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EULER APPROXIMATIONS OF ANTICIPATING QUASILINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We obtain the rate of convergence of Euler type approximations for an anticipating quasilinear stochastic differential equation involving the white noise integral.

Introduction

There are numerous applications of anticipating stochastic differential equations. A typical application is presented by models of a market of securities where there exists an "insider" possessing partial information about the future fluctuations of prices in the market. Thus there is a real need to develop a method of solving such stochastic differential equations (or, at least, of finding approximations of their solutions).

There are several known generalizations of the Itô integral for the case of anticipating integrands. The procedure of the evaluation of an approximate solution is clear for pathwise integrals (such as the forward stochastic integral or Stratonovich integral); namely, it is sufficient to approximate the flow generated by the nonanticipating equation and use the approximate initial value. Results concerning approximations for the Stratonovich integral can be found in [1, 2]. The forward integral can be defined with the help of an enlargement of filtration that reduces it to an integral with respect to a semimartingale. Approximations of solutions of stochastic differential equations with semimartingales are discussed in [5]. The case of the Skorokhod integral (or, which is the same, of the white noise integral) is not so easy, since it is not defined pathwise and is not a continuous operator in any space $\mathcal{L}^p$. This results not only in the discretization in time but also in a shift in ω at every step of the discretization.

1. Approximations of solutions obtained by the method of variation of parameters

Let $(\Omega, \mathcal{F}, \mathsf{P}) = (\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$ be a standard white noise space, $\cdot \diamondsuit \cdot$ be the Wick product,

$$B(t) = \langle \omega, \mathbb{1}_{[0,t]} \rangle$$

be the Wiener process (Brownian motion), $W(t) = \dot{B}(t)$ be a white noise, and

$$D_t F(\omega) = \lim_{\Delta \to 0} \Delta^{-1} \left(F(\omega + \mathbb{1}_{[t,t+\Delta]}) - F(\omega) \right)$$

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be the stochastic derivative. For $h\colon [0,\infty) \to \mathbb{R}$ we define the shift operator

$$T_h F(\omega) = F(\omega + h).$$

More details concerning the white noise space can be found in [3, Chapter 1]. Below we denote by C all constants whose values are not important for the purposes of this paper.

Consider the quasilinear stochastic differential equation

$$X(t) = X_0 + \int_0^t b(s, X(s), \omega)\, ds + \int_0^t \sigma(s) X(s) \diamondsuit W(s)\, ds \tag{1.1}$$

on the interval $[0,T]$, where $X_0 \in \mathbb{R}$, σ is a certain real bounded nonrandom function, and $b(s,x,\omega)$ is, generally speaking, an anticipating random function. Our goal is to construct discrete-time Euler type approximations of a given equation. At a first glance, a natural way is to take $\delta = T/N$ for a positive integer N, to consider a uniform partition $\tau_n = n\delta$, $n = 0, \dots, N$, of the interval $[0,T]$, and to construct approximations as follows:

$$\widetilde{X}(0) = X(0), \qquad \widetilde{X}(\tau_{n+1}) = \widetilde{X}(\tau_n) + b\big(t, \widetilde{X}(\tau_n), \omega\big)\delta + \sigma(s)\widetilde{X}(\tau)_n \diamondsuit \Delta B_n, \tag{1.2}$$

where $\Delta B_n = B(\tau_{n+1}) - B(\tau_n)$. However, there are some problems when following this idea. First, it is not easy to evaluate the Wick product on the right-hand side. Second, even if one were able to approximate the solution by using the equality

$$F \diamondsuit \Delta B_n = F \cdot \Delta B_n - \int_{\tau_n}^{\tau_{n+1}} D_s F\, ds,$$

then one would face a problem when proving that approximations (1.2) converge, namely, when estimating the stochastic integral, since it is unbounded in every space $\mathcal{L}^p$ (thus the image of a function under this operator may not belong to $\mathcal{L}^p$). Therefore this natural way leads to many complications. Thus we want to study other possible approaches to the numerical solution of the stochastic differential equation.

2. Approximations for the reduced nonrandom equation

The first approach is to apply the Gjessing formula (see, for example, [3, Theorem 2.10.7]) that allows one to exclude the stochastic integral from equation (1.1) and thus to reduce it to an ordinary pathwise differential equation.

More precisely, let

$$J_\sigma(t) = \exp^\diamondsuit \left\{ -\int_0^t \sigma(s)\, dB(s) \right\} = \exp\left\{ -\int_0^t \sigma(s)\, dB(s) - \frac{1}{2}\int_0^t \sigma^2(s)\, ds \right\}$$

be the Wick exponent, $\sigma_t = \sigma \mathbb{1}_{[0,t]}$, and let

$$Z(t) = J_\sigma(t) \diamondsuit X(t) = J_\sigma(t) T_{\sigma_t} X(t).$$

Then

$$Z(t) = X_0 + \int_0^t J_\sigma(s) b\left(s, J_\sigma^{-1}(s) Z(s), \omega + \sigma_s\right)\, ds \tag{2.1}$$

(the latter is an ordinary differential equation). Now we discretize the time in this equation in a way similar to (1.2). In other words, we define the approximations step by step as follows:

$$\widetilde{Z}(0) = X_0, \qquad \widetilde{Z}(\tau_{n+1}) = \widetilde{Z}(\tau_n) + J_{\widetilde{\sigma}}(\tau_n) b\big(\tau_n, J_{\widetilde{\sigma}}^{-1}(\tau_n)\widetilde{Z}(\tau_n), \omega + \widetilde{\sigma}_n\big)\delta, \tag{2.2}$$

where $\widetilde{\sigma}_n = \sum_{i=0}^{n-1} \widetilde{\sigma}^k$, $\widetilde{\sigma}^k = \sigma(\tau_k)\mathbb{1}_{[\tau_k, \tau_{k+1}]}$, and $\widetilde{\sigma} = \widetilde{\sigma}_N$. Put $n_s = \min\{n\colon \tau_n \le s\}$, $t_s = \tau_{n_s}$, and consider the continuous interpolation for (2.2):

$$\widetilde{Z}(t) = X_0 + \int_0^t J_{\widetilde{\sigma}}(t_s) b\big(t_s, J_{\widetilde{\sigma}}^{-1}(t_s)\widetilde{Z}(t_s), \omega + \widetilde{\sigma}_{n_s}\big)\, ds. \tag{2.3}$$

Now we define the approximations for the initial equation by

$$\widetilde{X}(t) = J_{-\widetilde{\sigma}}(t) \diamondsuit \widetilde{Z}(t) = J_{-\widetilde{\sigma}}(t) T_{-\widetilde{\sigma}\mathbb{1}_{[0,t]}} \widetilde{Z}(t). \tag{2.4}$$

Below we list the conditions on the coefficients that we use in the proof of the convergence of approximations.

(1) The coefficient b grows at most linearly with respect to x, and its derivative b'_x is bounded:

$$\begin{gathered} |b(t,x,\omega)| \le C(1+|x|), \\ |b'_x(t,x,\omega)| \le C; \end{gathered} \tag{2.5a}$$

(2) the coefficient b has the stochastic differential and its derivative grows at most linearly with respect to x:

$$|D_s b(t,x,\omega)| \le C(1+|x|); \tag{2.5b}$$

(3) the coefficient b, considered as a function of t, is Hölder of index $\frac{1}{2}$ and with constant growing at most linearly with respect to x:

$$|b(t,x,\omega) - b(s,x,\omega)| \le C(1+|x|)\,|t-s|^{1/2}; \tag{2.5c}$$

(4) the coefficient σ is Hölder of index $\frac{1}{2}$:

$$|\sigma(t) - \sigma(s)| \le C\,|t-s|^{1/2}. \tag{2.5d}$$

Note that condition (2.5a) and the boundedness of σ imply that a solution of equation (1.1) exists and, moreover, the solution belongs to all spaces $\mathcal{L}^p$, $p > 1$ (see [3, Theorem 3.6.1]).

Prior to stating the main result of this section we prove some auxiliary results.

Lemma 2.1. *If condition* (2.5d) *holds, then*

$$\mathsf{E}\left[\left(J_{\pm\sigma}(s) - J_{\pm\widetilde{\sigma}}(t_s)\right)^{2p}\right] \le C\delta^p$$

for all $p \ge 1$.

Proof. First we note that $J_{\pm\sigma}(s)$ is a solution of the stochastic differential equation

$$dJ_{\pm\sigma}(s) = \pm\sigma(s) J_{\pm\sigma}(s)\, dB(s)$$

and $J_{\pm\widetilde{\sigma}}(s)$ is the Euler approximation for this solution. Now Lemma 2.1 follows from elementary properties of solutions of the latter equation (see, for example, [4]). □

Lemma 2.2. *If condition* (2.5b) *holds, then*

$$|b(t,x,\omega) - b(t,x,\omega+h)| \le C(1+|x|) \int_0^T |h(s)|\, ds$$

for all $h \in \mathcal{L}^1[0,T]$.

Proof. We derive Lemma 2.2 directly from condition (2.5b), since

$$b(t,x,\omega) - b(t,x,\omega+h) = \int_0^T \int_0^1 D_t b(t,x,\omega+sh) h(t)\, ds\, dt.$$ □

Lemma 2.3. *If condition* (2.5a) *holds, then*

$$\left|e^{\alpha_1} b(t, e^{-\alpha_1}x, \omega) - e^{\alpha_2} b(t, e^{-\alpha_2}x, \omega)\right| \le C(1 + e^{\alpha_1} + e^{\alpha_2} + |x|)\,|\alpha_1 - \alpha_2|.$$

Proof. Lemma 2.3 follows from the mean value formula and condition (2.5a). □

Lemma 2.4. *If condition* (2.5) *holds, then*

$$\mathsf{E}\left[\left(Z(t)-\widetilde{Z}(t)\right)^{2p}\right]\le C_p\delta^p$$

for all $p\ge 1$.

Proof. First we note that $Z(t)$ and $X(t)$ belong to all spaces $\mathcal{L}^p$ according to [3, Theorem 3.6.1]. Moreover $\mathsf{E}\left[|Z(t)|^p\right]$ and $\mathsf{E}\left[|X(t)|^p\right]$ are bounded with respect to t. Then it follows from equation (2.1) and condition (2.5a) that

$$\mathsf{E}\left[\left(Z(t)-Z(s)\right)^{2p}\right]\le C_p\left|t-s\right|^{2p}. \tag{2.6}$$

Furthermore, relation (2.2) and condition (2.5a) imply

$$\left|\widetilde{Z}(\tau_{n+1})\right|\le(1+C\delta)\le e^{C\delta}\left|\widetilde{Z}(\tau_n)\right|,$$

whence $\mathsf{E}\big[\widetilde{Z}^{2p}(\tau_n)\big]\le C_p$. Then we obtain from equation (2.3) that

$$\mathsf{E}\big[\widetilde{Z}^{2p}(t)\big]\le C_p.$$

Now we estimate $\left|Z(t)-\widetilde{Z}(t)\right|$ by

$$\left|Z(t)-\widetilde{Z}(t)\right|\le A_1+A_2+A_3+A_4+A_5,$$

where

$$\begin{aligned}
A_1&=\left|\int_0^t J_{\widetilde{\sigma}}(t_s)\big(b(t_s,J_{\widetilde{\sigma}}^{-1}(t_s)Z(t_s),\omega+\widetilde{\sigma}_{n_s})-b(t_s,J_{\widetilde{\sigma}}^{-1}(t_s)\widetilde{Z}(t_s),\omega+\widetilde{\sigma}_{n_s})\big)\,ds\right|,\\
A_2&=\left|\int_0^t \big(J_{\widetilde{\sigma}}(t_s)b(t_s,J_{\widetilde{\sigma}}^{-1}(t_s)Z(t_s),\omega+\widetilde{\sigma}_{n_s})-J_{\sigma}(s)b(t_s,J_{\sigma}^{-1}(s)Z(t_s),\omega+\widetilde{\sigma}_{n_s})\big)\,ds\right|,\\
A_3&=\left|\int_0^t J_{\sigma}(s)\big(b(s,J_{\sigma}^{-1}(s)Z(t_s),\omega+\widetilde{\sigma}_{n_s})-b(t_s,J_{\sigma}^{-1}(s)Z(t_s),\omega+\widetilde{\sigma}_{n_s})\big)\,ds\right|,\\
A_4&=\left|\int_0^t J_{\sigma}(s)\big(b(s,J_{\sigma}^{-1}(s)Z(t_s),\omega+\widetilde{\sigma}_{n_s})-b(s,J_{\sigma}^{-1}(s)Z(t_s),\omega+\sigma_t)\big)\,ds\right|,\\
A_5&=\left|\int_0^t J_{\sigma}(s)\big(b(s,J_{\sigma}^{-1}(s)Z(s),\omega+\sigma_t)-b(s,J_{\sigma}^{-1}(s)Z(t_s),\omega+\sigma_t)\big)\,ds\right|.
\end{aligned}$$

It follows from Lemma (2.3) that

$$\begin{aligned}
A_2&\le C\int_0^t\big(1+J_{\sigma}(s)+J_{\widetilde{\sigma}}(t_s)+|Z(t_s)|\big)\left(\left|\int_0^s\big(\sigma(u)-\widetilde{\sigma}(u)\big)\,dB(u)\right|\right.\\
&\qquad\left.+\left|\sigma(t_s)\big(B(s)-B(t_s)\big)\right|+\frac{1}{2}\int_0^s\left|\sigma^2(u)-\widetilde{\sigma}^2(u)\right|\,du\right)ds\\
&\le C\int_0^t\big(1+J_{\sigma}(s)+J_{\widetilde{\sigma}}(t_s)+|Z(t_s)|\,ds\big)\\
&\qquad\times\left(\left|\int_0^s\big(\sigma(u)-\widetilde{\sigma}(u)\big)\,dB(u)\right|+|B(s)-B(t_s)|+\delta^{1/2}\right)ds.
\end{aligned}$$

Using condition (2.5d) we get

$$A_3\le C\int_0^t\big(J_{\sigma}(s)+|Z(t_s)|\big)\,ds\,\delta^{1/2}.$$

Lemma 2.2 implies that

$$A_4 \le C \int_0^t \big(J_\sigma(s) + |Z(t_s)|\big)\, ds\, \delta^{1/2}.$$

When estimating the terms A_1 and A_5 we apply the boundedness of the derivative b'_x:

$$A_1 \le C \int_0^t \left|Z(t_s) - \widetilde{Z}(t_s)\right| ds,$$

$$A_5 \le C \int_0^t |Z(s) - Z(t_s)|\, ds.$$

Then the Gronwall lemma yields

$$\begin{aligned}\left|Z(t) - \widetilde{Z}(t)\right| &\le C \int_0^T |Z(s) - Z(t_s)|\, ds \\ &\quad + C \int_0^t \big(1 + J_\sigma(s) + J_{\tilde\sigma}(t_s) + |Z(t_s)|\big) \\ &\qquad \times \left(\left|\int_0^s \big(\sigma(u) - \widetilde{\sigma}(u)\big)\, dB(u)\right| + |B(s) - B(t_s)| + \delta^{1/2}\right) ds.\end{aligned}$$

Raising to the power $2p$, taking the expectations, using the Jensen, Hölder, and Cauchy–Schwartz inequalities, and applying estimate (2.6) we get

$$\begin{aligned}\mathsf{E}\left[\big(Z(t) - \widetilde{Z}(t)\big)^{2p}\right] &\le C_p \left(\delta^p + \int_0^T \mathsf{E}\left[\left(\int_0^s (\sigma(u) - \sigma(t_u))\, dB(u)\right)^{2p}\right] ds + \delta^p\right) \\ &\le C_p \left(\delta^p + \int_0^T \left(\int_0^s (\sigma(u) - \sigma(t_u))^2\, du\right)^p ds\right) \le C_p \delta^p,\end{aligned}$$

since the moments of Z, J_σ, and $J_{\tilde\sigma}$ are bounded. This is what had to be proved. □

Now we are ready to state the main result.

Theorem 2.5. *Let conditions* (2.5) *hold. Then the approximations* $\widetilde{X}$ *defined in* (2.4) *converge in the mean square sense to the solution* X *of equation* (1.1). *Moreover, the rate of convergence is* $\frac{1}{2}$*:*

$$\mathsf{E}\left[\big(X(t) - \widetilde{X}(t)\big)^2\right] \le C\delta.$$

Proof. First we estimate $|T_h Z(t) - Z(t)|$ as follows:

$$|T_h Z(t) - Z(t)| \le A_1 + A_2 + A_3,$$

where

$$A_1 = \int_0^t T_h J_\sigma(s) \left|b(s, (T_h J_\sigma^{-1}) T_h Z(s), \omega + h + \sigma_s) - b(s, (T_h J_\sigma^{-1}) Z(s), \omega + h + \sigma_s)\right| ds,$$

$$A_2 = \int_0^t T_h J_\sigma(s) \left|b(s, (T_h J_\sigma^{-1}) Z(s), \omega + h + \sigma_s) - b(t, (T_h J_\sigma^{-1}(s)) Z(s), \omega + \sigma_s)\right| ds,$$

$$A_3 = \int_0^t \left|T_h J_\sigma(s) b(t, (T_h J_\sigma^{-1}(s)) Z(s), \omega + \sigma_s) - J_\sigma(s) b(t, J_\sigma^{-1}(s) Z(s), \omega + \sigma_s)\right| ds.$$

Condition (2.5a) implies that $A_1 \le C \int_0^t |T_h Z(s) - Z(s)|\, ds$. Lemma 2.2 yields

$$A_2 \le C \int_0^T (T_h J_\sigma(s) + |Z(s)|)\, ds \int_{\mathbb{R}} |h(s)|\, ds,$$

while Lemma 2.3 gives

$$A_3 \le C \int_0^T (1 + T_h J_\sigma(s) + J_\sigma(s) + |Z(s)|) \left| \int_0^s h(u)\sigma(u)\, du \right| ds.$$

Applying the Gronwall lemma we obtain

$$|T_h Z(t) - Z(t)| \le C \int_0^T (1 + T_h J_\sigma(s) + J_\sigma(s) + |Z(s)|) \left(|h(s)| + \left| \int_0^s h(u)\, dB(u) \right| \right) ds.$$

Raising to the power $2p$ for $p \ge 1$, taking the expectations, and using the Hölder inequality we get

$$\mathsf{E}\left[\big(T_h Z(t) - Z(t)\big)^{2p} \right] \le C_p \int_0^T |h|^{2p}(s)\, ds,$$

since the moments of Z and J_σ are bounded. Now

$$\begin{aligned} \mathsf{E}\left[\big(X(t) - \widetilde{X}(t)\big)^2 \right] \le 3\Big(& \mathsf{E}\left[\big(J_{-\widetilde{\sigma}}(t) T_{-\widetilde{\sigma}\mathbb{1}_{[0,t]}} \big(Z(t) - \widetilde{Z}(t)\big)\big)^2 \right] \\ & + \mathsf{E}\left[\big(\big(J_{-\sigma}(t) - J_{-\widetilde{\sigma}}(t)\big) T_{-\widetilde{\sigma}\mathbb{1}_{[0,t]}} Z(t)\big)^2 \right] \\ & + \mathsf{E}\left[\big(J_{-\sigma}(t) \big(T_{\sigma_t} - T_{\widetilde{\sigma}\mathbb{1}_{[0,t]}}\big) Z(t)\big)^2 \right] \Big). \end{aligned}$$

To estimate the terms on the right-hand side we apply the Cauchy–Schwarz inequality two times, then use Lemma 2.4 for the first term and Lemma 2.1 for the second term. For the third term we apply the Girsanov theorem and the boundedness of moments of Z. Theorem 2.5 is proved. □

3. Approximations for the solution obtained by an analog of the "splitting-up" method

First we provide heuristic arguments that lead to another construction of approximations. Assume that equation (1.1) possesses a unique stochastically differentiable solution whose stochastic derivative is square integrable. This, in particular, implies that the integral with respect to the white noise on the right-hand side of equation (1.1) coincides with the Skorokhod integral that can be evaluated by a known formula:

$$\int_0^t \sigma(s) X(s) \diamondsuit W(s)\, ds = \int_0^t \sigma(s) X(s)\, d^- B(s) - \int_0^t \sigma(s) D_s^- X(s)\, ds,$$

where $\int \dots d^- B(s)$ is the forward stochastic integral, and $D_s^- X(s) = \lim_{\varepsilon \to 0+} D_s X(s - \varepsilon)$ (see, for example, [6, Section 3.1]). Thus we obtain the equation

$$X(t) = X_0 + \int_0^t b(s, X(s), \omega) + \int_0^t \sigma(s) X(s)\, d^- B(s) - \int_0^t \sigma(s) D_s^- X(s)\, ds, \tag{3.1}$$

which can be viewed as a stochastic differential equation with an unbounded linear operator $A(s) = -\sigma(s) D_s$ on the right-hand side. This operator generates the evolutionary family of shifts on $L_p(\Omega)$:

$$U(t,s) F(\omega) = F\left(w - \sigma(\cdot)\mathbb{1}_{[s,t]}(\cdot)\right) = T_{-\sigma\mathbb{1}_{[s,t]}} F(\omega).$$

Similarly to the method of variation of parameters we obtain the equation

$$X(t) = X_0 + \int_0^t U(t,s) b(s, X(s), \omega)\, ds + \int_0^t U(t,s) \sigma(s) X(s)\, d^- B(s). \tag{3.2}$$

The latter equation can be checked by differentiation.

Under some assumptions, the forward stochastic integral $\int f(s)\,d^-B(s)$ is a limit of forward integral sums $\sum f(s_i)(s_{i+1}-s_i)$. This suggests constructing the approximations of equation (1.1) as follows:

$$\begin{gathered} X^\delta(0)=0,\\ X^\delta(\tau_{n+1})=U^\delta(\tau_{n+1},\tau_n)\left[X^\delta(\tau_n)+b(\tau_n,X^\delta(\tau_n),\omega)\delta+\sigma(\tau_n)\Delta B_n\right], \end{gathered} \tag{3.3}$$

where $U^\delta(\tau_{n+1},\tau_n)F(\omega)=F(\omega-\widetilde{\sigma}^n(\cdot))$.

Remark 3.1. The approximations constructed above can be viewed as an analog of the "splitting-up" method well known in the theory of ordinary differential equations. We split equation (3.1) into two equations:

$$\begin{gathered} dX_1(t)=a(t,X_1(t))\,dt+b(t,X_1(t),\omega)\,d^-B(t),\\ dX_2(t)=\sigma(t)D_t^-X_2(t)\,dt. \end{gathered}$$

Now we use the recursive procedure, namely, put $X^\delta(0)=X_0$ and use the approximation $X^\delta(\tau_n)$ of the solution at the node n of the partition as the initial value for the first equation. Then we construct approximations for the first equation by solving it numerically in the interval $[\tau_n,\tau_{n+1}]$:

$$X_1^\delta(\tau_{n+1})=X^\delta(\tau_n)+b\big(\tau_n,X^\delta(\tau_n),\omega\big)\delta+b\big(\tau_n,X^\delta(\tau_n)\big)\Delta B_n.$$

This approximation is taken as the initial condition for the second equation; the approximations for the second equation can be constructed under the condition that the function σ is constant on $[\tau_n,\tau_{n+1}]$:

$$X^\delta(\tau_{n+1})=X_2^\delta(\tau_{n+1})=U^\delta(\tau_{n+1},\tau_n)X_1^\delta(\tau_{n+1}).$$

We rewrite approximations (3.3) as follows:

$$X^\delta(\tau_{n+1})=T_{-\widetilde{\sigma}^n}\left[X^\delta(\tau_n)+b(\tau_n,X^\delta(\tau_n),\omega)\delta+\sigma(\tau_n)X^\delta(\tau_n)\Delta B_n\right]. \tag{3.4}$$

As above put

$$Z^\delta(\tau_n)=J_{\widetilde{\sigma}}(\tau_n)\diamondsuit X^\delta(\tau_n).$$

Applying the Gjessing formula [3, Theorem 2.10.7], we make sure that

$$\begin{aligned} Z^\delta(\tau_{n+1})&=J_{\widetilde{\sigma}^n}(\tau_{n+1})Z^\delta(\tau_n)\\ &\quad+J_{\widetilde{\sigma}^n}(\tau_{n+1})J_{\widetilde{\sigma}}(\tau_n)T_{\widetilde{\sigma}_n}\big(b(\tau_n,X^\delta(\tau_n),\omega)\delta+\sigma(\tau_n)X^\delta(\tau_n)\Delta B_n\big), \end{aligned}$$

whence

$$\begin{aligned} Z^\delta(\tau_{n+1})&=J_{\widetilde{\sigma}^n}(\tau_{n+1})Z^\delta(\tau_n)\big(1+\sigma(\tau_n)\Delta B_n\big)\\ &\quad+J_{\widetilde{\sigma}}(\tau_{n+1})b\big(\tau_n,J_{\widetilde{\sigma}}^{-1}(\tau_n)Z^\delta(\tau_n),\omega+\widetilde{\sigma}_n\big)\delta. \end{aligned} \tag{3.5}$$

The expression on the right-hand side of (3.5) is similar to that of (2.2). Consider the continuous interpolation Z^δ, namely, put

$$\begin{aligned} Z^\delta(t)&=J_{\widetilde{\sigma}^n}(t)Z^\delta(\tau_n)\big(1+\sigma(\tau_n)(B(t)-B(\tau_n))\big)\\ &\quad+J_{\widetilde{\sigma}}(t)b\big(\tau_n,J_{\widetilde{\sigma}}^{-1}(\tau_n)Z^\delta(\tau_n),\omega+\widetilde{\sigma}_n\big)(t-\tau_n) \end{aligned} \tag{3.6}$$

for $t\in[\tau_n,\tau_{n+1}]$. The construction of the interpolation X^δ is obvious.

Below is the main result of this section.

Theorem 3.2. *Let conditions* (2.5) *hold. Then the approximations* X^δ *defined in* (3.4) *converge to the solution* X *of equation* (1.1). *Moreover,*

$$\mathsf{E}\left[\big(X(t)-X^\delta(t)\big)^2\right]\le C\delta.$$

Proof. As in the proof of the preceding theorem it is sufficient to prove that

$$\mathsf{E}\left[(Z(t)-Z^\delta(t))^{2p}\right] \le C_p\delta^p.$$

This bound follows from the inequality $\mathsf{E}\big[(\widetilde{Z}(t)-Z^\delta(t))^{2p}\big] \le C_p\delta^p$ in view of Lemma 2.4. Using equality (3.5) we easily obtain that

$$\left|Z^\delta(\tau_{n+1})\right| \le J_{\widetilde{\sigma}^n}(\tau_{n+1})\big(|1+\sigma(\tau_n)\Delta B_n| + C\delta\big)\left|Z^\delta(\tau_n)\right|,$$

whence

$$\begin{aligned}\left|Z^\delta(\tau_{n+1})\right|^{2q} &\le J^{2q}_{\widetilde{\sigma}^n}(\tau_{n+1})\big((1+C_q\delta)(1+\sigma(\tau_n)\Delta B_n)^{2q} + C\delta\big)\left|Z^\delta(\tau_n)\right|^{2q}\\ &\le \exp\left\{C_q\big(\delta+\sigma(\tau_n)\Delta B_n\big)\right\}\left|Z^\delta(\tau_n)\right|^{2q}\end{aligned}$$

for $q \ge 1$, since

$$(x+\delta)^{2q} \le (1+\delta)^{2q-1}x^{2q} + (1+1/\delta)^{2q-1}\delta^{2q} \le (1+C_q\delta)x^{2q} + C_q\delta.$$

Then

$$\big(Z^\delta(\tau_n)\big)^{2q} \le C_q \exp\left\{\int_0^{\tau_n}\sigma(t_s)\,dB(s)\right\}|x_0|$$

and thus $\mathsf{E}\big[\big(Z^\delta(\tau_n)\big)^{2q}\big] \le C_q$. In its turn the latter bound implies that

$$\mathsf{E}\left[\big(Z^\delta(s)-Z^\delta(t_s)\big)^{2p}\right] \le C_p\delta^p$$

by equality (3.6). Thus it remains to prove that $\mathsf{E}\big[\big(\widetilde{Z}(\tau_n)-Z^\delta(\tau_n)\big)^{2p}\big] \le C_p\delta^p$. Put

$$\begin{aligned}\alpha_n &= 1 - J_{\widetilde{\sigma}^n}(\tau_{n+1})(1+\sigma(\tau_n)\Delta B_n),\\ \beta_n &= J_{\widetilde{\sigma}^n}(\tau_{n+1}) - 1.\end{aligned}$$

Explicit calculations show that

$$\begin{aligned}\mathsf{E}\left[\alpha_n^{2q}\right] &\le C_q\delta^{3q},\\ \mathsf{E}\left[\beta_n^{2q}\right] &\le C_q\delta^q.\end{aligned}$$

Let $d_{n+1} := \left|\widetilde{Z}(\tau_{n+1}) - Z^\delta(\tau_{n+1})\right| = |a_1+a_2+a_3+a_4|$, where

$$\begin{aligned}a_1 &= \widetilde{Z}(\tau_n) - Z^\delta(\tau_n), \qquad a_2 = \alpha_n Z^\delta(\tau_n),\\ a_3 &= J_{\widetilde{\sigma}}(\tau_n)\left(b\big(t, J^{-1}_{\widetilde{\sigma}}(\tau_n)\widetilde{Z}(\tau_n), \omega-\widetilde{\sigma}_n\big) - b\big(t, J^{-1}_{\widetilde{\sigma}}(\tau_n)Z^\delta(\tau_n), \omega-\widetilde{\sigma}_n\big)\right)\delta,\\ a_4 &= \beta_n J_{\widetilde{\sigma}}(\tau_n) b\big(\tau_n, J^{-1}_{\widetilde{\sigma}}(\tau_n)Z^\delta(\tau_n), \omega-\widetilde{\sigma}_n\big)\delta.\end{aligned}$$

Using the inequality $(x+y)^{2p} \le (1+C_p\delta)x^{2p} + (1+1/\delta)^{2p-1}y^{2p}$ we obtain

$$\begin{aligned}&\mathsf{E}\left[\big(Z(\tau_{n+1}) - Z^\delta(\tau_{n+1})\big)^{2p}\right]\\ &\quad\le (1+C_p\delta)\,\mathsf{E}\left[a_1^{2p}\right] + C_p(1+1/\delta)^{2p-1}\left(\mathsf{E}\left[a_2^{2p}\right] + \mathsf{E}\left[a_3^{2p}\right] + \mathsf{E}\left[a_4^{2p}\right]\right)\\ &\quad\le (1+C_p\delta)\,\mathsf{E}\left[\big(Z(\tau_n) - Z^\delta(\tau_n)\big)^{2p}\right]\\ &\qquad + C_p(1+1/\delta)^{2p-1}\Big(\delta^{3p} + \mathsf{E}\left[\big(Z(\tau_n)-Z^\delta(\tau_n)\big)\right]^{2p}\delta^{2p} + \delta^{3p}\Big)\\ &\quad\le (1+C_p\delta)\,\mathsf{E}\left[\big(Z(\tau_n) - Z^\delta(\tau_n)\big)^{2p}\right] + C_p\delta^{p+1}.\end{aligned}$$

Now we apply the induction to get

$$\mathsf{E}\left[\big(Z(\tau_n)-Z^\delta(\tau_n)\big)\right]^{2p} \le C_p\delta^p e^{C_pT} \le C_p\delta^p. \qquad \square$$

Remark 3.3. Following the same method one can prove that

$$\mathsf{E}\left[\left|X(t)-\widetilde{X}(t)\right|^s\right] \le C_s\delta^{s/2},$$
$$\mathsf{E}\left[\left|X(t)-X^\delta(t)\right|^s\right] \le C_s\delta^{s/2}$$

for $s \ge 1$. Theorems 2.5 and 3.2 above are stated for the case of $s = 2$ to simplify the calculations in the proof and to give the results in the "classical" form.

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