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BOUNDED LAW OF THE ITERATED LOGARITHM FOR SUMS OF INDEPENDENT RANDOM VECTORS NORMALIZED BY MATRICES

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Abstract. Let $(X_n, n \geq 1)$ be a sequence of independent centered random vectors in $\mathbf{R}^d$ with finite moments of order $p \in (2, 3]$ and let $(A_n, n \geq 1)$ be a sequence of $m \times d$ matrices. We find explicit conditions under which

$$\limsup_{n\to\infty} c_n \left\| A_n \sum_{i=1}^{n} X_i \right\| < \infty$$

almost surely, where $(c_n, n \geq 1)$ is some sequence of positive numbers.

1. Introduction

Throughout the paper $\mathbf{R}^d$ denotes the Euclidean space of vector columns equipped with the norm

$$\|x\| = \left(x^\top x\right)^{1/2};$$

the symbol $\top$ stands for the transposition. The symbols $\|A\|$, $|A|$, and $\operatorname{tr} A$ for an arbitrary matrix A mean its Euclidean norm, determinant, and trace, respectively. Put $\chi(t) = (\ln\ln t)^{1/2}$ for $t \geq e^e$ and $\chi(t) = 1$ for $t < e^e$.

Let $(X_n, n \geq 1)$ be a sequence of independent centered random vectors in $\mathbf{R}^d$. Put

$$S_n = \sum_{i=1}^{n} X_i, \qquad B_n = \mathsf{E}\left(S_n S_n^\top\right), \qquad n \geq 1.$$

Assume that $|B_{n_0}| \neq 0$ for some $n_0 \geq 1$ and denote by $B_n^{-1/2}$ the square root of the inverse matrix B_n^{-1}, $n \geq n_0$. The following result is proved in [1].

Theorem 1.1. *Let*

$$\lim_{n\to\infty} \|B_n\| = \infty; \tag{1}$$

$$\limsup_{n\to\infty} \|B_{n+1}\| / \|B_n\| < \infty; \tag{2}$$

there are positive constants c and τ such that

$$\left\| B_n^{-1/2} B_m^{1/2} \right\| \leq c \left(|B_m| \,/\, |B_n| \right)^\tau \tag{3}$$

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for all $n \geq m \geq n_0$ *and*

$$\sum_{i=1}^{\infty} \chi^{-3}\left(\|B_i\|\right) \mathsf{E}\left\|B_i^{-1/2} X_i\right\|^3 < \infty.$$

Then

$$\limsup_{n\to\infty} \chi^{-1}\left(\|B_n\|\right) \left\|B_n^{-1/2} S_n\right\| = \sqrt{2} \quad \textit{almost surely.}$$

Below we prove that if conditions (2) and (3) are excluded from the set of assumptions of Theorem 1.1, then the bounded law of the iterated logarithm holds instead of the law of the iterated logarithm, that is,

$$\limsup_{n\to\infty} \chi^{-1}\left(\|B_n\|\right) \left\|B_n^{-1/2} S_n\right\| = L \quad \text{almost surely,} \tag{4}$$

where L is some nonrandom constant of the interval $[0, +\infty)$. We also show that the bounded law of the iterated logarithm (4) holds if a more general condition, $\mathsf{E}\,\|X_n\|^p < \infty$ for some $p \in (2, 3]$ and all $n \geq 1$, is substituted for the condition $\mathsf{E}\,\|X_n\|^3 < \infty$, $n \geq 1$.

2. Main results

First we obtain a general result.

Theorem 2.1. *Let* $(A_n, n \geq 1)$ *be a sequence of* $m \times d$ *matrices. Assume that there are two sequences of positive numbers* $(\varphi_n, n \geq 1)$ *and* $(f_n, n \geq 1)$ *such that* $(f_n, n \geq 1)$ *is increasing to infinity and*

- *for all* $n > k \geq k_0 \geq 1$,

$$\sum_{i=k+1}^{n} \mathsf{E}\,\|A_n X_i\|^2 \leq \varphi_n \left(1 - \frac{f_k}{f_n}\right); \tag{5}$$

$$\lim_{n\to\infty} \left(\varphi_n^{1/2} \chi\left(f_n\right)\right)^{-1} \|A_n\| = 0; \tag{6}$$

- *for some* $p \in (2, 3]$,

$$\sum_{i=1}^{\infty} \sup_{n\geq i} \left[\left(\varphi_n^{1/2} \chi\left(f_n\right)\right)^{-p} \mathsf{E}\,\|A_n X_i\|^p\right] < \infty. \tag{7}$$

Then

$$\limsup_{n\to\infty} \left(\varphi_n^{1/2} \chi\left(f_n\right)\right)^{-1} \|A_n S_n\| = L \quad \textit{almost surely,}$$

where L *is some nonrandom constant of the interval* $[0, +\infty)$.

We need two auxiliary results to prove Theorem 2.1. By $\mathfrak{N}$ we denote the set of nondecreasing sequences of natural numbers whose limit is infinite. The members of $\mathfrak{N}$ are not necessarily strictly increasing sequences.

Lemma 2.1 ([2]). *Let* $(\tilde{A}_n, n \geq 1)$ *be a sequence of* $m \times d$ *matrices such that*

$$\lim_{n\to\infty} \left\|\tilde{A}_n\right\| = 0; \tag{8}$$

for any sequence $(n_j, j \geq 1) \in \mathfrak{N}$, *there exists* $p \in (2, 3]$ *such that*

$$\sum_{j=2}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} \mathsf{E}\left\|\tilde{A}_{n_j} X_i\right\|^p < \infty; \tag{9}$$

for any sequence $(n_j, j \geq 1) \in \mathfrak{N}$, *there exists* $\varepsilon > 0$ *such that*

$$\sum_{j=2}^{\infty} \exp\left[-\varepsilon\Big(\sum_{i=n_{j-1}+1}^{n_j} \mathsf{E}\big\|\tilde{A}_{n_j} X_i\big\|^2\Big)^{-1}\right] < \infty. \tag{10}$$

Then

$$\limsup_{n\to\infty}\big\|\tilde{A}_n S_n\big\| = L \quad \textit{almost surely},$$

where L *is some nonrandom constant of the interval* $[0, +\infty)$.

Remark 2.1. All the expressions in Lemmas 2.1 and 2.2 are well defined if we agree that $e^{-1/0} = 0$.

Lemma 2.2. *Let* $(a_j, j \geq 1)$ *be a nondecreasing sequence of positive numbers whose limit is infinite and* $a_1 \geq e^e$. *Then*

$$\sum_{j=2}^{\infty} \exp\left[-\varepsilon \cdot \chi^2(a_j) \cdot \left(1 - \frac{a_{j-1}}{a_j}\right)^{-1}\right] < \infty$$

for all $\varepsilon > 2$.

Proof. Put

$$\upsilon_j = \exp\left[-\varepsilon\chi^2(a_j)\left(1 - \frac{a_{j-1}}{a_j}\right)^{-1}\right], \qquad j \geq 2.$$

It is clear that $e^{-\varepsilon/(ab)} \leq (b/\varepsilon)e^{-\varepsilon/a}$ for all $\varepsilon > 0$ and $a, b > 0$ such that $a + b \leq 1$. Thus

$$\begin{aligned}
\upsilon_j &= \exp\left[\frac{-\varepsilon/2}{\chi^2(a_j) \cdot \frac{1}{2}(1 - a_{j-1}/a_j)}\right] \leq \frac{1}{\varepsilon}\left(1 - \frac{a_{j-1}}{a_j}\right)\exp\left(-\frac{1}{2}\varepsilon\chi^2(a_j)\right) \\
&= \frac{1}{\varepsilon}\frac{(\ln a_j)^{-\varepsilon/2}}{a_j}(a_j - a_{j-1}) = \frac{1}{\varepsilon}\int_{a_{j-1}}^{a_j} a_j^{-1}(\ln a_j)^{-\varepsilon/2}\,dt \\
&\leq \frac{1}{\varepsilon}\int_{a_{j-1}}^{a_j} t^{-1}(\ln t)^{-\varepsilon/2}\,dt
\end{aligned}$$

for $j \geq j_0$ where j_0 is such that $a_{j_0} \geq \exp(e^2)$. Then

$$\sum_{j=j_0+1}^{\infty} \upsilon_j \leq \sum_{j=j_0+1}^{\infty} \frac{1}{\varepsilon}\int_{a_{j-1}}^{a_j} t^{-1}(\ln t)^{-\varepsilon/2}\,dt = \frac{1}{\varepsilon}\int_{a_{j_0}}^{\infty} t^{-1}(\ln t)^{-\varepsilon/2}\,dt < \infty \quad \text{for } \varepsilon > 2.$$

Lemma 2.2 is proved. □

Proof of Theorem 2.1. We verify that the assumptions of Lemma 2.1 hold for $\tilde{A}_n = \big(\varphi_n^{1/2}\chi(f_n)\big)^{-1}A_n$, $n \geq 1$.

Condition (8) follows from (6). Now we check (9):

$$\begin{aligned}
\sum_{j=2}^{\infty}\sum_{i=n_{j-1}+1}^{n_j} \mathsf{E}\big\|\tilde{A}_{n_j}X_i\big\|^p &\leq \sum_{j=2}^{\infty}\sum_{i=n_{j-1}+1}^{n_j} \sup_{n\geq i}\Big[\big(\varphi_n^{1/2}\chi(f_n)\big)^{-p}\,\mathsf{E}\,\|A_nX_i\|^p\Big] \\
&\leq \sum_{i=1}^{\infty} \sup_{n\geq i}\Big[\big(\varphi_n^{1/2}\chi(f_n)\big)^{-p}\,\mathsf{E}\,\|A_nX_i\|^p\Big] < \infty
\end{aligned}$$

by condition (7).

Finally we check (10). According to (5) we get

$$\sum_{i=n_{j-1}+1}^{n_j} \mathsf{E}\left\|\tilde{A}_{n_j} X_i\right\|^2 = \left(\varphi_{n_j}\chi^2\left(f_{n_j}\right)\right)^{-1} \sum_{i=n_{j-1}+1}^{n_j} \mathsf{E}\left\|A_{n_j} X_i\right\|^2$$
$$\leq \left(\varphi_{n_j}\chi^2\left(f_{n_j}\right)\right)^{-1} \varphi_{n_j}\left(1-\frac{f_{n_{j-1}}}{f_{n_j}}\right) = \chi^{-2}\left(f_{n_j}\right)\left(1-\frac{f_{n_{j-1}}}{f_{n_j}}\right).$$

Thus condition (10) follows from Lemma 2.2 with $a_j = f_{n_j}$, $j \geq j_0 \geq 1$. Theorem 2.1 is proved. □

Theorem 2.2. *Assume that $|B_{n_0}| \neq 0$ for some $n_0 \geq 1$. Let condition* (1) *hold and*

$$\sum_{i=1}^{\infty} \chi^{-p}\left(\|B_i\|\right) \mathsf{E}\left\|B_i^{-1/2} X_i\right\|^p < \infty \tag{11}$$

for some $p \in (2,3]$. Then the bounded law of the iterated logarithm (4) *holds.*

Proof. We derive Theorem 2.2 from Theorem 2.1 for $A_n = B_n^{-1/2}$, $n \geq n_0$. Let I be the unit $d \times d$ matrix. First we check condition (5):

$$\sum_{i=k+1}^{n} \mathsf{E}\left\|B_n^{-1/2} X_i\right\|^2 = \operatorname{tr}\left(I - B_n^{-1/2} B_k B_n^{-1/2}\right) = d - \operatorname{tr}\left(B_n^{-1/2} B_k B_n^{-1/2}\right).$$

By $\lambda_1, \lambda_2, \ldots, \lambda_d$ we denote the eigennumbers of the matrix $B_n^{-1/2} B_k B_n^{-1/2}$. It is obvious that all of them are positive. Since

$$B_n^{-1/2} B_k B_n^{-1/2} = B_n^{-1/2}\left(B_k - B_n\right) B_n^{-1/2} + I,$$

it follows that $\lambda_i \leq 1$, $i = 1, 2, \ldots, d$. Then

$$\sum_{i=k+1}^{n} \mathsf{E}\left\|B_n^{-1/2} X_i\right\|^2 = d - \left(\lambda_1 + \lambda_2 + \cdots + \lambda_d\right) \leq d - d\left(\lambda_1 \cdot \lambda_2 \cdots \lambda_d\right)$$
$$= d\left(1 - \left|B_n^{-1/2} B_k B_n^{-1/2}\right|\right) = d\left(1 - \frac{|B_k|}{|B_n|}\right).$$

Therefore condition (5) holds for $\varphi_n = d$ and $f_n = |B_n|$, $n \geq n_0$, since the sequence $\left(|B_n|, n \geq 1\right)$ is increasing to infinity by condition (1).

Condition (6) holds, since $|B_n| \to \infty$ as $n \to \infty$ and the sequence $\left(\|B_n^{-1/2}\|, n \geq n_0\right)$ is bounded.

Finally we check condition (7). Since the matrix $B_n - B_m$ is nonnegative definite for all $m \leq n$, we have $\left\|B_n^{-1/2} x\right\| \leq \left\|B_m^{-1/2} x\right\|$ for all $x \in \mathbf{R}^d$. The sequence $\left(\chi\left(|B_n|\right), n \geq 1\right)$ is increasing; hence

$$\sup_{n \geq i}\left[\chi^{-p}\left(|B_n|\right) \mathsf{E}\left\|B_n^{-1/2} X_i\right\|^p\right] \leq \chi^{-p}(|B_i|) \mathsf{E}\left\|B_i^{-1/2} X_i\right\|^p. \tag{12}$$

Since $\chi\left(|B_i|\right) \sim \chi\left(\|B_i\|\right)$ as $i \to \infty$, condition (7) follows from (11) and (12). Theorem 2.2 is proved. □

3. Concluding remarks

Sufficient conditions for the bounded law of the iterated logarithm for sums of independent centered random vectors normalized by operators are studied. We show that the bounded law of the iterated logarithm holds if some usual (for the law of the iterated logarithm) assumptions posed on the covariance matrices of random vectors are omitted.

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