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SOME REMARKS ON THE ORDINAL STRONG LAW OF LARGE NUMBERS

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Abstract. We prove that the ordinal law of large numbers and the law of large numbers in the norm are equivalent for Banach lattices that do not contain uniformly the space l_1^n.

1. Introduction

The law of large numbers is studied in detail for Banach spaces (see [1, 2]). Mainly, the proofs are given for the convergence in the norm of the underlying space (that is, for the b-convergence).

Let B be a Banach space equipped with a norm $\|\cdot\|$ and let X_i, $i \geq 1$, be a sequence of independent copies of a random element X assuming values in B,

$$S_n = \sum_{i=1}^{n} X_i.$$

It is known that the b-law of large numbers, that is, the relation

$$\lim_{n\to\infty} \frac{\|S_n\|}{n} = 0 \quad \text{a.s.,} \tag{1}$$

where "a.s." stands for "almost surely", holds for a random element X with $\mathsf{E}\, X = 0$ in the case of a separable Banach space if and only if

$$\mathsf{E}\,\|X\| < \infty \tag{2}$$

(Kolmogorov, Mourier; see, for example, [1]).

Another convergence, the ordinal convergence or o-convergence, can also be studied in Banach spaces along with the convergence in norm.

Recall that a sequence of elements (x_n) in a Banach lattice B with a module $|\cdot|$ is called o-convergent to an element x, $x = o\text{-}\lim_{n\to\infty} x_n$, if there exists a sequence (v_n) such that $|x - x_n| < v_n$ and $v_n \downarrow 0$, that is, $v_1 \geq v_2 \geq \cdots$ and $\inf_{n\geq 1} v_n = 0$ ([3, 4]).

We say that a random element X with $\mathsf{E}\, X = 0$ satisfies the ordinal law of large numbers (o-law of large numbers) if

$$o\text{-}\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad \text{a.s.} \tag{3}$$

The ordinal law of large numbers is studied in the paper [5]; the case of nonidentically distributed random elements is also treated in [5].

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The condition $\mathsf{E}\,\|X\|_u < \infty$ is sufficient for the o-law of large numbers (3) for a separable σ-complete Banach lattice, where $\|x\|_u = \inf\{\lambda > 0\colon |x| \le \lambda u\}$ is a norm generated by some element $u \in B_+$.

Sufficient conditions for the o-law of large numbers in q-concave Banach lattices can be expressed in terms of the mean deviation of order q denoted by $\mathfrak{S}_q X$ or the mean ψ-deviation $\mathfrak{S}_\psi X$ if $1 < q < \infty$ or $q = 1$, respectively (see [5]). These conditions are also sufficient for the convergence of the corresponding moments in the ordinal law of large numbers.

It is clear that the above conditions posed on the random element X are strong enough and, in general, cannot be necessary. For example, if $X = \xi \cdot x$, where $x \in B$ and ξ is a random variable in $\mathbf{R}^1$ such that $\mathsf{E}\,|\xi| < \infty$, then the o-law of large numbers (3) holds despite the fact that

$$\mathfrak{S}_q \xi = \infty$$

for $q > 1$.

For the Banach lattice c_0 the o-convergence is equivalent to the b-convergence.

It is known for the space $C[0,1]$ that

$$b\text{-}\lim_{n\to\infty} x_n = x \Longrightarrow o\text{-}\lim_{n\to\infty} x_n = x,$$

although the converse implication is wrong. On the other hand, if the o-law of large numbers (3) holds in a Banach lattice, then condition (2) is satisfied (see the proof of Theorem 1).

Therefore condition (2) is necessary for the o-law of large numbers (3) in a separable Banach lattice. Moreover both the o-law of large numbers (3) and b-law of large numbers (1) are equivalent to condition (2) in the spaces c_0 and $C[0,1]$. This is not the case for a general separable Banach lattice.

An example of a random element X assuming values in the Banach lattice l_1 is constructed in [5] such that both equality (1) and inequality (2) hold for the random element X, while the o-law of large numbers (3) does not hold for it.

A natural problem arises to describe Banach lattices for which the o-law of large numbers (3) is equivalent to the b-law of large numbers (1) and condition (2).

A partial answer to this question is given in this paper; we describe a wide class of Banach lattices with the latter property (this class includes, for example, the spaces L_p and l_p, $1 < p < \infty$).

2. Main result

Let n be an integer number and $\varepsilon > 0$. By l_1^n we denote the space $\mathbf{R}^n$ equipped with the norm

$$\sum_{i=1}^n |a_i|, \qquad a = (a_1, \dots, a_n) \in \mathbf{R}^n.$$

We say that a Banach space B contains a subspace that is $(1+\varepsilon)$-isomorphic to l_1^n if there are elements $x_1, \dots, x_n \in B$ such that

$$\sum_{i=1}^n |a_i| \le \left\| \sum_{i=1}^n a_i x_i \right\| \le (1+\varepsilon) \sum_{i=1}^n |a_i|$$

for all $a = (a_1, \dots, a_n) \in \mathbf{R}^n$. We say that B contains uniformly l_1^n if it contains subspaces that are $(1+\varepsilon)$-isomorphic to l_1^n for all n and $\varepsilon > 0$ ([1, p. 237]).

Theorem 1. *Let B be a separable Banach lattice that does not contain uniformly l_1^n. Let X be a random element assuming values in B and such that $\mathsf{E}\,X = 0$. Then the* *following conditions are equivalent:*

(i) *the b-law of large numbers* (1) *holds for X;*
(ii) *the ordinal law of large numbers* (3) *holds for X;*
(iii) *condition* (2) *holds.*

Corollary 1. *Let X be a random element assuming values in the space L_p (l_p) for $1 < p < \infty$ and such that $\mathsf{E}\,X = 0$. Then conditions* (i)–(iii) *of Theorem* 1 *are equivalent.*

Remark 1. An attempt is made in [5] to construct a random element X in the space l_p, $1 < p < \infty$, for which condition (2) holds but the ordinal law of large numbers (3) does not hold. As became clear later, the corresponding reasoning (the proof of assertion (ii) of Theorem 3 in [5]) contains a gap. Moreover, it follows from Corollary 1 above that there is no example of such kind in the space l_p, $1 < p < \infty$. Thus one of the aims of this paper is to correct some of the author's earlier results.

Let $1 \le p < \infty$. A Banach lattice B is called p-convex if there exists a constant $D^{(p)} = D^{(p)}(B)$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \le D^{(p)} \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}$$

for all n and all elements $(x_i)_1^n \subset B$. Analogously, a Banach lattice B is called q-concave if there is a constant $D_{(q)} = D_{(q)}(B)$ such that

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \le D_{(q)} \left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} \right\|$$

for all n and all elements $(x_i)_1^n \subset B$.

Corollary 2. *Let B be a separable σ-complete and p-convex Banach lattice, $1 < p < \infty$. Let X be a random element assuming values in B. Then condition* (2) *is equivalent to*

$$o\text{-}\lim_{n\to\infty} \frac{X_n}{n} = 0 \quad a.s. \tag{4}$$

Proof of Theorem 1. The equivalence of conditions (i) and (iii) is a known classical result (see, for example, [1]).

Let us show that (ii) and (iii) are equivalent. If the ordinal law of large numbers (3) holds, then

$$\sup_{n\ge1} \frac{\|X_n\|}{n} \le 2 \sup_{n\ge1} \frac{\|S_n\|}{n} \le 2 \left\| \sup_{n\ge1} \frac{|S_n|}{n} \right\| < \infty \quad \text{a.s.}$$

Thus there exists a number t_0 such that

$$\mathsf{P}\left(\sup_{n\ge1} \frac{\|X_n\|}{n} \ge t_0 \right) \le \frac{1}{2}.$$

Hence

$$\mathsf{P}\left(\sup_{n\ge1} \frac{\|X_n\|}{n} \ge t \right) \ge \frac{1}{2} \sum_{n\ge1} \mathsf{P}(\|X_n\| \ge tn)$$

for $t \ge t_0$ by Lemma 2.6 of the book [1]. The convergence of the latter series implies condition (2) (see [6, Chapter VII, §8, Lemma 2]).

It remains to prove that (iii) $\Rightarrow$ (ii). According to a known Pisier result ([7], [1, Chapter 9]), a Banach lattice that does not contain uniformly l_1^n is of the Rademacher type p for some $p > 1$. Thus it is q-concave and p-convex for some $q < \infty$ and $p > 1$ ([3, 8]).

Any separable σ-complete Banach lattice is ordinally isomorphic to some Banach ideal space. Since any q-concave Banach lattice is σ-complete, one can assume, without loss of generality, that B is a separable q-concave Banach ideal space defined on some measurable space (T, Λ, μ) with $\mu(T) = 1$ (see [3, 4]).

For a space of this type, the conditions

$$\mu\Big(t \in T\colon \lim_{n\to\infty} x_n(t) = x(t)\Big) = 1 \tag{5}$$

and

$$\begin{gathered}\text{there exists } y = (y(t),\ t \in T) \in B \quad \text{such that}\\ \mu\big(t \in T\colon |x_n(t)| \le y(t)\big) = 1\end{gathered} \tag{6}$$

are sufficient for the o-convergence of a sequence (x_n) to x [8].

Let

$$X_n = (X_n(t), t \in T), \qquad S_n = (S_n(t), t \in T),$$
$$\tau = \left\| \sup_{n\ge1} \frac{1}{n} \left| \sum_{i=1}^{n} X_i \right| \right\|.$$

To check conditions (5) and (6) one needs to show that

$$\mu\left(t \in T\colon \lim_{n\to\infty} \frac{S_n(t)}{n} = 0\right) = 1 \quad \text{a.s.}, \tag{7}$$

$$\tau < \infty \quad \text{a.s.} \tag{8}$$

Relation (7) is a straightforward corollary of the Fubini theorem and the Kolmogorov law of large numbers in $\mathbf{R}^1$ since

$$\mathsf{E}\,\|X\| \ge \big\|\mathsf{E}\,|X|\big\|.$$

Now we obtain relation (8). First of all, we assume that X is a symmetric random element. Then $X = \varepsilon\hat{X}$, where $\hat{X}$ and ε are independent, $\hat{X}$ is a copy of X, and ε is a symmetric Bernoulli random variable.

Let (X_n) be a sequence of independent copies of X. Put

$$\bar{X}_n = \hat{X}_n I\big(\|\hat{X}_n\| \le n\big), \qquad \tilde{X}_n = \hat{X}_n I\big(\|\hat{X}_n\| > n\big),$$
$$\bar{S}_n = \sum_{i=1}^{n} \varepsilon_i \bar{X}_i, \qquad \tilde{S}_n = \sum_{i=1}^{n} \varepsilon_i \tilde{X}_i.$$

It is clear that $X_n = \varepsilon_n(\bar{X}_n + \tilde{X}_n)$ and

$$\tau \le \left\| \sup_{n\ge1} \frac{|\bar{S}_n|}{n} \right\| + \left\| \sup_{n\ge1} \frac{|\tilde{S}_n|}{n} \right\| \quad \text{a.s.} \tag{9}$$

It is known that condition (2) implies the convergence of the series

$$\sum_{n\ge1} \mathsf{P}(\|X_n\| \ge n)$$

(see [6, Chapter VII, §8, Lemma 2]). The Borel–Cantelli lemma yields that with probability one only a finite number of random elements $\tilde{X}_n$ is nonzero, that is,

$$\left\| \sup_{n\ge1} \frac{|\tilde{S}_n|}{n} \right\| < \infty \quad \text{a.s.}$$

Thus estimate (8) holds if the first term on the right-hand side of (9) is bounded. To prove the boundedness we show that

$$\mathsf{E}_{\hat{X}} \left\| \sup_{n\geq 1} \frac{|\bar{S}_n|}{n} \right\|^q < \infty \quad \text{a.s.,} \tag{10}$$

where the symbol $\mathsf{E}_{\hat{X}}(\xi)$ denotes the mathematical expectation of a variable ξ in a given sequence $(\hat{X}_n)$.

To prove inequality (10) we need two auxiliary results.

Lemma 1 ([9])**.** *Let B be a separable q-concave Banach ideal space for $1 \leq q < \infty$. Let $Y = (Y(t), t \in T)$ be a random element assuming values in B. Then*

$$(\mathsf{E}\,\|Y\|^q)^{1/q} \leq D_{(q)} \big\| (\mathsf{E}\,|Y(t)|^q)^{1/q} \big\|.$$

Lemma 2. *Let (a_i) be a sequence of real numbers. Then*

$$\max_{1\leq k\leq n} \frac{1}{k} \left| \sum_{i=1}^k a_i \right| \leq 2 \max_{1\leq k\leq n} \left| \sum_{i=1}^k \frac{a_i}{i} \right|.$$

Lemma 2 is a particular case of an inequality proved in [10].

It follows from Lemmas 1 and 2 that

$$\begin{aligned} \left(\mathsf{E}_{\hat{X}} \left\| \sup_{n\geq 1} \frac{|\bar{S}_n|}{n} \right\|^q \right)^{1/q} &\leq D_{(q)} \left\| \left(\mathsf{E}_{\hat{X}} \sup_{n\geq 1} \left| \frac{\bar{S}_n(t)}{n} \right|^q \right)^{1/q} \right\| \\ &\leq 2D_{(q)} \left\| \left(\mathsf{E}_{\hat{X}} \sup_{n\geq 1} \left| \sum_{i=1}^n \frac{\varepsilon_i \bar{X}_i(t)}{i} \right|^q \right)^{1/q} \right\|. \end{aligned} \tag{11}$$

We continue the estimation by applying the Lévy ([1, p. 48]) and Khinchine ([11, p. 251]) inequalities for symmetric Bernoulli random variables:

$$\left(\mathsf{E}_{\hat{X}} \sup_{n\geq 1} \left| \sum_{i=1}^n \frac{\varepsilon_i \bar{X}_i(t)}{i} \right|^q \right)^{1/q} \leq \left(2\,\mathsf{E}_{\hat{X}} \left| \sum_{n=1}^\infty \frac{\varepsilon_n \bar{X}_n(t)}{n} \right|^q \right)^{1/q} \leq C_q \left(\sum_{n=1}^\infty \left| \frac{\bar{X}_n(t)}{n} \right|^2 \right)^{1/2}. \tag{12}$$

In a Banach lattice of type p we have

$$\left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\| \leq C(B) \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$

(see [8]). Accounting for (11) and (12) we get

$$\left(\mathsf{E}_{\hat{X}} \left\| \sup_{n\geq 1} \frac{|\bar{S}_n|}{n} \right\|^q \right)^{1/q} \leq C(B) \left(\sum_{n=1}^\infty \frac{\|\bar{X}_n\|^p}{n^p} \right)^{1/p}. \tag{13}$$

To complete the proof of Theorem 1 one needs to show that the series on the right-hand side of inequality (13) converges. We have

$$\begin{aligned} \sum_{n=1}^\infty \frac{\mathsf{E}\,\|\bar{X}_n\|^p}{n^p} &= \sum_{n=1}^\infty \frac{1}{n^p} \int_0^\infty \mathsf{P}\big(\|\bar{X}_n\|^p > t\big)\, dt \\ &= \sum_{n=1}^\infty \frac{1}{n^p} \int_0^{n^p} \mathsf{P}(\|X_n\|^p I(\|X_n\| \leq n) > t)\, dt \\ &\leq \sum_{n=1}^\infty \frac{1}{n^p} \int_0^{n^p} \mathsf{P}(\|X\|^p > t)\, dt = \int_0^\infty \mathsf{P}(\|X\|^p > t) \beta_p(t)\, dt, \end{aligned} \tag{14}$$

where $\beta_p(t) = \sum_{n^p \geq t} n^{-p}$. Since

$$\sum_{n \geq t} \frac{1}{n^p} \sim \frac{p-1}{t^{p-1}}$$

for $p > 1$ and as $t \to \infty$, the precise asymptotics of β_p is as follows:

$$\beta_p(t) \sim \frac{p-1}{t^{1-1/p}}.$$

The convergence of the integral on the right-hand side of (14) is equivalent to

$$\int_0^\infty \mathsf{P}(\|X\|^p > t) t^{1/p-1}\, dt < \infty. \tag{15}$$

Using the known equality ([6, Chapter 5, §7, Lemma 1])

$$\mathsf{E}\,|\xi|^\alpha = \alpha \int_0^\infty \mathsf{P}(|\xi| > t) t^{\alpha-1}\, dt$$

with $\alpha = 1/p$ and $\xi = \|X\|^p$, we prove that condition (15) is equivalent to condition (2). Without loss of generality, we may assume that $1 < p \leq 2 \leq q < \infty$; thus (13)–(15) imply

$$\mathsf{E} \left\| \sup_{n \geq 1} \frac{|\bar{S}_n|}{n} \right\|^p \leq C(B)\, \mathsf{E}\,\|X\|. \tag{16}$$

Therefore the implication (iii) ⇒ (ii) in Theorem 1 is proved for symmetric random elements.

The general case can be reduced to the particular case of symmetric random elements with the help of the standard symmetrization procedure. Put

$$\bar{X}_n^{(s)} = X_n I(\|X_n\| \leq n) - X_n' I(\|X_n'\| \leq n),$$
$$\bar{S}_n^{(s)} = \sum_{i=1}^n \bar{X}_i^{(s)},$$

where (X_n') is an independent copy of the sequence (X_n).

As in the case of symmetric random elements the main point of the proof is to show inequality (8). Repeating the above reasoning for $\bar{S}_n^{(s)}$ (relations (11)–(15)) we get an estimate similar to (16):

$$\left(\mathsf{E} \left\| \sup_{n \geq 1} \frac{|\bar{S}_n^{(s)}|}{n} \right\|^p \right)^{1/p} \leq C(B)(\mathsf{E}\,\|X\|)^{1/p}.$$

Using the latter inequality and known moment estimates for Banach spaces ([11, Chapter 5, Lemma 3.4]) we have

$$\begin{aligned} \left(\mathsf{E} \left\| \sup_{n \geq 1} \frac{|\bar{S}_n|}{n} \right\|^p \right)^{1/p} &\leq \left\| \sup_{n \geq 1} \frac{|\sum_{k=1}^n \mathsf{E}\, X_k I(\|X_k\| \leq k)|}{n} \right\| + \left(\mathsf{E} \left\| \sup_{n \geq 1} \frac{|\bar{S}_n^{(s)}|}{n} \right\|^p \right)^{1/p} \\ &\leq \|\,\mathsf{E}\,|X|\| + C(B)(\mathsf{E}\,\|X\|)^{1/p}, \end{aligned}$$

whence (8) follows by (9). □

Corollary 1 follows from Theorem 1, since the spaces L_p and l_p for $1 < p < \infty$ do not contain uniformly l_1^n.

The proof of Corollary 2 is, in fact, given in the proof of Theorem 1. Indeed, the assumption that the Banach lattice B is q-concave is used there only to obtain estimates (11) and (12). The corresponding estimate for Corollary 2 is as follows:

$$\sup_{n\ge 1}\left|\frac{\bar{X}_n}{n}\right| \le \left(\sum_{n=1}^{\infty}\left|\frac{\bar{X}_n}{n}\right|^2\right)^{1/2}.$$

Note that this estimate holds for an arbitrary Banach lattice. Having obtained the latter estimate one needs to repeat the corresponding reasoning of the proof of Theorem 1.

The method used above allows one to prove the ordinal law of large numbers for nonidentically distributed random variables. The following is one of the possible results of this type.

Proposition 1. *Let B be a separable Banach ideal space on (T,Λ,μ) that does not contain uniformly l_1^n. Let (X_n) be a sequence of independent random elements assuming values in B and such that*

$$\mathsf{E}\,X_n = 0, \qquad S_n = \sum_{i=1}^{n} X_i, \qquad \psi(t) = |t|\ln^{1+\varepsilon}(1+|t|), \qquad \varepsilon > 0.$$

If $\sup_{n\ge 1}\mathsf{E}\,\psi(X_n(t)) < \infty$ almost everywhere on T and $\sup_{n\ge 1}\mathsf{E}\,\psi(\|X_n\|) < \infty$, then the ordinal law of large numbers (3) *holds.*

We do not provide the proof of this result.

3. Example

The random element X constructed in the paper [5] as a counterexample to the o-law of large numbers in l_1 is of the following form:

(17) $\qquad X = (a_i\xi_i), \qquad \xi_i$ are independent copies of a random variable ξ.

For this random element X (as well as for ξ) condition (2) holds. Nevertheless

$$\mathsf{E}\,\|X\|\ln(1+\|X\|) = \infty.$$

If

$$\mathsf{E}\,\|X\|^m < \infty \tag{18}$$

for some $m > 1$ and a random element X is of the form (17), then the o-law of large numbers (3) holds. Are the moment conditions like (18) sufficient for the o-law of large numbers in general Banach lattices?

The following example shows that this is not the case in general. Let a random element X assuming values in l_1 be such that condition (18) holds for all $m > 1$ and

$$\left\|\sup_{n\ge 1}\left|\frac{X_n}{n}\right|\right\|_{l_1} = \infty \quad \text{a.s.}, \tag{19}$$

where (X_n) is a sequence of independent copies of the random element X.

It is clear that the ordinal law of large numbers (3) does not hold for the random element X constructed in this way.

Put $L(t) = \ln t$ for $t > 2$ and $L(t) = 1$ for $t \le 2$. Let

$$\theta = \sum_{k\ge 1}\frac{1}{kL^2(k)}, \qquad p_k = \frac{1}{\theta k L^2(k)}, \quad k \ge 1.$$

It is clear that $\sum_{k\ge 1} p_k = 1$.

Let (ξ_k) be a sequence of independent random variables assuming only three values and such that

$$\mathsf{P}(\xi_k = +1) = p_k/2, \qquad \mathsf{P}(\xi_k = -1) = p_k/2, \qquad \mathsf{P}(\xi_k = 0) = 1 - p_k.$$

Then the random element $X = (\xi_k)$ is such that

$$\mathsf{E}\,|\xi_k| = p_k, \qquad \mathsf{E}\,\|X\|_{l_1} = \sum_{k\geq 1} p_k = 1. \tag{20}$$

To show that condition (18) holds for the random element X uniformly in n we apply the following Skorokhod inequality for sums of independent bounded random variables (see [12, Chapter 1, §3, Theorem 4]).

Lemma 3. *Let $\eta_1, \dots, \eta_n$ be a sequence of independent random variables,*

$$S_n = \sum_{i=1}^n \eta_i$$

and

$$|\eta_i| \leq 1 \quad a.s., \qquad i = 1, 2, \dots, n.$$

If there exists a constant a such that

$$\mathsf{P}(|S_n| > a) \leq \frac{1}{8e},$$

then

$$\mathsf{E}\,|S_n|^m \leq L_m(a+1)^m$$

for $m > 0$.

Conditions (20) and Markov's inequality imply

$$\mathsf{P}(\|X\|_{l_1} > 8e) \leq \frac{1}{8e}.$$

By Lemma 3, condition (18) holds for the random element X whatever $m > 0$ is.

Now we check equality (19). Let $X_n = (\xi_{nk})$ where (ξ_{nk}) are independent copies of the sequence (ξ_k). We have

$$\left\| \sup_{n\geq 1} \left| \frac{X_n}{n} \right| \right\|_{l_1} = \sum_{k\geq 1} \sup_{n\geq 1} \left| \frac{\xi_{nk}}{n} \right|. \tag{21}$$

By construction, $|\xi_{nk}| \leq 1$ almost surely. Thus the series on the right-hand side of equality (21) converges if and only if

$$\sum_{k\geq 1} \mathsf{E} \sup_{n\geq 1} \left| \frac{\xi_{nk}}{n} \right| < \infty \tag{22}$$

(see [13, Chapter III, Theorem 6]).

Then we apply an estimate of the paper [14]:

$$C_1 \mathfrak{S}_\psi(|\xi_k|) \leq \mathsf{E} \sup_{n\geq 1} \left| \frac{\xi_{nk}}{n} \right| \leq C_2 \mathfrak{S}_\psi(|\xi_k|),$$

where $\psi(t) = |t| \ln(1 + |t|)$ and $\mathfrak{S}_\psi(\xi)$ is the mean ψ-deviation of the random variable ξ (see [14]). In other words, $\mathfrak{S}_\psi(\xi)$ is the Orlicz norm of the random variable ξ:

$$\mathfrak{S}_\psi(\xi) = \sup(x \in K_\psi),$$
$$K_\psi = (\mathsf{E}\,\eta\xi \colon\ \mathsf{E}\,\psi^*(\eta) \leq 1),$$

where

$$\psi^*(t) = \sup_{s \in \mathbf{R}^1} (st - \psi(s))$$

is the dual function to the N-function $\psi(t)$ [4]. The latter inequalities imply that condition (22) is equivalent to

$$\sum_{k \geq 1} \mathfrak{S}_\psi(|\xi_k|) < \infty. \tag{23}$$

The norm $\mathfrak{S}_\psi \xi$ is equivalent to the norm

$$\|\xi\|_1 = \inf \left(\lambda > 0\colon\ \mathsf{E}\, \psi\left(\frac{\xi}{\lambda}\right) \leq 1 \right)$$

(see [4, Chapter 4, §3]). Now the norm $\||\xi_k|\|_1$ can easily be estimated from below by using the definition of the random variable ξ_k. Indeed,

$$\||\xi_k|\|_1 = \inf \left(\lambda > 0\colon \frac{1}{\lambda} \ln\left(1 + \frac{1}{\lambda}\right) \frac{1}{\theta k L^2(k)} \leq 1 \right).$$

Since $\theta > 1$, we put $\lambda = (\theta k \ln(k))^{-1}$ and obtain for $k > 2$ that

$$\frac{1}{\lambda} \ln\left(1 + \frac{1}{\lambda}\right) \frac{1}{\theta k L^2(k)} \geq 1.$$

Thus

$$\||\xi_k|\|_1 \geq \frac{1}{\theta k \ln(k)}$$

for $k > 2$, whence we conclude that inequality (23) does not hold since the norms $\| \cdot \|_1$ and $\mathfrak{S}_\psi(\cdot)$ are equivalent. This completes the proof of equality (19).

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