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WAVELET ORTHOGONAL APPROXIMATION OF FRACTIONAL GENERALIZED RANDOM FIELDS ON BOUNDED DOMAINS

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Abstract. We consider a class of generalized random fields defined on bounded domains, which admit a white-noise linear filter representation in terms of linear operators of fractional order. We obtain two-sided estimates of the eigenvalues defining the pure point spectra of the associated class of covariance operators. We next derive an orthonormal basis of the reproducing kernel Hilbert space from an orthonormal basis of wavelet functions. An alternative orthogonal expansion to the Karhunen–Loève expansion is then obtained in terms of wavelet functions. The results derived can be applied to compute, in particular, the mean-square solution to fractional-order integro-differential equations, and to approximate least-squares linear estimates for the class of random fields considered.

1. Introduction

The theory of generalized random fields defined on Sobolev spaces of fractional order provides a suitable framework to study random fields with (positive or negative) fractional regularity order, particularly those defined as the mean-square solutions of fractional-order integro-differential equations (see [50, 51]). We refer to these as *fractional generalized random fields* (FGRFs). In the case where these random fields are defined on $\mathbf{R}^d$, their second-order structure can be characterized in terms of the continuous spectrum of the covariance operator, or equivalently, the continuous spectrum of the operator generating the bilinear form defining the inner product of the associated reproducing kernel Hilbert space (RKHS) (see, for example, [9, 42], in the ordinary case, and [5], in the generalized case). In the stationary case, the continuous spectrum of the covariance operator coincides with the spectral density of the random field. In the bounded domain case, under certain conditions, the second-order structure can be characterized in terms of the pure point spectrum of the covariance operator.

This paper is concerned with generalized random fields defined on fractional Sobolev spaces on bounded domains characterized, in the weak sense, by their fractional-order pure point spectra. Specifically, we assume that the norm of their RKHS is equivalent to the norm defined on a fractional Sobolev space of a certain order. This condition is equivalent to the existence of a dual generalized random field, and to the existence of a white-noise linear filter representation (see [50, 51]). Indeed, such a representation is derived from the covariance factorization of the dual random field. We prove that the

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spectral properties of the corresponding class of covariance operators are equivalent to those presented by compact embeddings between fractional Sobolev spaces on bounded domains (see [28, 55] and [56, pp. 162–168]).

Wavelet-based methods have been used to represent random fields and solutions of integral and differential equations ([2, 18, 36, 53]). In particular, the discrete wavelet transform of a d-dimensional random field induces a $2d$-dimensional discrete wavelet transform of its covariance function and a wavelet-based representation of the corresponding covariance operator (see [34, 35, 39, 58]). The associated random wavelet coefficients provide a scale-space local description of the second-order properties of the random field. However, a certain degree of redundancy is present in the information given by such random coefficients since they are not uncorrelated.

In the ordinary case, an alternative approach to remove redundancy, based on the diagonalization of the covariance function using wavelets, has been proposed in [9, 60] for the unbounded domain case, and in [4] for the bounded domain case. In this paper, we study such wavelet-based orthogonal approximations for FGRFs defined on a bounded domain. The orthonormal bases constructed for the RKHSs of a FGRF and its dual provide a non-redundant description of the second-order structure of the FGRF. Such bases then allow the definition of an orthogonal expansion in terms of wavelets for the FGRFs considered. This type of expansion provides an alternative to the Karhunen–Loève expansion useful in the cases where the second-kind Fredholm integral equation defining the covariance eigenfunction system is not explicitly solvable.

2. Preliminaries

In this section, we introduce some preliminary definitions on multiresolution approximations and the basic elements and results derived in [50, 51] in relation to the fractional generalized random field theory.

Definition 2.1. Let $\{\phi_m : m \in \mathbb{N}\}$ be a spanning set of a Hilbert space H of functions. We say that the system $\{\phi_m : m \in \mathbb{N}\}$ constitutes a Riesz basis of H if

(i) there exist constants $C' > C > 0$ such that, for every sequence of scalars $\{c_m : m \in \mathbb{N}\}$,

$$C \sum_m |c_m|^2 \leq \left\| \sum_m c_m \phi_m \right\|_H^2 \leq C' \sum_m |c_m|^2, \tag{2.1}$$

(ii) the vector space of finite sums $\sum_m c_m \phi_m$ (on which (2.1) is tested) is dense in H.

If $\{\phi_m : m \in \mathbb{N}\}$ is a Riesz basis of H, it has a unique dual Riesz basis

$$\{\phi^m : m \in \mathbb{N}\} \subset H^*$$

such that

$$\phi_m(\phi^n) = \langle \phi_m, (\phi^n)^* \rangle_H = \delta_{m-n} \quad \text{for all } m, n \in \mathbb{N}, \tag{2.2}$$

where * stands for the duality between Hilbert spaces. Every function $f \in H$ can then be represented, in the weak sense, as

$$f(\psi) = \sum_m f(\phi^m)\, \phi_m(\psi) = \sum_m f^*(\phi_m)\, \phi^m(\psi^*), \tag{2.3}$$

for all $\psi \in H^*$. In the case where $H = L^2(T)$, $T \subset \mathbf{R}^d$, and f is defined in the strong sense, Eq. (2.3) can be rewritten as

$$
\begin{aligned}
f(\mathbf{x}) &= \sum_m \langle f, \phi^m \rangle_{L^2(T)} \phi_m(\mathbf{x}) \\
&= \sum_m \langle f, \phi_m \rangle_{L^2(T)} \phi^m(\mathbf{x}) \quad \text{for all } \mathbf{x} \in T.
\end{aligned} \tag{2.4}
$$

The concept of multiresolution approximation of $L^2(\mathbf{R}^d)$ provides a multiscale and local description of functions belonging to this space (see, for example, [10, 19, 33, 38]).

Definition 2.2. A *multiresolution approximation* of $L^2(\mathbf{R}^d)$ is defined as an increasing sequence $\{V_j \colon j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbf{R}^d)$ satisfying the following properties:

(i) $\bigcap_{-\infty}^{\infty} V_j = \{0\}$, $\bigcup_{-\infty}^{\infty} V_j$ is dense in $L^2(\mathbf{R}^d)$;
(ii) $f(\mathbf{x}) \in V_j$ if and only if $f(2\mathbf{x}) \in V_{j+1}$, for all $\mathbf{x} \in \mathbf{R}^d$ and $j \in \mathbb{Z}$;
(iii) $f(\mathbf{x}) \in V_0$ if and only if $f(\mathbf{x} - \mathbf{k}) \in V_0$, for all $\mathbf{x} \in \mathbf{R}^d$ and $\mathbf{k} \in \mathbb{Z}^d$;
(iv) there exists a function $g(\cdot) \in V_0$ such that the sequence $\left\{g(\mathbf{x} - \mathbf{k}) \colon \mathbf{k} \in \mathbb{Z}^d\right\}$ is a Riesz basis of V_0.

Different approaches have been followed in the construction of an orthonormal basis of wavelet functions from a multiresolution analysis of $L^2(\mathbf{R}^d)$. In particular, from a Riesz basis $\left\{g(\mathbf{x} - \mathbf{k}) \colon \mathbf{k} \in \mathbb{Z}^d\right\}$ of V_0, an orthonormal basis of wavelet functions can be constructed using the spectral theory of $L^2([0, 2\pi)^d)$-functions (see, for example, [38]).

In this paper, we consider generalized and ordinary random fields defined on a bounded domain $T \subset \mathbf{R}^d$. In the derivation of a mean-square orthogonal expansion for these random fields, in terms of wavelets, we will start from a multiresolution approximation of $L^2(T)$. We will assume that T satisfies the conditions that allow the construction of such an approximation (see, for example, [15, 17], in the one-parameter case, and [30, 16], in the multiparameter case). More specifically, in [30], the existence of orthonormal wavelet bases in $L^2(T)$ is established for any given open subset T of $\mathbb{R}^d$, with any given degree of smoothness $m \geq 1$. In the special case where T satisfies the outer cone condition, multiscale decompositions of functions in Hölder–Zygmund and fractional Sobolev spaces defined on T are derived, and characterizations of these spaces are carried out considering appropriate mixed norm summability conditions on the corresponding wavelet coefficients. In [16], a multiresolution approximation of $L^2(T)$, with T being a general domain with sufficiently regular boundary, is constructed, preserving the important features of one-dimensional multiresolution analysis, including local polynomial reproduction and locally supported stable bases. Multiscale decompositions and characterizations of function spaces on a domain T, satisfying the assumed regularity conditions on its boundary (for example, satisfying the uniform cone condition) are also obtained, in particular, for fractional Besov spaces.

We next outline some definitions and fundamental results from [50, 51] in relation to generalized random fields defined on fractional Sobolev spaces on bounded domains.

Let $\mathcal{D}(\mathbf{R}^d)$ be the space of C^∞-functions with compact support contained in $\mathbf{R}^d$, and let $\mathcal{S}(\mathbf{R}^d)$ be the space of C^∞-functions with rapid decay at infinity. The duals of these spaces are respectively the space of distributions, $\mathcal{D}'(\mathbf{R}^d)$, and the space of tempered distributions, $\mathcal{S}'(\mathbf{R}^d)$. For a domain $T \subseteq \mathbf{R}^d$, we denote by $\mathcal{D}(T)$ the space of infinitely differentiable functions with compact support contained in T, and by $\mathcal{D}'(T)$ the space of distributions on T (see, for example, [20, 55]).

Definition 2.3. For $s \in \mathbf{R}$, $H^s(\mathbf{R}^d)$ is the space of tempered distributions u such that

$$\left(1 + |\boldsymbol{\xi}|^2\right)^{s/2} \hat{u} \in L^2(\mathbf{R}^d), \qquad \boldsymbol{\xi} \in \mathbf{R}^d.$$

Definition 2.4. Let T be an open bounded C^∞-domain in $\mathbf{R}^d$. For $s \in \mathbf{R}$, the following spaces are defined on T:

$$\overline{H}^s(T) = \left\{ f \in H^s\left(\mathbf{R}^d\right); \operatorname{supp} f \subseteq \overline{T} \right\},$$
$$H^s(T) = \left\{ u \in \mathcal{D}'(T); \text{ there exists } U \in H^s\left(\mathbf{R}^d\right) \text{ with } u = U_T \right\},$$

where U_T denotes the restriction of U to T. With the quotient norm

$$\|u\|_{H^s(T)} = \inf_{\{U; U_T = u\}} \|U\|_{H^s(\mathbf{R}^d)},$$

$H^s(T)$ is a Hilbert space ([20, p. 118]).

Remark 2.1. Note that

$$\overline{H}^s(T) = \overline{\mathcal{D}(T)}^{\|\cdot\|_{H^s(\mathbf{R}^d)}} \subseteq \overline{\mathcal{D}(T)}^{\|\cdot\|_{H^s(T)}}.$$

The spaces given by Definition 2.4 are related by duality (see [55, p. 332]). Specifically, $\left[\overline{H}^s(T)\right]^* = H^{-s}(T)$, $s \in \mathbf{R}$, and $\left[H^{-s}(T)\right]^* = \left[[\overline{H}^s(T)]^*\right]^* = \overline{H}^s(T)$, with H^* denoting the dual of the Hilbert space H. Moreover, the spaces $\overline{H}^s(T)$ and $H^{-s}(T)$ can be isomorphically related in terms of positive and negative s-powers of the negative Laplacian on the bounded domain T (the negative T-Laplacian) $(-\Delta)_T$ (see [55, pp. 335–336], and [56, pp. 214–215]).

We will denote by U_α the space $\overline{H}^\alpha(T)$, and by V_α the space $H^{-\alpha}(T)$.

Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a complete probability space, and let $\mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$ be the Hilbert space of real-valued zero-mean random variables defined on $(\Omega, \mathcal{A}, \mathsf{P})$ with finite second-order moments and with the inner product defined by

$$\langle X, Y \rangle_{\mathcal{L}^2(\Omega)} = \mathsf{E}[XY], \qquad X, Y \in \mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P}).$$

Definition 2.5. For $\alpha \in \mathbf{R}$, a random function X_α from U_α into $\mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$ is said to be an *α-generalized random field* (α-GRF) if it is linear and continuous in the mean-square sense with respect to the U_α-topology.

Remark 2.2. The FGRF X_α defines the weak-sense restriction to the domain T of a random distribution on $\mathbf{R}^d$ for $\alpha \geq 0$, and of an ordinary random field on $\mathbf{R}^d$ for $\alpha < 0$.

The second-order regularity and singularity properties of an α-GRF are characterized in terms of two associated Hilbert spaces: the Hilbert space of random variables

$$H(X_\alpha) = \overline{\mathrm{sp}}^{\mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})} \left\{ X_\alpha(\varphi) \colon \varphi \in U_\alpha \right\},$$

and the RKHS $\mathcal{H}(X_\alpha)$ generated by the covariance function B_α of X_α, constituted by the functions of V_α satisfying the following condition:

$$f \in \mathcal{H}(X_\alpha) \leftrightarrow f(\phi) = \mathsf{E}\left[X X_\alpha(\phi)\right] \quad \text{for all } \phi \in U_\alpha \quad \text{and for a certain } X \in H(X_\alpha).$$

Specifically, $\mathcal{H}(X_\alpha)$ is the closed span in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$ of the random variables X in $H(X_\alpha)$ defining the functions of V_α satisfying the above condition.

From the Kernel Theorem (see [29]), the covariance function B_α of X_α admits the representation

$$\begin{aligned} B_\alpha(\varphi, \phi) &= \mathsf{E}\left[X_\alpha(\varphi) X_\alpha(\phi)\right] = \left\langle [R_\alpha \varphi]^*, \phi \right\rangle_{U_\alpha} \\ &= \int_T R_\alpha \varphi(\mathbf{z}) \phi(\mathbf{z})\, d\mathbf{z}, \qquad \varphi, \phi \in U_\alpha, \end{aligned} \tag{2.5}$$

in terms of a symmetric positive continuous linear operator R_α from U_α into V_α. In the ordinary case, R_α is an integral operator with kernel given by the covariance function of the corresponding ordinary random field $\mathcal{X}_\alpha$ defined, in the weak sense, by X_α. We therefore refer to this operator as a covariance operator.

The following concept of duality defines a class of FGRFs with RKHS norm equivalent to the norm of the space V_α. This condition also implies the ellipticity of the covariance operator R_α of X_α.

Definition 2.6. For $\alpha \in \mathbf{R}$, we say that the generalized random field

$$\widetilde{X}_\alpha \colon [U_\alpha]^* \to \mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$$

is the *dual* relative to U_α of the α-GRF $X_\alpha \colon U_\alpha \to \mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$ if

(i) $H(X_\alpha) = H\big(\widetilde{X}_\alpha\big)$, and

(ii) $\big\langle X_\alpha(\phi), \widetilde{X}_\alpha(g)\big\rangle_{H(X_\alpha)} = \langle \phi, g^* \rangle_{U_\alpha} = \int_T \phi(\mathbf{z}) g(\mathbf{z})\, d\mathbf{z}$, for $\phi \in U_\alpha$ and $g \in V_\alpha$, with g^* being the dual element of g with respect to the U_α-topology.

The dual $\widetilde{X}_\alpha$ has support contained in T, and its fractional regularity order is α. We consider the space $H\big(\widetilde{X}_\alpha\big)$ defined as the closed span in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathsf{P})$ of

$$\left\{ \widetilde{X}_\alpha(g) \colon g \in V_\alpha \right\},$$

and the RKHS $\mathcal{H}\big(\widetilde{X}_\alpha\big)$ of $\widetilde{X}_\alpha$ constituted by the functions of U_α defined from the elements of $H\big(\widetilde{X}_\alpha\big)$ as follows: For each $Y \in H\big(\widetilde{X}_\alpha\big)$, Y defines the function $\phi_Y \in U_\alpha$ of $\mathcal{H}\big(\widetilde{X}_\alpha\big)$ as

$$\phi_Y(g) = \mathsf{E}\left[Y \widetilde{X}_\alpha(g) \right] \quad \text{for all } g \in V_\alpha.$$

Obviously, the spaces $H(X_\alpha)$ and $\mathcal{H}\big(X_\alpha\big)$, and correspondingly $H\big(\widetilde{X}_\alpha\big)$ and $\mathcal{H}\big(\widetilde{X}_\alpha\big)$, are respectively related by the isometric isomorphisms

$$J \colon H(X_\alpha) \to \mathcal{H}\left(X_\alpha\right) \quad \text{and} \quad J' \colon H\left(\widetilde{X}_\alpha\right) \to \mathcal{H}\left(\widetilde{X}_\alpha\right),$$

which are defined as

$$\begin{aligned} Y &\to JY, \quad \text{with } (JY)(\phi) = \mathsf{E}\, Y X_\alpha(\phi) \quad \text{for all } \phi \in U_\alpha, \\ Z &\to J'Z, \quad \text{with } (J'Z)(g) = \mathsf{E}\, Z \widetilde{X}_\alpha(g) \quad \text{for all } g \in V_\alpha. \end{aligned}$$

As $\mathcal{H}(X_\alpha) \subseteq V_\alpha$ and $\mathcal{H}\big(\widetilde{X}_\alpha\big) \subseteq U_\alpha$, the following operators can be defined:

$$\begin{aligned} K &\colon \mathcal{H}(X_\alpha) \to V_\alpha, \quad \text{with } g \to Kg = g, \\ K' &\colon \mathcal{H}\left(\widetilde{X}_\alpha\right) \to U_\alpha, \quad \text{with } \phi \to K'\phi = \phi. \end{aligned}$$

Under the duality condition, the operators K and K' are bicontinuous (see [51]). Thus, the operators S_α and S'_α, defined as

$$\begin{aligned} S_\alpha &:= KJ \colon H(X_\alpha) \to V_\alpha, \\ S'_\alpha &:= K'J' \colon H\left(\widetilde{X}_\alpha\right) \to U_\alpha, \end{aligned}$$

are isomorphisms. The covariance operators R_α and $\widetilde{R}_\alpha$ then admit the following factorizations in terms of S_α and S'_α.

Proposition 2.1 (Covariance factorization). *Let X_α be a FGRF. Assume that the dual random field $\widetilde{X}_\alpha$ exists. Then, the covariance operator R_α of X_α and the covariance operator $\widetilde{R}_\alpha$ of $\widetilde{X}_\alpha$ can be factorized respectively as*

$$R_\alpha = S_\alpha \left(S'_\alpha\right)^{-1}, \tag{2.6}$$

$$\widetilde{R}_\alpha = S'_\alpha S_\alpha^{-1}. \tag{2.7}$$

Proof. See [50]. □

In the study of the spectral properties of R_α and $\widetilde{R}_\alpha$ (see Proposition 3.1), we will consider the entropy numbers $\{e_n(R_\alpha)\}_{n\in\mathbb{N}}$ of R_α, defined as

$$e_k(R_\alpha) = \inf\left\{\varepsilon > 0\colon R_\alpha(\mathcal{U}_{U_\alpha}) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon\mathcal{U}_{V_\alpha}) \text{ for some } b_1, \dots, b_{2^{k-1}} \in V_\alpha\right\}, \tag{2.8}$$

with $\mathcal{U}_H$ denoting the unit ball in the Hilbert space H. From Carl's inequality (see [28]), $\{e_n(R_\alpha)\}_{n\in\mathbb{N}}$ provides an upper bound sequence for the modulus of the eigenvalues $\{\lambda_n(R_\alpha)\}_{n\in\mathbb{N}}$ of R_α, that is,

$$|\lambda_k(R_\alpha)| \leq 2^{1/2} e_k(R_\alpha), \qquad k \in \mathbb{N}. \tag{2.9}$$

In the case where the operator R_α is a positive compact and selfadjoint operator on $L^2(T)$, its sequence of eigenvalues coincides with its sequence of approximation numbers $\{a_n(R_\alpha)\}_{n\in\mathbb{N}}$, given by (see [56, pp. 191–193])

$$a_k(R_\alpha) = \inf\left\{\|R_\alpha - L\|\colon L \in \mathcal{L}(L^2(T)), \operatorname{rank} L < k\right\}, \qquad k \in \mathbb{N},$$

where $\operatorname{rank} L$ is the dimension of the range of L, and $\mathcal{L}(L^2(T))$ denotes the space of linear and bounded operators on $L^2(T)$.

From Eqs. (2.6) and (2.7), $R_\alpha = \widetilde{R}_\alpha^{-1}$. Let $\{Y_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ be two orthonormal bases of the spaces $H(X_\alpha) = H\big(\widetilde{X}_\alpha\big)$ and U_α, respectively. We define the isometric isomorphism

$$\mathcal{I}\colon \left(U_\alpha, \langle\cdot,\cdot\rangle_{U_\alpha}\right) \to \Big(H(X_\alpha), \langle\cdot,\cdot\rangle_{H(X_\alpha)}\Big) \tag{2.10}$$

by

$$\varphi_n \to \mathcal{I}\varphi_n = Y_n \quad \text{for all } n \in \mathbb{N}.$$

The operator L on U_α given by

$$L := S'_\alpha \mathcal{I}$$

then defines an isomorphism satisfying

$$\begin{aligned} \langle X_\alpha L(\phi), X_\alpha L(\varphi)\rangle_{H(X_\alpha)} &= \langle X_\alpha S'_\alpha \mathcal{I}(\phi), X_\alpha S'_\alpha \mathcal{I}(\varphi)\rangle_{H(X_\alpha)} \\ &= \langle \mathcal{I}(\phi), \mathcal{I}(\varphi)\rangle_{H(X_\alpha)} \\ &= \langle \phi, \varphi\rangle_{U_\alpha} \quad \text{for all } \phi, \varphi \in U_\alpha. \end{aligned} \tag{2.11}$$

From Eq. (2.11),

$$X_\alpha(L\phi) \underset{\text{m.s.}}{=} \varepsilon_{L^2(T)}\Big((I-\Delta)_T^{\alpha/2}\phi\Big) \quad \text{for all } \phi \in U_\alpha, \tag{2.12}$$

where $\varepsilon_{L^2(T)}$ represents a generalized random field satisfying

$$\mathsf{E}\left[\varepsilon_{L^2(T)}(f)\varepsilon_{L^2(T)}(g)\right] = \langle f, g\rangle_{L^2(T)} \quad \text{for all } f, g \in L^2(T),$$

and $(I-\Delta)_T^{\alpha/2}$ denotes the fractional power of order $\alpha/2$ of the operator $(I-\Delta)_T$, given in terms of the negative T-Laplacian.

Similarly, $\widetilde{X}_\alpha$ satisfies

$$\widetilde{X}_\alpha\,(L'g) = \varepsilon_{L^2(T)}\left((I-\Delta)_T^{-\alpha/2} g\right) \quad \text{for all } g \in V_\alpha, \tag{2.13}$$

where $L' = R_\alpha L I_{V_\alpha}$, with $I_{V_\alpha}\colon V_\alpha \to U_\alpha$ being the isometric isomorphism defined by the Riesz Representation Theorem.

The formal inversion of Eqs. (2.12) and (2.13) respectively leads to the following identities:

$$X_\alpha(\phi) = \varepsilon_{L^2(T)}\left[(I-\Delta)_T^{\alpha/2} L^{-1}\phi\right] \quad \text{for all } \phi \in U_\alpha, \tag{2.14}$$

and

$$\widetilde{X}_\alpha\,(g) = \varepsilon_{L^2(T)}\left[(I-\Delta)_T^{-\alpha/2}(L')^{-1} g\right] \quad \text{for all } g \in V_\alpha. \tag{2.15}$$

The covariance operators R_α and $\widetilde{R}_\alpha$ of X_α and $\widetilde{X}_\alpha$ then admit the representations

$$R_\alpha = \left[(I-\Delta)_T^{\alpha/2} L^{-1}\right]^* \left[(I-\Delta)_T^{\alpha/2} L^{-1}\right] = \mathcal{T}\mathcal{T}^*, \tag{2.16}$$

$$\widetilde{R}_\alpha = \left[(I-\Delta)_T^{-\alpha/2}\,(L')^{-1}\right]^* \left[(I-\Delta)_T^{-\alpha/2}\,(L')^{-1}\right] = \widetilde{\mathcal{T}}\widetilde{\mathcal{T}}^*, \tag{2.17}$$

in terms of the isomorphisms $\mathcal{T}\colon L^2(T) \to V_\alpha$ and $\widetilde{\mathcal{T}}\colon L^2(T) \to U_\alpha$.

In the ordinary case, $\alpha < -d/2$, the representation (2.12) of X_α becomes

$$L^*\mathcal{X}_\alpha(\mathbf{z}) = (I-\Delta)_T^{\alpha/2}\varepsilon(\mathbf{z}) \quad \text{for all } \mathbf{z} \in T, \tag{2.18}$$

where $\{\mathcal{X}_\alpha(\mathbf{z}), \mathbf{z} \in T\}$ is the ordinary random field associated with the α-GRF X_α, and L^* is the adjoint of L. The right-hand side of (2.18) can be interpreted as fractional integration of generalized white noise.

In the case where $\alpha > d/2$, the representation (2.13) admits an ordinary solution $\widetilde{\mathcal{X}}_\alpha$ satisfying

$$(L')^*\,\widetilde{\mathcal{X}}_\alpha(\mathbf{z}) = (I-\Delta)_T^{-\alpha/2}\widetilde{\varepsilon}(\mathbf{z}), \tag{2.19}$$

where, as before, $(L')^*$ represents the adjoint of the operator L', and the right-hand side of (2.19) is interpreted as fractional integration of generalized white noise.

Further specific examples of fractional-order differential and integral equations are provided in [50, 51]. For example, from the equivalent definitions of the spaces

$$V_s = H^s(T), \qquad s > 0,$$

as restrictions of the Lebesgue–Sobolev spaces $L^{2,s}(\mathbf{R}^n)$, $s > 0$, and as restrictions of the Bessel-potential spaces $\mathcal{L}_{2,s}(\mathbf{R}^n)$, $s > 0$ (see [55]), the following family of fractional-order differential models can be obtained from Eq. (2.12):

$$a(\mathbf{z})(I-\Delta)_T^{\beta/2}(-\Delta)_T^{\gamma/2}\mathcal{X}_\alpha(\mathbf{z}) = \varepsilon(\mathbf{z}), \qquad \beta + \gamma = -\alpha > 0, \tag{2.20}$$

where $(-\Delta)_T^{\gamma/2}$ represents the fractional power of order $\gamma/2$ of the negative T-Laplacian, and $a(\mathbf{z})$ is a tempered distribution.

3. Spectral properties and wavelet-based orthogonal expansion of FGRFs

The spectral properties of the class of covariance operators introduced, under the duality condition, are characterized in this section. Specifically, we obtain two-sided estimates for the eigenvalues of the operators R_α and $\widetilde{R}_\alpha$. We derive a biorthogonal decomposition of the spaces U_α and V_α, in terms of two dual Riesz bases, obtained by applying to a given orthonormal wavelet basis of $L^2(T)$ the linear operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, respectively defined in Eqs. (2.16) and (2.17). We then prove that these bases are orthonormal bases of the RKHSs $\mathcal{H}(X_\alpha)$ and $\mathcal{H}(\widetilde{X}_\alpha)$, respectively, and provide orthogonal representations, i.e., with uncorrelated coefficients, of X_α and $\widetilde{X}_\alpha$.

The uncorrelatedness of the random coefficients defining the wavelet-based orthogonal expansion derived guarantees a non-redundant description of the second-order properties of the model considered. Furthermore, the dual Riesz bases constructed belong to dual fractional Sobolev spaces where the covariance operator of a FGRF and its inverse are bounded. Therefore, in terms of these bases, a stable inversion (stability of the corresponding inverse integral operator) of the Wiener–Hopf equation, defining the solution to the least-squares linear (direct and inverse) estimation problems associated with the class of models considered, can be obtained. Truncation of the wavelet-based orthogonal expansions derived also provides a finite-dimensional approximation of these problems, which allows the computation of their solutions. Applications in environmental sciences can be found in [11, 12, 13].

3.1. **Spectral properties.** We first characterize the spectral properties of the operators R_α and $\widetilde{R}_\alpha$.

Proposition 3.1. *Assume that the dual random field $\widetilde{X}_\alpha$ of X_α exists. Then, the following inequalities hold for the eigenvalues $\{\lambda_n(R_\alpha)\}_{n\in\mathbb{N}}$ of R_α, and $\{\lambda_n(\widetilde{R}_\alpha)\}_{n\in\mathbb{N}}$ of $\widetilde{R}_\alpha$:*

(i) *For $\alpha < 0$,*

$$|\lambda_k(R_\alpha)| \leq C_2 k^{2\alpha/d}, \qquad k \in \mathbb{N}, \tag{3.1}$$

$$|\lambda_k(\widetilde{R}_\alpha)| \geq \widetilde{C}_1 k^{-2\alpha/d}, \qquad k \in \mathbb{N}, \tag{3.2}$$

and for $\alpha > 0$,

$$|\lambda_k(R_\alpha)| \geq C_1' k^{2\alpha/d}, \qquad k \in \mathbb{N}, \tag{3.3}$$

$$|\lambda_k(\widetilde{R}_\alpha)| \leq \widetilde{C}_2' k^{-2\alpha/d}, \qquad k \in \mathbb{N}. \tag{3.4}$$

(ii) *In the case where T is a compact set, we also have, for $\alpha < 0$,*

$$|\lambda_k(R_\alpha)| \geq C_1 k^{2\alpha/d}, \qquad k \in \mathbb{N}, \tag{3.5}$$

$$|\lambda_k(\widetilde{R}_\alpha)| \leq \widetilde{C}_2 k^{-2\alpha/d}, \qquad k \in \mathbb{N}, \tag{3.6}$$

and for $\alpha > 0$,

$$|\lambda_k(R_\alpha)| \leq C_2' k^{2\alpha/d}, \qquad k \in \mathbb{N}, \tag{3.7}$$

$$|\lambda_k(\widetilde{R}_\alpha)| \geq \widetilde{C}_1' k^{-2\alpha/d}, \qquad k \in \mathbb{N}. \tag{3.8}$$

Proof. (i) For $\alpha < 0$, the entropy numbers $\{e_n(id_T)\}_{n\in\mathbb{N}}$ of the embedding

$$id_T\colon H^{-\alpha}(T) \to L^2(T)$$

satisfy

$$e_k(id_T) \leq c k^{\alpha/d}, \qquad k \in \mathbb{N},$$

for certain positive constants c. Then,

$$\begin{aligned} e_k\left(\mathcal{T}\colon L^2(T) \to L^2(T)\right) &= e_k\left(id_T \cdot \left(\mathcal{T}\colon L^2(T) \to H^{-\alpha}(T)\right)\right) \\ &\leq e_k(id_T) e_1\left(\mathcal{T}\colon L^2(T) \to H^{-\alpha}(T)\right) \leq C k^{\alpha/d}, \qquad k \in \mathbb{N}. \end{aligned}$$

Therefore, from Carl's inequality (2.9), $\mathcal{T}$ defines a compact operator on $L^2(T)$, and, from Eqs. (2.16) and (2.17), Eqs. (3.1) and (3.2) hold.

Similarly, for $\alpha > 0$, Eqs. (3.3) and (3.4) are obtained from the following identities and inequalities:

$$\begin{aligned} e_k\left(\widetilde{\mathcal{T}}\colon L^2(T) \to L^2(T)\right) &= e_k\left((id_T\colon \bar{H}^{\alpha}(T) \to L^2(T)) \cdot (\widetilde{\mathcal{T}}\colon L^2(T) \to \bar{H}^{\alpha}(T))\right) \\ &\leq e_k\left(id_T\colon \bar{H}^{\alpha}(T) \to L^2(T)\right) e_1\left(\widetilde{\mathcal{T}}\colon L^2(T) \to \bar{H}^{\alpha}(T)\right) \\ &\leq \widetilde{C}_2' k^{-\alpha/d}, \qquad k \in \mathbb{N}. \end{aligned} \tag{3.9}$$

(ii) Since, for $\alpha < 0$, R_α is a compact and selfadjoint operator on $L^2(T)$, we have

$$a_k\left(R_\alpha\colon L^2(T) \to L^2(T)\right) = |\lambda_k(R_\alpha)| \quad \text{for all } k \in \mathbb{N}, \tag{3.10}$$

where $\left\{a_n\left(R_\alpha\colon L^2(T) \to L^2(T)\right)\right\}_{n\in\mathbb{N}}$ denotes the sequence of approximation numbers of R_α on $L^2(T)$. Since T is a compact set, we may assume that

$$\int_T d\mathbf{z} = 1.$$

We can find an optimal covering of T given by M_j disjoint balls of radius $\rho = 2^{-j}$, for a certain $j \in \mathbb{N} \setminus \{0\}$, where M_j satisfies

$$c_1 2^{jd} \leq M_j \leq c_2 2^{jd},$$

with c_1 and c_2 satisfying $0 < c_1 \leq c_2 < \infty$. Then, considering an appropriate orthonormal basis of wavelet functions of $L^2(T)$ with support contained in T, for each $j \in \mathbb{N} \setminus \{0\}$, we can define a bounded linear operator K_j which associates with each function $f \in L^2(T)$ the M_j-tuple $(f_{j:1}, \ldots, f_{j:M_j})$, defined by

$$f_{j:\boldsymbol{\lambda}} = \int_T f(\mathbf{z})\psi_{j:\boldsymbol{\lambda}}(\mathbf{z})\, d\mathbf{z}, \qquad \boldsymbol{\lambda} = 1, \ldots, M_j,$$

of the linear space $l_2^{M_j}$, constituted by M_j-vectors $\mathbf{y} = (y_1, \ldots, y_{M_j})$ with the norm

$$\|\mathbf{y}/l_2^{M_j}\| = \left(\sum_{j=1}^{M_j} |y_j|^2\right)^{1/2}. \tag{3.11}$$

Thus, for each $j \in \mathbb{N} \setminus \{0\}$, K_j is defined from $L^2(T)$ into $l_2^{M_j}$. Here, we assume that, for each scale j, the wavelet functions $\left\{\psi_{j:\boldsymbol{\lambda}_j}\colon \boldsymbol{\lambda}_j = 1, \ldots, M_j\right\}$ have disjoint supports. Moreover, for each $j \in \mathbb{N} \setminus \{0\}$, we can also define a bounded linear operator H_j which associates with each M_j-tuple $\mathbf{x} = (x_{j:1}, \ldots, x_{j:M_j})$ of the space $2^{-j2\alpha} l_2^{M_j}$ the function g of $H^{-2\alpha}(T)$ given by

$$g(\mathbf{z}) = \sum_{\boldsymbol{\lambda}=1}^{M_j} x_{j:\boldsymbol{\lambda}}\psi_{j:\boldsymbol{\lambda}}(\mathbf{z}),$$

where

$$\left(\sum_{\boldsymbol{\lambda}=1}^{M_j} |2^{-j2\alpha} x_{j:\boldsymbol{\lambda}}|^2\right)^{1/2} < \infty.$$

Finally, we can define the embedding $id_j\colon 2^{-j2\alpha} l_2^{M_j} \to l_2^{M_j}$ as

$$id_j = K_j \cdot id_T \cdot H_j, \tag{3.12}$$

where id_T denotes the compact embedding from $H^{-2\alpha}(T)$ into $L^2(T)$. Therefore, from the properties of approximation numbers (see [56, pp. 191–192]), we have, for each $k \in \mathbb{N}$,

$$\begin{aligned} a_k(id_j) \leq C a_k(id_T) &= C a_k\left((R_\alpha\colon L^2(T) \to L^2(T)) \cdot (\widetilde{R}_\alpha\colon H^{-2\alpha}(T) \to L^2(T))\right) \\ &\leq C' a_k\left(R_\alpha\colon L^2(T) \to L^2(T)\right) = C'|\lambda_k(R_\alpha)|. \end{aligned} \tag{3.13}$$

Since $a_k(id_j) \sim 2^{2\alpha j}$ (see [56, p. 192]), and considering $k = M_j - 1 \sim 2^{jd}$, we obtain

$$2^{2\alpha j} \leq C''|\lambda_{c2^{jd}}(R_\alpha)|, \qquad j \in \mathbb{N},$$

and hence

$$|\lambda_k(R_\alpha)| \geq C_1 k^{2\alpha/d}, \qquad k \in \mathbb{N}.$$

Thus, we have Eqs. (3.5) and (3.6). In a similar way, Eqs. (3.7) and (3.8) are obtained using the properties of approximation numbers and the fact that $\widetilde{R}_\alpha$ is, for $\alpha > 0$, a compact and selfadjoint operator on $L^2(T)$, with $\widetilde{R}_\alpha^{-1} = R_\alpha$ bounded from $\bar{H}^{2\alpha}(T)$ into $L^2(T)$. □

3.2. **Multiresolution-like approximation.** We consider an orthonormal basis of a space $L^2(T)$, with T being a C^∞-bounded domain of $\mathbf{R}^d$, constituted by an orthonormal scaling function basis $\{\phi_{\mathbf{k}} \colon \mathbf{k} \in \Gamma_0^T\}$ of the coarsest scale space $V_0^T \subset L^2(T)$, and orthonormal wavelet bases $\{\psi_{j:\boldsymbol{\lambda}} \colon \boldsymbol{\lambda} \in \Lambda_j^T\}$, $j \geq 0$, of the spaces $W_j^T \subset L^2(T)$, $j \geq 0$, respectively, with

$$L^2(T) = V_0^T \bigoplus_{j\geq 0} W_j^T. \tag{3.14}$$

Here, Γ_j^T and Λ_j^T, $j \in \mathbb{Z}$, are finite index sets, since we are considering scaling and wavelet functions with compact support contained in T.

In the following proposition, we describe the properties of the sequences of subspaces of V_α, $\mathcal{T}\left(V_j^T\right)$, $j \in \mathbb{Z}$, and $\mathcal{T}(W_j^T)$, $j \in \mathbb{Z}$, and the properties of the sequences of subspaces of U_α, $\widetilde{\mathcal{T}}(V_j^T)$, $j \in \mathbb{Z}$, and $\widetilde{\mathcal{T}}(W_j^T)$, $j \in \mathbb{Z}$, where V_j^T, $j \in \mathbb{Z}$, are respectively generated by $\{\phi_{j:\mathbf{k}} \colon \mathbf{k} \in \Gamma_j^T\}$, $j \in \mathbb{Z}$, and provide a multiresolution approximation of $L^2(T)$, and $W_j^T \subset L^2(T)$, $j \in \mathbb{Z}$, are respectively generated by $\{\psi_{j:\boldsymbol{\lambda}} \colon \boldsymbol{\lambda} \in \Lambda_j^T\}$, $j \in \mathbb{Z}$.

Proposition 3.2. *Let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be the operators defined in Eqs.* (2.16) *and* (2.17)*, respectively. Then, the following assertions hold:*

(i) The sequences of spaces $\{M_j = \mathcal{T}\left(V_j^T\right) : j \in \mathbb{Z}\}$ and $\{M^j = \widetilde{\mathcal{T}}\left(V_j^T\right) : j \in \mathbb{Z}\}$ are increasing sequences of closed and disjoint subspaces whose unions are respectively dense in the spaces V_α and U_α.

(ii) The systems $\{\varphi_{j:\mathbf{k}} := \mathcal{T}\left(\phi_{j:\mathbf{k}}\right) : \mathbf{k} \in \Gamma_0^T\}$ and $\{\varphi^{j:\mathbf{k}} := \widetilde{\mathcal{T}}\left(\phi_{j:\mathbf{k}}\right) : \mathbf{k} \in \Gamma_0^T\}$ respectively generate the subspaces M_j and M^j, for each $j \in \mathbb{Z}$. Furthermore,

$$M_{j+1} = M_j + N_j, \qquad j \in \mathbb{Z},$$
$$M^{j+1} = M^j + N^j, \qquad j \in \mathbb{Z},$$

where $N_j = \mathcal{T}(W_j^T)$ is generated by $\{\gamma_{j:\boldsymbol{\lambda}} := \mathcal{T}\left(\psi_{j:\boldsymbol{\lambda}}\right) : \boldsymbol{\lambda} \in \Lambda_j^T\}$, and $N^j = \widetilde{\mathcal{T}}\left(W_j^T\right)$ is generated by $\{\gamma^{j:\boldsymbol{\lambda}} := \widetilde{\mathcal{T}}\left(\psi_{j:\boldsymbol{\lambda}}\right) : \boldsymbol{\lambda} \in \Lambda_j^T\}$.

(iii) The systems $\{\varphi_{\mathbf{k}} := \mathcal{T}\left(\phi_{\mathbf{k}}\right) : \mathbf{k} \in \Gamma_0^T\} \cup \{\gamma_{j:\boldsymbol{\lambda}} := \mathcal{T}\left(\psi_{j:\boldsymbol{\lambda}}\right) : \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\}$ and $\{\varphi^{\mathbf{k}} := \widetilde{\mathcal{T}}\left(\phi_{\mathbf{k}}\right) : \mathbf{k} \in \Gamma_0^T\} \cup \{\gamma^{j:\boldsymbol{\lambda}} := \widetilde{\mathcal{T}}\left(\psi_{j:\boldsymbol{\lambda}}\right) : \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\}$ are dual Riesz bases with respect to $L^2(T)$. Thus, for each $g \in V_\alpha$ and $\phi \in U_\alpha$,

$$\begin{aligned}\langle g, \phi^* \rangle_{V_\alpha} &= \sum_{\mathbf{k}\in\Gamma_0^T} g\left(\varphi^{\mathbf{k}}\right)\phi\left(\varphi_{\mathbf{k}}\right) + \sum_{j\geq 0}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} g\left(\gamma^{j:\boldsymbol{\lambda}}\right)\phi\left(\gamma_{j:\boldsymbol{\lambda}}\right) \\ &= \sum_{\mathbf{k}\in\Gamma_0^T} g^*\left(\varphi_{\mathbf{k}}\right)\phi^*\left(\varphi^{\mathbf{k}}\right) + \sum_{j\geq 0}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} g^*\left(\gamma_{j:\boldsymbol{\lambda}}\right)\phi^*\left(\gamma^{j:\boldsymbol{\lambda}}\right) = \langle g^*, \phi \rangle_{U_\alpha},\end{aligned} \tag{3.15}$$

where * represents, as before, duality between Hilbert spaces.

The two decompositions

$$V_\alpha = M_0 \widetilde{\bigoplus_{j\geq 0}} N_j \quad \textit{and} \quad U_\alpha = M^0 \widetilde{\bigoplus_{j\geq 0}} N^j \tag{3.16}$$

are biorthogonal, with $\widetilde{\bigoplus}$ *denoting the non-orthogonal direct sum of subspaces.*

Remark 3.1. In the case where the operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are homogeneous the biorthogonal decomposition defined in the above proposition provides a multiresolution-like approximation to V_α and U_α (see, for example, [22]).

Proof. Under the existence of the dual random field $\widetilde{X}_\alpha$, the operators $\mathcal{T}\colon L^2(T) \to V_\alpha$ and $\widetilde{\mathcal{T}}\colon L^2(T) \to U_\alpha$ are isomorphisms. The points (i) and (ii) then follow. Also we have that the function systems in (iii) satisfy conditions (i) and (ii) in Definition 2.1. Moreover,

$$\begin{aligned} \left\langle \varphi_{\mathbf{k}}, [\varphi^{\mathbf{l}}]^* \right\rangle_{V_\alpha} &= \left\langle \mathcal{T}(\phi_{\mathbf{k}}), \left[\widetilde{\mathcal{T}}(\phi_{\mathbf{l}})\right]^* \right\rangle_{V_\alpha} = \left\langle \mathcal{T}^{-1}\mathcal{T}[\phi_{\mathbf{k}}], \phi_{\mathbf{l}} \right\rangle_{L^2(T)} = \langle \phi_{\mathbf{k}}, \phi_{\mathbf{l}} \rangle_{L^2(T)} \\ &= \delta_{\mathbf{k},\mathbf{l}} \quad \text{for all } \mathbf{k}, \mathbf{l} \in \Gamma_0^T. \end{aligned} \tag{3.17}$$

Similarly, it can be proved that the systems $\left\{\mathcal{T}(\psi_{j:\boldsymbol{\lambda}}) := \gamma_{j:\boldsymbol{\lambda}}\colon \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\right\}$ and

$$\left\{\widetilde{\mathcal{T}}(\psi_{j:\boldsymbol{\lambda}}) := \gamma^{j:\boldsymbol{\lambda}}\colon \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\right\}$$

are biorthonormal.

For each $g \in V_\alpha$,

$$\begin{aligned} g(\varphi) &= g_0(\varphi) + \sum_{j\geq 0} g_j(\varphi) \\ &= \sum_{\mathbf{k}\in\Gamma_0^T} \mathcal{T}^{-1} g(\phi_{\mathbf{k}}) \mathcal{T}\phi_{\mathbf{k}}(\varphi) + \sum_{j\geq 0} \sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \mathcal{T}^{-1} g(\psi_{j:\boldsymbol{\lambda}}) \mathcal{T}\psi_{j:\boldsymbol{\lambda}}(\varphi) \\ &= \sum_{\mathbf{k}\in\Gamma_0^T} g\left(\varphi^{\mathbf{k}}\right) \varphi_{\mathbf{k}}(\varphi) + \sum_{j\geq 0} \sum_{\boldsymbol{\lambda}\in\Lambda_j^T} g\left(\gamma^{j:\boldsymbol{\lambda}}\right) \gamma_{j:\boldsymbol{\lambda}}(\varphi) \\ &= \sum_{\mathbf{k}\in\Gamma_0^T} g^*(\varphi_{\mathbf{k}}) \varphi^{\mathbf{k}}(\varphi^*) + \sum_{j\geq 0} \sum_{\boldsymbol{\lambda}\in\Lambda_j^T} g^*(\gamma_{j:\boldsymbol{\lambda}}) \gamma^{j:\boldsymbol{\lambda}}(\varphi^*) = g^*(\varphi^*), \end{aligned} \tag{3.18}$$

for all $\varphi \in U_\alpha$, with $g_0 \in M_0$ and $g_j \in N_j$, for all $j \geq 0$, and where g^* and φ^* denote the dual elements of g and φ, respectively. Similarly, for each $\phi \in U_\alpha$,

$$\begin{aligned} \phi(f) &= \phi^0(f) + \sum_{j\geq 0} \phi^j(f) \\ &= \sum_{\mathbf{k}\in\Gamma_0^T} \widetilde{\mathcal{T}}^{-1} \phi(\phi_{\mathbf{k}}) \widetilde{\mathcal{T}}\phi_{\mathbf{k}}(f) + \sum_{j\geq 0} \sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \widetilde{\mathcal{T}}^{-1} \phi(\psi_{j:\boldsymbol{\lambda}}) \widetilde{\mathcal{T}}\psi_{j:\boldsymbol{\lambda}}(f) \\ &= \sum_{\mathbf{k}\in\Gamma_0^T} \phi(\varphi_{\mathbf{k}}) \varphi^{\mathbf{k}}(f) + \sum_{j\geq 0} \sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \phi(\gamma_{j:\boldsymbol{\lambda}}) \gamma^{j:\boldsymbol{\lambda}}(f) \\ &= \sum_{\mathbf{k}\in\Gamma_0^T} \phi^*\left(\varphi^{\mathbf{k}}\right) \varphi_{\mathbf{k}}(f^*) + \sum_{j\geq 0} \sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \phi^*\left(\gamma^{j:\boldsymbol{\lambda}}\right) \gamma_{j:\boldsymbol{\lambda}}(f^*) = \phi^*(f^*), \end{aligned} \tag{3.19}$$

for all $f \in V_\alpha$, with $\phi^0 \in M^0$ and $\phi^j \in N^j$, for all $j \geq 0$, and where ϕ^* and f^* denote the dual elements of ϕ and f, respectively.

From Eqs. (3.18) and (3.19), Eq. (3.15) is obtained, and the biorthogonality of the decomposition in Eq. (3.16) follows. □

In the following section, we apply Proposition 3.2 to obtain, under the existence of the dual field $\widetilde{X}_\alpha$ of X_α, a wavelet-based orthogonal expansion of X_α (resp. $\widetilde{X}_\alpha$) in terms of the two dual Riesz bases defined in Proposition 3.2.

3.3. **Generalized wavelet-based orthogonal expansion.** The following theorem provides an alternative to the Karhunen–Loève expansion to construct orthonormal bases of the RKHS of a random field. More specifically, the Karhunen–Loève expansion provides the following orthonormal bases of the RKHSs $\mathcal{H}(X_\alpha)$ and $\mathcal{H}(\widetilde{X}_\alpha)$, respectively:

$$\varphi_m = (\lambda_m(R_\alpha))^{1/2}\phi_m, \qquad m \in \mathbb{N},$$
$$\varphi^m = (\lambda_m(\widetilde{R}_\alpha))^{1/2}\phi_m = (\lambda_m(R_\alpha))^{-1/2}\phi_m, \qquad m \in \mathbb{N}, \tag{3.20}$$

where $\{\lambda_n(R_\alpha)\}_{n\in\mathbb{N}}$ and $\left\{\lambda_n(\widetilde{R}_\alpha)\right\}_{n\in\mathbb{N}}$ respectively denote the eigenvalues of R_α and $\widetilde{R}_\alpha$, and $\{\phi_n\}_{n\in\mathbb{N}}$ denotes the associated system of eigenfunctions (see [2], for the generalized case). The following result provides orthonormal bases for the spaces $\mathcal{H}(X_\alpha)$ and $\mathcal{H}(\widetilde{X}_\alpha)$ constructed from an orthonormal basis of wavelet functions.

Theorem 3.1. *Assume that the conditions of Proposition* 3.2 *hold. Then, the dual Riesz bases* $\left\{\varphi_{\mathbf{k}} := \mathcal{T}(\phi_{\mathbf{k}}) : \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\gamma_{j:\boldsymbol{\lambda}} := \mathcal{T}(\psi_{j:\boldsymbol{\lambda}}) : \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\right\}$ *and*

$$\left\{\varphi^{\mathbf{k}} := \widetilde{\mathcal{T}}(\phi_{\mathbf{k}}) : \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\gamma^{j:\boldsymbol{\lambda}} := \widetilde{\mathcal{T}}(\psi_{j:\boldsymbol{\lambda}}) : \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\right\}$$

are respectively orthonormal bases of the RKHSs of X_α *and* $\widetilde{X}_\alpha$.

Proof. From the duality condition (see Definition 2.6), the inverses S_α^{-1} and $(S'_\alpha)^{-1}$ of the isomorphisms $S_\alpha = KJ\colon H(X_\alpha) \to V_\alpha$ and $S'_\alpha = K'J'\colon H(\widetilde{X}_\alpha) \to U_\alpha$ (see Section 2) can be respectively written as

$$S_\alpha^{-1}(g) = \widetilde{X}_\alpha(g) \quad \text{for all } g \in V_\alpha, \tag{3.21}$$
$$(S'_\alpha)^{-1}(\phi) = X_\alpha(\phi) \quad \text{for all } \phi \in U_\alpha. \tag{3.22}$$

Eqs. (3.21) and (3.22) respectively provide the random variables defining each element of the RKHSs of X_α and $\widetilde{X}_\alpha$. Hence, for all $g, f \in \mathcal{H}(X_\alpha)$, and for all $\phi, \varphi \in \mathcal{H}(\widetilde{X}_\alpha)$,

$$\langle g, f\rangle_{\mathcal{H}(X_\alpha)} = \left\langle \widetilde{X}_\alpha(g), \widetilde{X}_\alpha(f)\right\rangle_{H(X_\alpha)} = \widetilde{B}_\alpha(g,f), \tag{3.23}$$
$$\langle \phi, \varphi\rangle_{\mathcal{H}(\widetilde{X}_\alpha)} = \langle X_\alpha(\phi), X_\alpha(\varphi)\rangle_{H(\widetilde{X}_\alpha)} = B_\alpha(\phi,\varphi). \tag{3.24}$$

Therefore,

$$\langle \varphi_{\mathbf{k}}, \varphi_{\mathbf{l}}\rangle_{\mathcal{H}(X_\alpha)} = \widetilde{B}_\alpha(\varphi_{\mathbf{k}}, \varphi_{\mathbf{l}}) = \delta_{\mathbf{k},\mathbf{l}}, \qquad \mathbf{k}, \mathbf{l} \in \Gamma_0^T,$$
$$\left\langle \gamma_{j:\boldsymbol{\lambda}}, \gamma_{p:\boldsymbol{\lambda}'}\right\rangle_{\mathcal{H}(X_\alpha)} = \delta_{j,p}\cdot\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}, \qquad j,p \in \mathbb{N},\ \boldsymbol{\lambda} \in \Lambda_j^T,\ \boldsymbol{\lambda}' \in \Lambda_p^T,$$
$$\langle \varphi_{\mathbf{k}}, \gamma_{j:\boldsymbol{\lambda}}\rangle_{\mathcal{H}(X_\alpha)} = 0, \qquad \mathbf{k} \in \Gamma_0^T,\ \boldsymbol{\lambda} \in \Lambda_j^T,\ j \geq 0. \tag{3.25}$$

Similarly, from Eq. (3.24), $\left\{\varphi^{\mathbf{k}}\colon \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\gamma^{j:\boldsymbol{\lambda}}\colon \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\right\}$ is an orthonormal system in $\mathcal{H}(\widetilde{X}_\alpha)$.

Furthermore, from Eqs. (2.14)–(2.17),

$$X_\alpha(\phi) = \varepsilon_{L^2(T)}(\mathcal{T}^*\phi) \quad \text{for all } \phi \in U_\alpha, \tag{3.26}$$
$$\widetilde{X}_\alpha(g) = \varepsilon_{L^2(T)}(\widetilde{\mathcal{T}}^*g) \quad \text{for all } g \in V_\alpha. \tag{3.27}$$

From Eqs. (3.21) and (3.27), for each $g \in \mathcal{H}(X_\alpha)$, the following identities hold:

$$
\begin{aligned}
\langle g, \varphi_{\mathbf{k}} \rangle_{\mathcal{H}(X_\alpha)} &= \mathsf{E}\left[\widetilde{X}_\alpha(g)\widetilde{X}_\alpha(\varphi_{\mathbf{k}})\right] = \mathsf{E}\left[\varepsilon_{L^2(T)}\left(\widetilde{\mathcal{T}}^* g\right)\varepsilon_{L^2(T)}\left(\widetilde{\mathcal{T}}^*\varphi_{\mathbf{k}}\right)\right] \\
&= \left\langle \widetilde{\mathcal{T}}^* g, \widetilde{\mathcal{T}}^*\varphi_{\mathbf{k}} \right\rangle_{L^2(T)} = \left\langle \mathcal{T}^{-1} g, \mathcal{T}^{-1}\mathcal{T}\phi_{\mathbf{k}} \right\rangle_{L^2(T)} \\
&= \left\langle \mathcal{T}^{-1} g, \phi_{\mathbf{k}} \right\rangle_{L^2(T)} = \varphi^{\mathbf{k}}(g) \quad \text{for all } \mathbf{k} \in \Gamma_0^T,
\end{aligned} \tag{3.28}
$$

$$
\langle g, \gamma_{j:\boldsymbol{\lambda}} \rangle_{\mathcal{H}(X_\alpha)} = \gamma^{j:\boldsymbol{\lambda}}(g) \quad \text{for all } \boldsymbol{\lambda} \in \Lambda_j,\ j \geq 0. \tag{3.29}
$$

Similarly, from Eqs. (3.22) and (3.26), for each $\phi \in \mathcal{H}(\widetilde{X}_\alpha)$, we have

$$
\begin{aligned}
\left\langle \phi, \varphi^{\mathbf{k}} \right\rangle_{\mathcal{H}(\widetilde{X}_\alpha)} &= \mathsf{E}\left[X_\alpha(\phi)X_\alpha(\varphi^{\mathbf{k}})\right] = \mathsf{E}\left[\varepsilon_{L^2(T)}\left(\mathcal{T}^*\phi\right)\varepsilon_{L^2(T)}\left(\mathcal{T}^*\varphi^{\mathbf{k}}\right)\right] \\
&= \left\langle \mathcal{T}^*\phi, \mathcal{T}^*\varphi^{\mathbf{k}} \right\rangle_{L^2(T)} = \left\langle \widetilde{\mathcal{T}}^{-1}\phi, \widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{T}}\phi_{\mathbf{k}} \right\rangle_{L^2(T)} \\
&= \left\langle \widetilde{\mathcal{T}}^{-1}\phi, \phi_{\mathbf{k}} \right\rangle_{L^2(T)} = \varphi_{\mathbf{k}}(\phi) \quad \text{for all } \mathbf{k} \in \Gamma_0^T,
\end{aligned} \tag{3.30}
$$

$$
\left\langle \phi, \gamma^{j:\boldsymbol{\lambda}} \right\rangle_{\mathcal{H}(X_\alpha)} = \gamma_{j:\boldsymbol{\lambda}}(\phi) \quad \text{for all } \boldsymbol{\lambda} \in \Lambda_j,\ j \geq 0. \tag{3.31}
$$

From Eqs. (3.21)–(3.31), the systems

$$
\left\{\varphi_{\mathbf{k}} \colon \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\gamma_{j:\boldsymbol{\lambda}} \colon \boldsymbol{\lambda} \in \Lambda_j, j \geq 0\right\}
$$

and

$$
\left\{\varphi^{\mathbf{k}} \colon \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\gamma^{j:\boldsymbol{\lambda}} \colon \boldsymbol{\lambda} \in \Lambda_j, j \geq 0\right\}
$$

constitute orthonormal bases of the spaces $\mathcal{H}(X_\alpha)$ and $\mathcal{H}(\widetilde{X}_\alpha)$, respectively, if and only if, for each $g \in \mathcal{H}(X_\alpha)$, and for each $\phi \in \mathcal{H}(\widetilde{X}_\alpha)$,

$$
\mathsf{E}\left[\widetilde{X}_\alpha(g) - \sum_{\mathbf{k}\in\Gamma_0^T} \varphi^{\mathbf{k}}(g)\widetilde{X}_\alpha(\varphi_{\mathbf{k}}) + \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \gamma^{j:\boldsymbol{\lambda}}(g)\widetilde{X}_\alpha(\gamma_{j:\boldsymbol{\lambda}})\right]^2 \to 0 \quad \text{as } M \to \infty,
$$

$$
\mathsf{E}\left[X_\alpha(\phi) - \sum_{\mathbf{k}\in\Gamma_0^T} \varphi_{\mathbf{k}}(\phi)X_\alpha(\varphi^{\mathbf{k}}) + \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \gamma_{j:\boldsymbol{\lambda}}(\phi)X_\alpha(\gamma^{j:\boldsymbol{\lambda}})\right]^2 \to 0 \quad \text{as } M \to \infty.
$$

From Parseval's identity, the above limits hold since

$$
\begin{aligned}
&\mathsf{E}\left[\widetilde{X}_\alpha(g) - \sum_{\mathbf{k}\in\Gamma_0^T} \varphi^{\mathbf{k}}(g)\widetilde{X}_\alpha(\varphi_{\mathbf{k}}) - \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \gamma^{j:\boldsymbol{\lambda}}(g)\widetilde{X}_\alpha(\gamma_{j:\boldsymbol{\lambda}})\right]^2 \\
&\quad = \widetilde{B}_\alpha(g,g) - \left[\sum_{\mathbf{k}\in\Gamma_0^T}\left[\varphi^{\mathbf{k}}(g)\right]^2 + \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T}\left[\gamma^{j:\boldsymbol{\lambda}}(g)\right]^2\right] \\
&\quad = \left\|\widetilde{\mathcal{T}}_\alpha^*(g)\right\|_{L^2(T)}^2 - \sum_{\mathbf{k}\in\Gamma_0^T}\left[\widetilde{\mathcal{T}}_\alpha^*(g)(\phi_{\mathbf{k}})\right]^2 - \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T}\left[\widetilde{\mathcal{T}}_\alpha^*(g)(\psi_{j:\boldsymbol{\lambda}})\right]^2,
\end{aligned} \tag{3.32}
$$

$$\mathsf{E}\left[X_\alpha(\phi) - \sum_{\mathbf{k}\in\Gamma_0^T} \varphi_{\mathbf{k}}(\phi) X_\alpha\left(\varphi^{\mathbf{k}}\right) - \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \gamma_{j:\boldsymbol{\lambda}}(\phi) X_\alpha\left(\gamma^{j:\boldsymbol{\lambda}}\right)\right]^2$$

$$= B_\alpha(\phi,\phi) - \left[\sum_{\mathbf{k}\in\Gamma_0^T} [\varphi_{\mathbf{k}}(\phi)]^2 + \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} [\gamma_{j:\boldsymbol{\lambda}}(\phi)]^2\right] \tag{3.33}$$

$$= \|\mathcal{T}_\alpha^*(\phi)\|^2_{L^2(T)} - \sum_{\mathbf{k}\in\Gamma_0^T} [\mathcal{T}_\alpha^*(\phi)(\phi_{\mathbf{k}})]^2 - \sum_{j=0}^{M}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} [\mathcal{T}_\alpha^*(\phi)(\psi_{j:\boldsymbol{\lambda}})]^2. \quad \square$$

Corollary 3.1. *Assume that the conditions of Theorem* 3.1 *hold. Then,* X_α *can be represented by the following expansion, in the mean-square sense:*

$$X_\alpha(\phi) = \sum_{\mathbf{k}\in\Gamma_0^T} X_\alpha\left(\varphi^{\mathbf{k}}\right)\varphi_{\mathbf{k}}(\phi) + \sum_{j\geq 0}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} X_\alpha\left(\gamma^{j:\boldsymbol{\lambda}}\right)\gamma_{j:\boldsymbol{\lambda}}(\phi) \tag{3.34}$$

for all $\phi \in U_\alpha$*. All the random coefficients* $\left\{X_\alpha\left(\varphi^{\mathbf{k}}\right) : \mathbf{k}\in\Gamma_0^T\right\}$ *and*

$$\left\{X_\alpha\left(\gamma^{j:\boldsymbol{\lambda}}\right) : \boldsymbol{\lambda}\in\Lambda_j^T, j\geq 0\right\}$$

are uncorrelated.

Parallelly, the dual field $\widetilde{X}_\alpha$ *admits the following representation, in the mean-square sense:*

$$\widetilde{X}_\alpha(g) = \sum_{\mathbf{k}\in\Gamma_0^T} \widetilde{X}_\alpha\left(\varphi_{\mathbf{k}}\right)\varphi^{\mathbf{k}}(g) + \sum_{j\geq 0}\sum_{\boldsymbol{\lambda}\in\Lambda_j^T} \widetilde{X}_\alpha\left(\gamma_{j:\boldsymbol{\lambda}}\right)\gamma^{j:\boldsymbol{\lambda}}(g), \tag{3.35}$$

for all $g\in V_\alpha$*, with uncorrelated random coefficients.*

Remark 3.2. From Eqs. (2.14)–(2.17), we have

$$\begin{aligned}
X_\alpha(\varphi^{\mathbf{k}}) &= \varepsilon_{L^2(T)}\left((I-\Delta)^{\alpha/2}L^{-1}\varphi^{\mathbf{k}}\right)\\
&= \varepsilon_{L^2(T)}\left((I-\Delta)^{\alpha/2}L^{-1}L(I-\Delta)^{-\alpha/2}\phi_{\mathbf{k}}\right)\\
&= \varepsilon_{L^2(T)}\left(\phi_{\mathbf{k}}\right), \qquad \mathbf{k}\in\Gamma_0^T,\\
\widetilde{X}_\alpha(\varphi_{\mathbf{k}}) &= \varepsilon_{L^2(T)}\left((I-\Delta)^{-\alpha/2}(L')^{-1}\varphi_{\mathbf{k}}\right)\\
&= \varepsilon_{L^2(T)}\left((I-\Delta)^{-\alpha/2}(L')^{-1}L'(I-\Delta)^{\alpha/2}\phi_{\mathbf{k}}\right)\\
&= \varepsilon_{L^2(T)}\left(\phi_{\mathbf{k}}\right), \qquad \mathbf{k}\in\Gamma_0^T,\\
X_\alpha\left(\gamma^{j:\boldsymbol{\lambda}}\right) &= \varepsilon_{L^2(T)}\left(\psi_{j:\boldsymbol{\lambda}}\right), \qquad \boldsymbol{\lambda}\in\Lambda_j,\ j\geq 0,\\
\widetilde{X}_\alpha\left(\gamma_{j:\boldsymbol{\lambda}}\right) &= \varepsilon_{L^2(T)}\left(\psi_{j:\boldsymbol{\lambda}}\right), \qquad \boldsymbol{\lambda}\in\Lambda_j,\ j\geq 0.
\end{aligned} \tag{3.36}$$

The wavelet-based orthogonal expansions given in Corollary 3.1 then provide a *wavelet-vaguelette* decomposition, in the mean-square sense, of the generalized linear filters (2.14) and (2.15) respectively relating X_α and $\widetilde{X}_\alpha$ with $\varepsilon_{L^2(T)}$ (see [22] on *wavelet-vaguelette* decompositions for the inversion of deterministic linear filters).

Proof. The orthonormality of the random coefficients in expansions (3.34) and (3.35) follows from the orthonormality of the bases $\left\{\varphi^{\mathbf{k}} : \mathbf{k}\in\Gamma_0^T\right\}\cup\left\{\gamma^{j:\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\Lambda_j^T\right\}$ and

$$\left\{\varphi_{\mathbf{k}} : \mathbf{k}\in\Gamma_0^T\right\}\cup\left\{\gamma_{j:\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\Lambda_j^T\right\}$$

in the RKHSs of $\widetilde{X}_\alpha$ and X_α, respectively. The convergence in the mean-square sense of such expansions also follows from the proof of Theorem 3.1. $\square$

The results derived also provide a *multiresolution-like* approximation of

$$H(X_\alpha) = H(\widetilde{X}_\alpha)$$

in terms of the *random wavelet-like* basis

$$\left\{\varepsilon_{L^2(T)}\left(\phi_{\mathbf{k}}\right), \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\varepsilon_{L^2(T)}\left(\psi_{j:\boldsymbol{\lambda}}\right) : \boldsymbol{\lambda} \in \Lambda_j, j \geq 0\right\},$$

which leads to a multiscale description, in the mean-square sense, of any random function whose random components belong to this space. Under the duality condition, the generalized random function X_α has mean-square coefficients $\left\{\varphi_{\mathbf{k}} \colon \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\gamma_{j:\boldsymbol{\lambda}} \colon \boldsymbol{\lambda} \in \Lambda_j^T\right\}$ with respect to such a random basis. Similarly, $\widetilde{X}_\alpha$ has mean-square coefficients

$$\left\{\varphi^{\mathbf{k}} \colon \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\gamma^{j:\boldsymbol{\lambda}} \colon \boldsymbol{\lambda} \in \Lambda_j^T, j \geq 0\right\}$$

with respect to $\left\{\varepsilon_{L^2(T)}\left(\phi_{\mathbf{k}}\right) : \mathbf{k} \in \Gamma_0^T\right\} \cup \left\{\varepsilon_{L^2(T)}\left(\psi_{j:\boldsymbol{\lambda}}\right) : \boldsymbol{\lambda} \in \Lambda_j, j \geq 0\right\}$.

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