k\le N} e^{-R_{n+1,k}} g_k(0) \le e^{-r_{n+1}} g_{n+1}(0) \\ &= e^{-r_{n+1}} \,\mathsf{E}\, g_{n+1}(A_{n+1}(0, Y_{n+1})) \\ &\le e^{-r_{n+1}} \,\mathsf{E}\, g_{n+1}(A_{n+1}(x, Y_{n+1})) \\ &\le e^{-r_{n+1}} \,\mathsf{E}\, w_{N-(n+1)}(A_{n+1}(x, Y_{n+1})) \\ &= e^{-r_{n+1}} T_{n+1} w_{N-(n+1)}(x) \end{aligned} \tag{23}$$

for every $x \ge 0$.

By continuity of the functions $g_n(x)$ and $\tilde{T}_{n+1} w_{N-(n+1)}(x)$, there exists a $\delta > 0$ such that for every $x \in [0, \delta)$: $g_n(x) > e^{-r_{n+1}} \tilde{T}_{n+1} w_{N-(n+1)}(x)$. Hence, $d_n > 0$.

An example of a payoff function that satisfies condition $\tilde{\mathbf{D}}\mathbf{1}$ is

$$g_n(x) = a_n + b_n x$$

with $0 < a_1 < \cdots < a_N$ and $b_n > 0$ for all n. Here a_n can be interpreted as a riskless profit assured by the writer of the option.

Theorem 1 is applied for payoff functions with the following property:

$$g'_{n,(+)}(x) > 0, \qquad x > 0,\ n = 0, 1, \ldots, N. \tag{24}$$

However, property (24) is not fulfilled by, for example, the standard payoff function $g_n(x) = [x - K_n]^+$ with $K_n > 0$ and other payoff functions with the following property:

$$g_n(x) = 0, \qquad 0 \le x \le K_n,\ n = 0, 1, \ldots, N. \tag{25}$$

In this case condition $\tilde{\mathbf{B}}\mathbf{1}$ can fail to hold for $0 < x < K_n$. Hence, Theorem 1 needs to be modified.

We now consider payoff functions with the following additional property.

$\tilde{\mathbf{G}}\mathbf{2}$**:** *For $n = 0, 1, \ldots, N$, and each $0 \le x \le K_n$, $g_n(x) = 0$.*

For payoff functions with property $\tilde{\mathbf{G}}\mathbf{2}$ we modify condition $\tilde{\mathbf{B}}\mathbf{1}$ in the following way:

$\tilde{\mathbf{B}}\mathbf{2}$**:** *For every $n = 0, 1, \ldots, N-1$, and all $x \ge K_n$,*

$$\begin{aligned} g'_{n,(+)}(x) \ge e^{-r_{n+1}} \Big[&\mathsf{E}\, g'_{n+1,(+)}(A_{n+1}(x, Y_{n+1})) A'_{n+1,(+)}(x, Y_{n+1}) \\ &+ g'_{n+1,(+)}(K_{n+1}) \,\mathsf{E}\, A'_{n+1,(+)}(x, Y_{n+1}) I_{[0,K_{n+1})}(A_{n+1}(x, Y_{n+1})) \Big]. \end{aligned}$$

Note that under natural assumptions, for all $n = 0, \ldots, N-1$, at point $x = K_n$ we have

$$\mathsf{E}\, g'_{n+1,(+)}(A_{n+1}(x, Y_{n+1})) A'_{n+1,(+)}(x, Y_{n+1}) > 0.$$

In that case $\tilde{\mathbf{B}}_{\mathbf{2}}$ implies that $g_n(K_n) > 0$ for every $n = 0, \ldots, N-1$. Consider, for example, the payoff function $g_n(x) = [x - K_n]^+$.

Theorem 2. *Under conditions* $\mathbf{A1}$, $\tilde{\mathbf{A}}\mathbf{2}$, $\tilde{\mathbf{G}}\mathbf{1}$, $\tilde{\mathbf{G}}\mathbf{2}$, $\tilde{\mathbf{B}}\mathbf{2}$, $\tilde{\mathbf{E}}\mathbf{1}$ *and* $\tilde{\mathbf{E}}\mathbf{2}$, *for each*

$$n = 0, 1, \ldots, N-1,$$

there exists $0 \le d_n < \infty$ such that the optimal stopping domain has the one-threshold structure

$$\tilde{\Gamma}_n = [d_n, \infty).$$

Proof. The proof is based on the same idea as the proof of Theorem 3, i.e., for each moment $n = 0, 1, \dots, N-1$, we need to show that the optimal stopping domain $\tilde{\Gamma}_n$ fulfills: **(a)** $\tilde{\Gamma}_n \neq \varnothing$ and closed; **(b)** if $x' \in \tilde{\Gamma}_n$, then $x'' \in \tilde{\Gamma}_n$ for any $x' \le x'' < \infty$.

Consider moment $N-1$. As in the proof of Theorem 3, using $\tilde{\mathbf{E}}\mathbf{1}$, we can show that $\tilde{\Gamma}_{N-1}$ is a closed nonempty set, i.e., property **(a)** holds for the optimal stopping domain $\tilde{\Gamma}_{N-1}$.

Furthermore, $\tilde{\Gamma}_{N-1}$ is bounded below by K_{N-1}. Thus, there exists a point d_{N-1} such that

$$K_{N-1} \le d_{N-1} = \inf\bigl\{x\colon x \in \tilde{\Gamma}_{N-1}\bigr\} < \infty.$$

$\tilde{\Gamma}_{N-1}$ is closed, and therefore $d_{N-1} \in \tilde{\Gamma}_{N-1}$.

By condition $\tilde{\mathbf{B}}\mathbf{2}$ and Lemma 2, for any $x \ge K_{N-1}$,

$$\begin{aligned} g'_{N-1,(+)}(x) &\ge e^{-r_N}\,\mathsf{E}\, g'_{N,(+)}(A_N(x,Y_N))A'_{N,(+)}(x,Y_N) \\ &\quad + e^{-r_N} g'_{N,(+)}(K_N)\,\mathsf{E}\, A'_{N,(+)}(x,Y_N) I_{[0,K_N)}(A_N(x,Y_N)) \\ &\ge e^{-r_N}\,\mathsf{E}\, g'_{N,(+)}(A_N(x,Y_N))A'_{N,(+)}(x,Y_N) \\ &= \bigl(e^{-r_N}\tilde{T}_N w_0\bigr)'_{(+)}(x), \end{aligned} \tag{26}$$

since $g'_{N,(+)}(K_N) > 0$ and $A'_{N,(+)}(x,Y_N) > 0$ with probability 1.

Assume that $x' \in \tilde{\Gamma}_{N-1}$. By this assumption and relation (26), for any $x' \le x'' < \infty$,

$$\begin{aligned} g_{N-1}(x'') &= g_{N-1}(x') + \int_{x'}^{x''} g'_{N-1,(+)}(t)\,dt \\ &\ge e^{-r_N}\tilde{T}_N w_0(x') + \int_{x'}^{x''} e^{-r_N}\bigl(\tilde{T}_N w_0\bigr)'_{(+)}(t)\,dt \\ &= e^{-r_N}\tilde{T}_N w_0(x''). \end{aligned}$$

Thus, **(b)** holds for $\tilde{\Gamma}_{N-1}$.

Hence, we have proved that $\tilde{\Gamma}_{N-1}$ has the structure described in (8).

Consider now moment $N-2$. As in the proof of Theorem 1, using $\tilde{\mathbf{E}}\mathbf{1}$, $\tilde{\mathbf{E}}\mathbf{2}$ and the continuity of $g_{N-2}(x)$ and $\tilde{T}_{N-1}w_1$, we can show that $\tilde{\Gamma}_{N-2}$ is a closed nonempty set.

Since $\tilde{\Gamma}_{N-2}$ is bounded below, there exists a point d_{N-2} such that

$$K_{N-2} \le d_{N-2} := \inf\bigl\{x\colon x \in \tilde{\Gamma}_{N-2}\bigr\} < \infty.$$

$\tilde{\Gamma}_{N-2}$ is closed, and therefore $d_{N-2} \in \tilde{\Gamma}_{N-2}$. By Lemmas 1–3, we have

$$\bigl(\tilde{T}_{N-1}w_1\bigr)'_{(+)}(x) = \mathsf{E}\,[w_1(A_{N-1}(x,Y_{N-1}))]'_{(+)}\,.$$

We must consider two cases.

First, consider the case $A_{N-1}(x,Y_{N-1}) \ge d_{N-1}$. This is true if and only if $x \ge x_*$, where

$$x_* = \inf\{x\colon A_{N-1}(x,Y_{N-1}) \ge d_{N-1}\}.$$

Then for $x \ge x_*$,

$$[w_1(A_{N-1}(x,Y_{N-1}))]'_{(+)} = g'_{N-1,(+)}(A_{N-1}(x,Y_{N-1}))A'_{N-1,(+)}(x,Y_{N-1}). \tag{27}$$

Second, consider the case $0 \le A_{N-1}(x,Y_{N-1}) < d_{N-1}$. This is true if and only if $x < x_*$. Then for $x < x_*$,

$$w_1(A_{N-1}(x,Y_{N-1})) = e^{-r_N}\tilde{T}_N g_N(z)\big|_{z=A_{N-1}(x,Y_{N-1})}.$$

If $K_{N-1} \le A_{N-1}(x, Y_{N-1}) < d_{N-1}$, then by Lemma 2 and by condition $\tilde{\mathbf{B}}\mathbf{2}$,

$$\begin{aligned}
&\big[w_1(A_{N-1}(x, Y_{N-1}))\big]'_{(+)} \\
&\quad = e^{-r_N}(T_N g_N(z))'_{(+)} \cdot A'_{N-1,(+)}(x, Y_{N-1}) \\
&\quad = e^{-r_N}\,\mathsf{E}\big[g'_{N,(+)}(A_N(z, Y_N)) \cdot A'_{N,(+)}(z, Y_N)\big] \cdot A'_{N-1,(+)}(x, Y_{N-1}) \\
&\quad \le g'_{N-1,(+)}(A_{N-1}(x, Y_{N-1})) \cdot A'_{N-1,(+)}(x, Y_{N-1}),
\end{aligned} \tag{28}$$

where $z = A_{N-1}(x, Y_{N-1})$. If

$$0 \le A_{N-1}(x, Y_{N-1}) < K_{N-1},$$

then, by Lemma 2,

$$\begin{aligned}
&\big[w_1(A_{N-1}(x, Y_{N-1}))\big]'_{(+)} \\
&\quad = e^{-r_N}\big(\tilde{T}_N g_N(z)\big)'_{(+)}\big|_{z=A_{N-1}(x,Y_{N-1})} \cdot A'_{N-1,(+)}(x, Y_{N-1}) \\
&\quad \le e^{-r_N}\big(\tilde{T}_N g_N\big)'_{(+)}(K_{N-1}) \cdot A'_{N-1,(+)}(x, Y_{N-1}) \\
&\quad \le g'_{N-1,(+)}(K_{N-1}) \cdot A'_{N-1,(+)}(x, Y_{N-1}),
\end{aligned} \tag{29}$$

since $(\tilde{T}_N g_N)'_{(+)}(x)$ is nondecreasing in x and the last inequality holds by (26).

Thus, by (27), (28) and (29), for any $x \ge 0$,

$$\begin{aligned}
&[w_1(A_{N-1}(x, Y_{N-1}))]'_{(+)} \\
&\quad \le g'_{N-1,(+)}\left(A_{N-1}(x, Y_{N-1})\right) A'_{N-1,(+)}(x, Y_{N-1}) \\
&\qquad + g'_{N-1,(+)}(K_{N-1}) A'_{N-1,(+)}(x, Y_{N-1}) I_{[0,K_{N-1})}\left(A_{N-1}(x, Y_{N-1})\right).
\end{aligned}$$

Hence, by condition $\tilde{\mathbf{B}}\mathbf{2}$, for all $x \ge K_{N-2}$,

$$e^{-r_{N-1}}\,\mathsf{E}\left[w_1(A_{N-1}(x, Y_{N-1}))\right]'_{(+)} \le g'_{N-2,(+)}(x). \tag{30}$$

Assume that $x' \in \tilde{\Gamma}_{N-2}$. By this assumption and relation (30), for any $x' \le x'' < \infty$,

$$\begin{aligned}
g_{N-2}(x'') &= g_{N-2}(x') + \int_{x'}^{x''} g'_{N-2,(+)}(t)\,dt \\
&\ge e^{-r_{N-1}}\tilde{T}_{N-1} w_1(x') + \int_{x'}^{x''} e^{-r_{N-1}}(\tilde{T}_{N-1} w_1)'_{(+)}(t)\,dt \\
&= e^{-r_{N-1}}\tilde{T}_{N-1} w_1(x'').
\end{aligned}$$

Thus, **(b)** holds for $\tilde{\Gamma}_{N-2}$.

Hence, we have proved that $\tilde{\Gamma}_{N-2}$ has the structure described in (8).

By repeating the induction reasoning used for the case $N-2$, etc., it is possible to prove that the stopping domain $\tilde{\Gamma}_n$ has the structure given in (8) for all $n = N-3, \dots, 0$. □

Remark. Let $0 < K_n < \infty$ for all $n = 0, \dots, N$, but assume that condition $\tilde{\mathbf{G}}\mathbf{2}$ does not hold for the payoff functions g_n. Then the following will be known about the structure of the optimal stopping domains:

$$\tilde{\Gamma}_n \cap [K_n, \infty) = [d_n, \infty).$$

But we cannot say anything about the structure of the optimal stopping domain below K_n.

In conclusion we would like to make the following general remark. If we know the model of the market price of the option at the intermediate moments, then the problem of the optimal reselling of American type options (either put or call) is a separate and interesting problem. But if we do not possess such a model, but reselling is still possible, then the following advice to the holder of the option can be given. Find the stopping time τ due to the optimal stopping strategy considered in this paper, then compare the market price of the option at moment τ with the exercise payoff at this moment. If the market price is higher, then resell, otherwise exercise the option. Of course such a strategy could be nonoptimal provided the model of the market price of the option is given. But this strategy gives a lower bound of the expected profit in the reselling problem, and the strategy will be optimal in the min-max sense.

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