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# AN ESTIMATE OF THE MEAN SQUARE ERROR OF INTERPOLATION OF STOCHASTIC PROCESSES

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Abstract. For functions with an unbounded support of the spectrum, we obtain precise estimates of the error of interpolation by Whittaker–Kotelnikov–Shannon sums. We study the uniform and mean square errors of interpolation. Examples of extreme functions are given for which the estimate is precise. We obtain the rate of the mean square convergence of the error of interpolation for stochastic processes of the weak Cramér class and for processes generated by an orthogonal stochastic measure.

## 1. Introduction

One of the main problems of the theory of interpolation and approximation is the reconstruction of a continuous signal from its discrete readings and estimation of the losses due to the digitalization of the signal.

Let $\mathsf{X}$ be a normed space equipped with a norm $\|\cdot\|$. Many applications use an approximation of a function $f \in \mathsf{X}$ by sums

$$Y_{\mathfrak{J}}(f;x) = \sum_{n\in\mathfrak{J}} f(t_n) S(x,t_n), \qquad \mathfrak{J} \subset \mathbb{Z},$$

where $x \in \mathbb{R}$, and $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ and $S$ are the set of reading arguments and the reading function, respectively. Note that in applied problems, $\mathfrak{J}$ is a finite set.

The classical approach in interpolation theory is to estimate the truncation error of the interpolation $T_{\mathfrak{J}}(f;x)$ by some upper bound $\varphi_{\mathfrak{J}}(f)$:

$$\|T_{\mathfrak{J}}(f;\cdot)\| := \|f(\cdot) - Y_{\mathfrak{J}}(f;\cdot)\| \le \varphi_{\mathfrak{J}}(f).$$

Obtaining simple upper bounds for the error of deviation is an important topic of numerical applications. More details concerning this topic and related references can be found in [5, 8, 10].

Most of the earlier results on upper bounds for the error of interpolation deal with the case of a bounded support of the Fourier spectrum. We study the aliasing problem considered in [9] (also see [3, 5]). The difference between $f$ and its *nontruncated* series (if, for example, $\mathfrak{J} = \mathbb{Z}$ and $t_n = n$) is the subject of studies in aliasing problems. If the support of the Fourier spectrum of a function is bounded, then the corresponding difference vanishes. This means that the series reconstructs the function without error if the spectrum is bounded. A nonzero error appears for functions with an unbounded

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support of the spectrum, since the support of the spectrum of the function $f$ and that of the main series are different.

We continue the investigation of a model considered in [9] where functions with unbounded spectra are approximated by the truncated main series.

We deal with the one-dimensional case in the paper. A short survey of necessary results on uniform estimates [9] for nonrandom functions is given in Sections 2 and 3. In Section 3, we introduce some notation in the theory of stochastic processes of the weak Cramér class. In Section 4, we consider the stochastic interpolation of such processes.

We discuss the impossibility of the mean square approximation in Section 5 for the general aliasing problem and give examples of the aliasing error for some function classes. These results are applied in Section 6 to stochastic processes generated by orthogonal random measures.

## 2. Estimates for the aliasing error in the uniform metric

Let $f\colon \mathbb{R} \to \mathbb{C}$. Following [9, 11] we treat $f(\cdot)$ as the inverse Fourier transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixt}\, d\Phi(x), \tag{1}$$

where $\Phi(\cdot)$ has a bounded variation on $\mathbb{R}$, is continuous at points $(2k+1)\pi\omega$, $k \in \mathbb{Z}$, and is right continuous everywhere. If the derivative $\Phi'(\cdot)$ exists, we denote it by $\hat{f}(\cdot)$.

The aliasing error is

$$A(t,\omega) := f(t) - \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\omega}\right) \operatorname{sinc}(\omega t - n),$$

where

$$\operatorname{sinc}(t) := \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0; \\ 1, & t = 0. \end{cases}$$

The aliasing error appears because the spectra are different. Namely, the spectrum is $\mathbb{R}$ for $f(\cdot)$, while it is $[-\pi\omega; \pi\omega]$ for the main series. The classical result [11] says that the main series converges for all $t$ and

$$\|A(\cdot,\omega)\|_\infty \leq \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |d\Phi(x)|, \tag{2}$$

where

$$\|f(\cdot)\|_\infty := \inf\{a > 0\colon |f(t)| \leq a, t \in \mathbb{R}\}.$$

Only finitely many observations are available in practice. Thus we are interested in studying the aliasing error for the truncated series

$$A_N(t,\omega) := f(t) - \sum_{|n-\omega t|\leq N} f\left(\frac{n}{\omega}\right) \operatorname{sinc}(\omega t - n), \qquad N \in \mathbb{N}. \tag{3}$$

Denote by $BL_{\pi\omega}$ the class of all functions $f(\cdot)$ whose Fourier spectrum is concentrated on $[-\pi\omega; \pi\omega]$, and by $IBV$ we denote the class of nondecreasing right continuous functions that have bounded variation and are continuous at points $(2k+1)\pi\omega$, $k \in \mathbb{Z}$; here $\|\cdot\|_p$ is the norm in $L_p$.

Below are some results for nonrandom functions (see [9]) that we will need in this paper.

**Theorem 1.** *The set of extreme functions for* (2) *coincides with*

$$\mathfrak{L} := \left\{ f_0(t) + c \int_B e^{ix(t-t_0)}\, dG(x) \colon f_0 \in BL_{\pi\omega}, G \in IBV, c \in \mathbb{C}, t_0 = \frac{2k_0+1}{2n_0\omega}, \right.$$
$$\left. B := \bigcup_{m\in\mathbb{Z}} [(2m+1)2\pi\omega n_0 - \pi\omega; (2m+1)2\pi\omega n_0 + \pi\omega), n_0 \in \mathbb{N}, k_0 \in \mathbb{Z} \right\}.$$

*The equality in* (2) *is attained for functions of* $\mathfrak{L}$ *at the point* $t_0$.

**Theorem 2.**

$$\|A_N(\cdot,\omega)\|_\infty \le \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |d\Phi(x)| + \mathcal{C} \left( \sum_{n\in\mathbb{Z}} \left| f\left(\frac{n}{\omega}\right)\right|^2 \right)^{1/2}, \tag{4}$$

*where*

$$\mathcal{C} := \sqrt{1 - \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2}}.$$

*The estimate is precise, and the set of extreme functions coincides with*

$$\mathfrak{L}_N := \left\{ c\left( \beta \sum_{|n-l-\frac{1}{2}|>N} \operatorname{sinc}\left(n-l-\frac{1}{2}\right) \operatorname{sinc}(t\omega - n) \right.\right.$$
$$+ \sum_{n\in\mathbb{Z}} \hat{\hat{G}}\left(\frac{2l+1-2n}{2\omega}\right) \operatorname{sinc}(t\omega - n)$$
$$\left. + \sum_{m\in\mathbb{Z}} \exp\left\{ iA_m \left(t - \frac{2l+1}{2\omega}\right)\right\} \int_{-\pi\omega}^{\pi\omega} \exp\left\{ ix\left(t - \frac{2l+1}{2\omega}\right)\right\} dG(x+A_m) \right),$$
$$\left. l \in \mathbb{Z}, c \in \mathbb{C}, \beta \ge 0, A_m = (2m+1)2\pi\omega n_0 \right\},$$

*where* $G$ *is the same constant as in Theorem* 1*. The derivative*

$$\left( \sum_{m\in\mathbb{Z}} G(x+A_m) \right)' \chi_{[-\pi\omega,\pi\omega)}(x) \in L^2([-\pi\omega,\pi\omega]),$$
$$\hat{\hat{G}}(t) := \int_{-\pi\omega}^{\pi\omega} e^{-ixt}\, d\left( \sum_{m\in\mathbb{Z}} G(x+A_m) \right) \tag{5}$$

*exists. The upper bound in inequality* (4) *is attained for functions of* $\mathfrak{L}_N$ *at the point* $t_0 = (2l+1)/(2\omega)$.

**Theorem 3.** *Let* $f \in \mathcal{F} := \{f \colon f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \hat{f} \in L^1(\mathbb{R})\}$.

a) *If*

$$c(\omega) := \sum_{k\in\mathbb{Z}} \left( \int_{-\pi\omega+2k\pi\omega}^{\pi\omega+2k\pi\omega} |\hat{f}(x)|^2\, dx \right)^{1/2},$$

*then*

$$\|A_N(\cdot,\omega)\|_\infty \le \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |\hat{f}(x)|\, dx + \mathcal{C}\sqrt{\omega}\, c(\omega).$$

b) *If* $f'$ *exists and* $f' \in L^2(\mathbb{R})$, *then*

$$\|A_N(\cdot,\omega)\|_\infty \le \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |\hat{f}(x)|\, dx + \mathcal{C}\left( \sqrt{\omega}\|f\|_2 + \frac{\|f'\|_2}{\sqrt{\omega}} \right).$$

c) *If there exists a constant $M$ such that $|f(t)| \le M/t^\nu$ for $t \ne 0$ and $\nu > \frac{1}{2}$, then*

$$\|A_N(\cdot,\omega)\|_\infty \le \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |\hat{f}(x)|\,dx + \mathcal{C}\sqrt{|f(0)|^2 + 2M^2\omega^{2\nu}\zeta(2\nu)},$$

*where $\zeta(s)$ stands for the Riemann $\zeta$-function: $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$.*

## 3. Stochastic processes of the weak Cramér class

To apply the above results for the stochastic case we recall some notions of the theory of the processes of the *weak Cramér class.* More details and references can be found in [6, 9]).

Let $\mathcal{S}_\Lambda$ be the $\sigma$-ring of subsets of $\Lambda \subseteq \mathbb{R}$ and let $F\colon \mathcal{S}_\Lambda \times \mathcal{S}_\Lambda \mapsto \mathbb{C}$ be a positive definite bimeasure. By $\|F\|(A,B)$ we denote the Fréchet variation of $F$ on $(A,B)$.

Let $\{\xi(t), t \in \mathbb{R}\}$ be a second-order stochastic process defined in a certain probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. For the sake of simplicity we assume that $\mathsf{E}\,\xi(t) = 0$ and that the correlation function $\mathbf{B}(t,s) = \mathsf{E}\,\xi(t)\overline{\xi(s)}$ of the process $\xi$ is determined by the family $\{f(t,\lambda)\colon t \in \mathbb{R}, \lambda \in \Lambda\}$ of $\mathcal{S}_\Lambda$-measurable functions, namely,

$$\mathbf{B}(t,s) = \int_\Lambda \int_\Lambda^* f(t,\lambda)\overline{f(s,\mu)}\, F_\xi(d\lambda, d\mu), \tag{6}$$

where $*$ indicates that the integral in (6) exists in the strong Morse–Transue sense with respect to the bimeasure $F_\xi(d\lambda, d\mu)$ having a bounded variation. In such a case we say that the correlation function (6) belongs to the *weak Cramér class.* A stochastic process $\xi(t)$ belongs to the weak Cramér class if its correlation function belongs to the weak Cramér class. Then $\xi(t)$ admits the spectral representation

$$\xi(t) = \int_\Lambda f(t,\lambda)\, Z_\xi(d\lambda) \tag{7}$$

in the sense that the stochastic measure $Z_\xi\colon S_\Lambda \mapsto L^2(\Omega)$ exists and the integral in (7) exists in the Dunford–Schwartz sense. The converse assertion also holds.

The semivariations of the bimeasure $F_\xi$ and stochastic measure $Z_\xi$ are such that $\|Z_\xi\|(A)^2 = \|F_\xi\|(A,A)$ [6].

The Hilbert space of complex-valued functions defined on $\Lambda$ and that are square integrable (in the Morse–Transue sense) with respect to the measure $F_\xi$ is denoted by $L^2_{MT}(\Lambda; F_\xi)$. Note that the space $L^2_{MT}(\Lambda; F_\xi)$ is isometric to $L^2(\Lambda;\Omega)$, where $L^2(\Lambda;\Omega)$ consists of all stochastic measures

$$\zeta\colon S_\Lambda \times \mathfrak{F} \mapsto \mathbb{C}$$

that are $L^2(\Omega)$-finite. In other words, $\mathsf{E}\,|\zeta(A)|^2 < \infty$ for $A \in \mathcal{S}_\Lambda$.

## 4. The interpolation of stochastic processes of the weak Cramér class

We study the approximation of a stochastic process $\xi(t)$ of the weak Cramér class:

$$\xi(t) \approx Y_{\mathfrak{J},\omega}(\xi;t) = \sum_{n\in\mathfrak{J}} \operatorname{sinc}(t\omega - n)\xi\left(\frac{n}{\omega}\right) \quad \text{in the } L^2(\Omega) \text{ sense.}$$

We consider two cases: (i) $\mathfrak{J}_1 := \mathbb{Z}$ and (ii) $\mathfrak{J}_2 := \{n\colon |t\omega - n| \le N\}$. In both cases, we find optimal numbers $\varepsilon_i$, $i = 1,2$, such that $\mathsf{E}|\xi(t) - Y_{\mathfrak{J}_i,\omega}(\xi;t)|^2 < \varepsilon_i \|F_\xi\|(\Lambda,\Lambda)$.

In what follows we use the following notation:

$$\|g(\omega,\cdot)\|_{\infty,F_\xi} := \inf\bigl\{\alpha\colon \|F_\xi\|(A_\alpha, A_\alpha) = 0, A_\alpha = \{\lambda \in \Lambda\colon |g(\omega,\lambda)| \ge \alpha\}\bigr\}.$$

**Theorem 4.** *Let $\{\xi(t), t\in\mathbb{R}\}$ be a stochastic process of the weak Cramér class. Assume that $f(\cdot,\lambda)$ admits the representation (1) with respect to the first argument for some $\Phi_\lambda(\cdot)$ and all $\lambda$ except for some set of zero semivariation. Then the upper bound for the mean square truncation error $\mathfrak{T}_{\mathfrak{J},\omega}(\xi;t) := \mathsf{E}|\xi(t) - Y_{\mathfrak{J},\omega}(\xi;t)|^2$ is as follows:*

$$\|\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;\cdot)\|_\infty \le \frac{2}{\pi}\left\|\int_{|x|>\pi\omega}|d\Phi_\lambda(x)|\right\|^2_{\infty,F_\xi}\|F_\xi\|(\Lambda,\Lambda); \tag{8}$$

$$\|\mathfrak{T}_{\mathfrak{J}_2,\omega}(\xi;\cdot)\|_\infty \le \left\|\frac{2}{\sqrt{2\pi}}\int_{|x|>\pi\omega}|d\Phi_\lambda(x)| + \mathcal{C}\left(\sum_{n\in\mathbb{Z}}\left|f\left(\frac{n}{\omega},\lambda\right)\right|^2\right)^{1/2}\right\|^2_{\infty,F_\xi}\|F_\xi\|(\Lambda,\Lambda), \tag{9}$$

*where the norm $\|\cdot\|_{\infty,F_\xi}$ is considered with respect to the argument $\lambda$.*

*Inequalities (8) and (9) are sharp; the extreme processes for inequalities (8) and (9) are given by*

$$\eta^{(i)}_\varrho(t) = f^{(i)}(t)\varrho, \qquad i=1,2, \tag{10}$$

*where $\varrho\in L^2(\Omega)$ is an arbitrary random variable, $f^{(1)}(\cdot)\in\mathfrak{L}$, and $f^{(2)}(\cdot)\in\mathfrak{L}_N$.*

*Proof.* It follows from (7) and the definition of $Y_{\mathfrak{J}_1,\omega}(\xi;x)$ that

$$\xi(t) = \int_\Lambda f(t,\lambda)\,Z_\xi(d\lambda).$$

The process $\xi(t) - Y_{\mathfrak{J}_1,\omega}(\xi;t)$ admits the representation

$$\xi(x) - Y_{\mathfrak{J}_1,\omega}(\xi;x) = \int_\Lambda A(t,\omega,\lambda)\,Z_\xi(d\lambda) \quad \text{in the } L^2(\Omega) \text{ sense},$$

where

$$A(t,\omega,\lambda) = f(t,\lambda) - \sum_{n\in\mathbb{Z}} f\left(n/\omega,\lambda\right)\operatorname{sinc}(\omega t - n).$$

Since $L^2(\Lambda;\Omega)$ and the Hilbert space $L^2_{MT}(\Lambda;F_\xi)$ are isometric, we get

$$\begin{aligned}\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;t) &= \int_\Lambda\int_\Lambda^* A(t,\omega,\lambda)\overline{A(t,\omega,\mu)}\,F_\xi(d\lambda,d\mu)\\ &\le \|A(t,\omega,\cdot)\|^2_{\infty,F_\xi}\|F_\xi(\Lambda,\Lambda)\|.\end{aligned} \tag{11}$$

The inequality in (11) is obtained similarly to the proof of Theorem 2 in [8].

Using the estimate (2) we obtain

$$\|A(t,\omega,\cdot)\|_{\infty,F_\xi} \le \big\|\|A(\cdot,\omega,\lambda)\|_\infty\big\|_{\infty,F_\xi} \le \frac{2}{\sqrt{2\pi}}\left\|\int_{|x|>\pi\omega}|d\Phi_\lambda(x)|\right\|_{\infty,F_\xi},$$

whence

$$\|\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;\cdot)\|_\infty \le \frac{2}{\pi}\left\|\int_{|x|>\pi\omega}|d\Phi_\lambda(x)|\right\|^2_{\infty,F_\xi}\|F_\xi\|(\Lambda,\Lambda)$$

by inequality (11). Estimate (9) is obtained similarly.

Using the results of Section 2 we prove for a fixed $\lambda$ that the function $|A(t,\omega,\lambda)|$ attains its maximal value at the point $t=t_0$ if $f(t,\lambda)$ is multiplicative, that is,

$$f(t,\lambda) = f^{(1)}(t)g(\lambda),$$

where the function $f^{(1)}(\cdot)$ belongs to the class $\mathfrak{L}$ defined in Theorem 1. Putting $g(\lambda)\equiv 1$ we prove that $f(t,\lambda) = f^{(1)}(t)$. The stochastic process corresponding to such a function

$f(t,\lambda)$ is $\int_\Lambda f(t)\,Z_\xi(d\lambda)$. Thus (10) is an example of an extreme stochastic process for inequality (8) with a random variable $\varrho = \int_\Lambda Z_\xi(d\lambda)$ such that

$$\mathsf{E}|\varrho|^2 = \mathsf{E}Z_\xi(\Lambda)\overline{Z_\xi(\Lambda)} = \|F_\xi(\Lambda,\Lambda)\| < +\infty.$$

The extreme process for (9) is obtained from Theorem 2 following the same method. □

*Remark.* The extreme processes $\eta_\varrho^{(i)}(t)$, $i = 1,2$, are actually determined by a single random variable $\varrho$.

Posing additional conditions on the family of functions $f(t,\lambda)$ determining the stochastic process $\xi(t)$ one can improve the preceding theorem.

**Theorem 5.** *Let all the assumptions of Theorem* 4 *hold. Then*

a)

$$\|\mathfrak{T}_{\mathfrak{J}_2,\omega}(\xi;\cdot)\|_\infty \le \left\| \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |\hat f(x,\lambda)|\,dx + \mathcal{C}\sqrt{\omega}c(\omega,\lambda) \right\|^2_{\infty,F_\xi} \|F_\xi\|(\Lambda,\Lambda),$$

*where*

$$c(\omega,\lambda) := \sum_{k\in\mathbb{Z}} \left( \int_{-\pi\omega+2k\pi\omega}^{\pi\omega+2k\pi\omega} |\hat f(x,\lambda)|^2\,dx \right)^{1/2}$$

*and $\hat f(x,\lambda)$ is the Fourier transform of $f(t,\lambda)$ with respect to the first argument;*

b) *if the derivative*

$$f'_t(\cdot,\lambda) \in L^2(\mathbb{R})$$

*exists for all $\lambda$ except for some set of zero semivariation, then*

$$\|\mathfrak{T}_{\mathfrak{J}_2,\omega}(\xi;\cdot)\|_\infty \le \Bigg\| \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |\hat f(x,\lambda)|\,dx + \mathcal{C}\left( \sqrt{\omega}\|f(\cdot,\lambda)\|_2 + \frac{\|f'_t(\cdot,\lambda)\|_2}{\sqrt{\omega}} \right) \Bigg\|^2_{\infty,F_\xi} \|F_\xi\|(\Lambda,\Lambda);$$

c) *if, for all $\lambda$ except for some set of zero semivariation, there exists a constant $M$ such that $|f(t,\lambda)| \le M/t^\nu$ for $t \ne 0$ and $\nu > \frac{1}{2}$, then*

$$\|\mathfrak{T}_{\mathfrak{J}_2,\omega}(\xi;\cdot)\|_\infty \le \Bigg\| \frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |\hat f(x,\lambda)|\,dx + \mathcal{C}\sqrt{|f(0,\lambda)|^2 + 2M^2\omega^{2\nu}\zeta(2\nu)} \Bigg\|^2_{\infty,F_\xi} \|F_\xi\|(\Lambda,\Lambda).$$

*Proof.* Using Theorem 3 for the deterministic case and proceeding as in the proof of the preceding theorem we complete the proof of Theorem 5. □

**Example 1.** Let

$$f(\cdot,\lambda) \in PW^2_\sigma$$

for all $\lambda$ except for some set of zero semivariation, where $PW^2_\sigma$ is the Paley–Wiener class [5] of all complex-valued $L^2(\mathbb{R})$-functions, the Fourier spectrum of which belongs to $[-\sigma,\sigma]$. Let $\big\|\|f(\cdot,\lambda)\|_2\big\|_{\infty,F_\xi} < +\infty$. Then

$$\frac{2}{\sqrt{2\pi}} \int_{|x|>\pi\omega} |\,d\Phi_\lambda(x)| = 0$$

and

$$\left\| \sum_{n\in\mathbb{Z}} \left| f\left(\frac{n}{\omega}, \lambda\right)\right|^2 \right\|_{\infty,F_\xi} = \omega \big\| \|f(\cdot,\lambda)\|_2 \big\|^2_{\infty,F_\xi} < +\infty$$

for all $\omega \geq \sigma/\pi$. According to (9), $\omega$ and $N$ must tend to infinity in such a way that

$$\mathcal{C}\sqrt{\omega} \to 0 \tag{12}$$

to achieve the convergence to zero of the truncation error.

A question arises whether condition (12) (or another similar condition) is sufficient for the convergence of $\|\mathfrak{T}_{\mathfrak{J}_2,\omega}(\xi;\cdot)\|_\infty$ to zero? Unfortunately there is no sufficient condition for the convergence in the general case (see, for example, [9] concerning the deterministic case). Nevertheless one can obtain analogs of condition (12) that are sufficient for the convergence in some classes of functions.

Consider a subclass of stochastic processes of the weak Cramér class for which the "tails" of the Fourier spectrum with respect to the first argument tend to zero uniformly with respect to the second argument:

$$\mathfrak{G} := \left\{ \xi\colon \lim_{\omega\to\infty} \left\| \int_{|x|>\pi\omega} |\hat{f}(x,\cdot)|\, dx \right\|_{\infty,F_\xi} = 0 \right\}.$$

**Theorem 6.** *Let* $\xi \in \mathfrak{G}$. *Then*

$$\|\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;\cdot)\|_\infty \to 0 \quad \textit{as } \omega \to +\infty.$$

*Consider three cases of Theorem* 5 *for*

a) $\mathfrak{G}_R := \{\xi\colon \xi \in \mathfrak{G}, \|c(\omega,\cdot)\|_{\infty,F_\xi} \leq R_1\}$,
b) $\mathfrak{G}_R := \big\{\xi\colon \xi \in \mathfrak{G}, \big\|\|f(\cdot,\lambda)\|_2\big\|_{\infty,F_\xi} \leq R_1, \big\|\|f'_t(\cdot,\lambda)\|_2\big\|_{\infty,F_\xi} \leq R_2\big\}$, *and*
c) $\mathfrak{G}_R := \{\xi\colon \xi \in \mathfrak{G}, \|f(0,\cdot)\|_{\infty,F_\xi} \leq R_1\}$.

*Here* $R_1$ *and* $R_2$ *are some constants such that* $R_1, R_2 \in [0,+\infty)$.

*Let* $N,\omega \to +\infty$ *in such a way that* (12) *holds for the cases* a) *or* b) *and*

$$\mathcal{C}\omega^\nu \to 0, \qquad N,\omega \to +\infty,$$

*for the case* c).

*If* $\xi \in \mathfrak{G}_R$, *then*

$$\|\mathfrak{T}_{\mathfrak{J}_2,\omega}(\xi;\cdot)\|_\infty \to 0, \qquad N,\omega \to +\infty.$$

*Proof.* Applying Theorems 4 and 5 and recalling the definition of the classes $\mathfrak{G}$ and $\mathfrak{G}_R$ we complete the proof of Theorem 6. □

## 5. Estimates of the aliasing error in the mean square metric

As shown in [2], the results of [1, 7] are incorrect and estimates such as

$$\|A(\cdot,\omega)\|_2^2 \leq \text{const} \cdot \int_{|x|>\pi\omega} |\hat{f}(x)|^2\, dx \tag{13}$$

cannot be true at all.

However, the question of whether other upper bounds exist for $\|A(\cdot,\omega)\|_2$ is still open and the properties of such upper bounds are still unknown.

The left-hand side of (13) can be rewritten as follows:

$$
\begin{aligned}
\|A(\cdot,\omega)\|_2^2 &= \int_{\mathbb{R}} \left| f(t) - \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\omega}\right) \operatorname{sinc}(\omega t - n) \right|^2 dt \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \hat{f}(x) - \frac{1}{\omega} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\omega}\right) \chi_{[-\pi\omega,\pi\omega)}(x) e^{-inx/\omega} \right|^2 dx \qquad (14) \\
&= \frac{1}{2\pi} \left( \int_{|x|>\pi\omega} |\hat{f}(x)|^2\, dx + \int_{|x|<\pi\omega} \left| \hat{f}(x) - \frac{1}{\omega} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\omega}\right) e^{-inx/\omega} \right|^2 dx \right)
\end{aligned}
$$

(see [2]).

As $\omega$ increases, the first term in (14) decreases to 0. Is it true that the second term approaches 0, too? If this were the case one would get a result for the uniform metric telling us that the aliasing error decreases as the support of the spectrum is extended.

Let $f_\omega(\cdot)$ be a function and let $\hat{f}(x)\chi_{[-\pi\omega,\pi\omega)}(x)$ be its Fourier transform. Then $f_\omega(\cdot)$ has a bounded spectrum. Thus the Kotelnikov–Shannon theorem implies that

$$
f_\omega(x) = \sum_{n\in\mathbb{Z}} f_\omega\left(\frac{n}{\omega}\right) \operatorname{sinc}(\omega t - n).
$$

Then the second term in (14) can be rewritten as follows:

$$
\frac{1}{2\pi\omega^2} \int_{|x|<\pi\omega} \left| \sum_{n\in\mathbb{Z}} \left( f_\omega\left(\frac{n}{\omega}\right) - f\left(\frac{n}{\omega}\right) \right) e^{-inx/\omega} \right|^2 dx = \frac{1}{\omega} \sum_{n\in\mathbb{Z}} \left| f_\omega\left(\frac{n}{\omega}\right) - f\left(\frac{n}{\omega}\right) \right|^2 .
$$

Since the spectrum of $f_\omega(\cdot)$ is bounded, $\{f_\omega(n/\omega)\}_{n\in\mathbb{Z}} \in l_2$. The example constructed in [9] shows that there exists a function $f \in \mathcal{F}$ such that

$$
\sum_{n\in\mathbb{Z}} \left| f\left(\frac{n}{\omega_0}\right) \right|^2 = +\infty. \qquad (15)
$$

Thus (15) holds for all $\omega = k\omega_0$ substituted for $\omega_0$.

For such $\omega$,

$$
\|A(\cdot,\omega)\|_2 = +\infty
$$

and an estimate such as (2) whose right-hand side approaches zero as $\omega \to +\infty$ in the mean square norm cannot be obtained at all for the general case.

Nevertheless estimates of this kind are true in some classes of functions.

**Theorem 7.**

$$
\begin{aligned}
\|A(\cdot,\omega)\|_2 &= \frac{1}{\sqrt{2\pi}} \left( \int_{|x|>\pi\omega} |\hat{f}(x)|^2\, dx + \int_{|x|<\pi\omega} \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(x + 2k\pi\omega) \right|^2 dx \right)^{1/2} \\
&\le \frac{1}{\sqrt{2\pi}} \left( \int_{|x|>\pi\omega} |\hat{f}(x)|^2\, dx \right)^{1/2} + \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \left( \int_{-\pi\omega+2k\pi\omega}^{\pi\omega+2k\pi\omega} \left| \hat{f}(x) \right|^2 dx \right)^{1/2} .
\end{aligned}
$$

*Proof.* According to Theorem 3 of [9] the Fourier transform of the function

$$
\sum_{n\in\mathbb{Z}} f_\omega\left(\frac{n}{\omega}\right) \operatorname{sinc}(\omega t - n)
$$

is $\sum_{k\in\mathbb{Z}} \hat{f}(x+2k\pi\omega)\chi_{[-\pi\omega,\pi\omega)}(x)$. Thus one can rewrite (14) as follows:

$$\|A(\cdot,\omega)\|_2^2 = \frac{1}{2\pi}\left(\int_{|x|>\pi\omega} |\hat{f}(x)|^2\,dx + \int_{|x|<\pi\omega}\left|\hat{f}(x) - \sum_{k\in\mathbb{Z}} \hat{f}(x+2k\pi\omega)\right|^2 dx\right)$$
$$= \frac{1}{2\pi}\left(\int_{|x|>\pi\omega} |\hat{f}(x)|^2\,dx + \int_{|x|<\pi\omega}\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(x+2k\pi\omega)\right|^2 dx\right).$$

Now the upper bound follows from the well-known properties of the norm. □

According to Theorem 7, a necessary and sufficient condition for $\|A(\cdot,\omega)\|_2 \to 0$ is

$$\int_{|x|<\pi\omega}\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(x+2k\pi\omega)\right|^2 dx \to 0, \qquad \omega\to\infty. \tag{16}$$

Below we give some examples of classes of functions for which condition (16) holds.

**Example 2.** The integral in condition (16) equals 0 if $\omega > \sigma/\pi$ for the Paley–Wiener class $PW_\sigma^2$ of functions whose support of the spectrum belongs to $[-\sigma,\sigma]$.

**Example 3.** Consider functions such that $|\hat{f}(x)| \le K a^{|x|}$ where $K$ and $a$ are some constants, $a\in(0,1)$. Now, in contrast with the preceding case, the support of the spectrum is unbounded. We have

$$\int_{|x|<\pi\omega}\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(x+2k\pi\omega)\right|^2 dx \le \int_{|x|<\pi\omega}\left(\sum_{k\in\mathbb{Z}\setminus\{0\}}\left|\hat{f}(x+2k\pi\omega)\right|\right)^2 dx$$
$$= 4K^2\int_{|x|<\pi\omega}\left(\sum_{k\in\mathbb{N}} a^{x+2k\pi\omega}\right)^2 dx = \frac{2K^2(1+a^{2\pi\omega})}{\ln a(a^{2\pi\omega}-1)}a^{2\pi\omega} \to 0, \qquad \omega\to\infty.$$

**Example 4.** Consider functions such that $|\hat{f}(x)| \le K|x|^{-\alpha}$, where $K$ and $\alpha$ are some constants, $\alpha>1$. Again the support of the spectrum is unbounded in this case. By Theorem 7,

$$\begin{aligned}\int_{|x|<\pi\omega}\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(x+2k\pi\omega)\right|^2 dx &\le \left(2K\sum_{k\in\mathbb{N}}\left(\int_{-\pi\omega+2k\pi\omega}^{\pi\omega+2k\pi\omega}\frac{dx}{x^{2\alpha}}\right)^{1/2}\right)^2\\ &= \frac{4K^2}{(2\alpha-1)(2\pi\omega)^{2\alpha-1}}\left(\sum_{k\in\mathbb{N}}\left(\frac{1}{(k-\frac12)^{2\alpha-1}} - \frac{1}{(k+\frac12)^{2\alpha-1}}\right)^{1/2}\right)^2 \to 0\\ &\qquad \omega\to\infty,\end{aligned} \tag{17}$$

since

$$\frac{1}{(k-\frac12)^{2\alpha-1}} - \frac{1}{(k+\frac12)^{2\alpha-1}} \sim \frac{\text{const}}{k^{2\alpha}}, \qquad k\to\infty,$$

and the series in (17) converges.

## 6. The approximation of processes generated by orthogonal stochastic measures

Let $Z_\xi(\cdot)$ in representation (7) be an orthogonal stochastic measure [4]. Then

$$m(\triangle) := F_\xi(\triangle,\triangle)$$

is the structure function for $Z_\xi(\cdot)$.

**Theorem 8.**

$$\|\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;\cdot)\|_1 = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{|x|>\pi\omega} |\hat{f}(x,\lambda)|^2\,dx\,dm(\lambda) + \frac{1}{2\pi}\int_{\mathbb{R}}\int_{|x|<\pi\omega}\left|\sum_{k\in\mathbb{Z}\setminus\{0\}}\hat{f}(x+2k\pi\omega,\lambda)\right|^2 dx\,dm(\lambda).$$

*Proof.* Using the properties of the integral with respect to an orthogonal stochastic measure and by the definition of $\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;t)$ we have

$$\|\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;\cdot)\|_1 = \int_{\mathbb{R}}\int_{\mathbb{R}}\left|f(t,\lambda) - \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\omega},\lambda\right)\operatorname{sinc}(\omega t - n)\right|^2 dm(\lambda)\,dt = \int_{\mathbb{R}} \|A(\cdot,\omega,\lambda)\|_2^2\,dm(\lambda).$$

Thus Theorem 8 follows from Theorem 7. □

**Example 5.** Let $m(\mathbb{R}) < +\infty$, and let the functions of Examples 3 or 4 depend on $\lambda$. Assume that all the above upper bounds hold and either

$$|\hat{f}(x,\lambda)| \le K a^{|x|}$$

or

$$|\hat{f}(x,\lambda)| \le K|x|^{-\alpha}.$$

Then $\|\mathfrak{T}_{\mathfrak{J}_1,\omega}(\xi;\cdot)\|_1 \to 0$ if $\omega \to \infty$.

## 7. Concluding remarks

We obtained precise uniform upper bounds for the aliasing error of the interpolation of nonrandom functions with unbounded spectra and those for stochastic processes of the weak Cramér class whose kernel functions $f$ in the spectral representation (7) are represented in the form of (1). We proved that such mean square estimates are not available in the general case. We constructed examples of estimates for several classes of functions and found extreme functions.

An interesting problem for further consideration is to obtain analogous estimates for other metrics (in $L_p$, for example) and to consider not only the mean square convergence but also the almost sure convergence.

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