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LAWS OF ITERATED LOGARITHM FOR STOCHASTIC INTEGRALS OF GENERALIZED SUB-GAUSSIAN PROCESSES

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Abstract. We study the behavior of ϕ-sub-Gaussian martingales $(M_t)_{t>0}$ as $t \to 0$. Applications are given to the stochastic integral of a particular kind of process and to the double stochastic integral of it with respect to two independent Brownian motions.

1. Introduction

The aim of this paper is to study the behavior of ϕ-sub-Gaussian martingales $(M_t)_{t>0}$ as $t \to 0$. The class of ϕ-sub-Gaussian random variables X has been defined and studied in [1] and [4], where the importance of the Young–Fenchel transform ϕ^* of ϕ for the tail probabilities $\mathsf{P}(X > x)$ is enlightened. The results of the present paper show that ϕ^* plays a relevant role also in what concerns the behavior of $(M_t)_{t>0}$ in the neighborhood of 0; in fact, our Theorem 2.5 says that (M_t) behaves (in some "nonstrict" sense, which is made precise in the statement of Theorem 2.5) like the function $t \mapsto (\phi^*)^{-1}\big(\log\log(1/t)\big)$. The peculiar form of this function clearly reminds us of the classical Law of the Iterated Logarithm; in view of that, we prove a similar to (but more general than) Theorem 2.5, namely Theorem 2.1, in which we make use of the function $t \mapsto \log\log(1/t)$. Both our theorems have relevant applications, which we discuss in section 3 of the present paper. In particular, we study

i) the stochastic integral of a process (Y_t) such that its L^{2k} norms $\|Y_t\|_{L^{2k}}$ are "well controlled" (in the sense that they satisfy condition (9) of Theorem 3.4),
ii) the double stochastic integral of a process (X_t) (possessing well-controlled L^{2k} norms) with respect to two independent Brownian motions.

The last result mentioned here is contained in Corollary 3.8; its interest relies on the fact that it generalizes (in part) some classical results concerning the Lévy area process (see [5] and [6] as references) and the results of [7]; in particular, in [7] only iterated integrals of constant processes are considered, while our result, Corollary 3.8, holds for more general processes (X_t), as explained above.

2. Two general theorems

Theorem 2.1. *Let $M = (M_t)_{t>0}$ be a martingale. Assume that there exists a constant $A > 0$ and a positive function $h(t)$ defined on $(0, A]$, regularly varying at $t = 0$ such that*

$$\sup_{t\le A} \mathsf{E}\big[e^{h(t)|M_t|}\big] = B < +\infty. \tag{1}$$

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Then

$$\mathsf{P}\left(\limsup_{t\to 0}\frac{h(t)M_t}{\log\log(1/t)}\le 1\right)=1.$$

Proof. By the maximal inequality for martingales, for every pair of real numbers $\lambda>0$ and $\delta>0$ we have

$$\begin{aligned}\mathsf{P}\left(\sup_{s\le t}M_s>\frac{(1+\delta)}{h(t)}\log\log\frac{1}{t}\right)&\le\exp\left(-(1+\delta)\frac{\lambda}{h(t)}\log\log\frac{1}{t}\right)\mathsf{E}\big[e^{\lambda M_t}\big]\\&\le\exp\left(-(1+\delta)\frac{\lambda}{h(t)}\log\log\frac{1}{t}\right)\mathsf{E}\big[e^{\lambda|M_t|}\big].\end{aligned}$$

For $\lambda=h(t)$, $t\le A$, we get from the above

$$\mathsf{P}\left(\sup_{s\le t}M_s>\frac{(1+\delta)}{h(t)}\log\log\frac{1}{t}\right)\le B\exp\left(-(1+\delta)\log\log\frac{1}{t}\right).\tag{2}$$

Fix θ, with $0<\theta<1$, and take $t=\theta^n$, $n\in\mathbb{N}$, in (2). Then the series

$$\sum_n\exp\left(-(1+\delta)\log\log\frac{1}{\theta^n}\right)$$

is convergent, and the Borel–Cantelli lemma yields that, for P-almost every ω there exists $n_0=n_0(\omega)$ such that, for $n\ge n_0$, we have

$$\sup_{s\le\theta^n}M_s\le\frac{(1+\delta)}{h(\theta^n)}\log\log\frac{1}{\theta^n}.$$

Now, let $t\in(\theta^{n+1},\theta^n]$. For $n\ge n_0$ we have

$$M_t\le\sup_{s\le\theta^n}M_s\le\frac{(1+\delta)}{h(\theta^n)}\log\log\frac{1}{\theta^n}\le\frac{(1+\delta)}{h(t)}\left(\log\log\frac{1}{t}\right)\frac{h(t)}{h(\theta^n)}.\tag{3}$$

Since h is regularly varying, there exist a real number α and a slowly varying function H, defined in $(0,A]$ such that

$$h(t)=H(t)t^\alpha.$$

The relation $t\in(\theta^{n+1},\theta^n]$ is obviously equivalent to

$$\theta\le\frac{t}{\theta^n}\le 1.\tag{4}$$

From (3) and (4) we obtain

$$\begin{aligned}M_t&\le\frac{(1+\delta)}{h(t)}\left(\log\log\frac{1}{t}\right)\frac{H(t)}{H(\theta^n)}\left(\frac{t}{\theta^n}\right)^\alpha\\&\le\frac{(1+\delta)}{h(t)}\left(\log\log\frac{1}{t}\right)\frac{H(t)}{H(\theta^n)}\big(1\vee\theta^\alpha\big).\end{aligned}\tag{5}$$

We now prove that, for every $\varepsilon>0$, there exists ν such that, for $n>\nu$, we have

$$\frac{H(t)}{H(\theta^n)}\le 1+\varepsilon,\qquad\theta^{n+1}<t\le\theta^n.\tag{6}$$

By (5) our statement is an easy consequence of the following lemma.

Lemma 2.2. *Let H be slowly varying as $t\to 0$. Let a and b be two positive constants, with $0<a<b$. Then*

$$\lim_{\substack{x\to 0,\,y\to 0\\a\le y/x\le b}}\frac{H(x)}{H(y)}=1.$$

Proof of the lemma. We shall use a famous characterization of slowly varying functions. Our reference is [2], where, actually, slowly varying functions at ∞ are studied. It is nevertheless evident that $H(x)$ is slowly varying at 0 if and only if $\tilde{H}(y) \doteq H(1/y)$ is slowly varying at ∞. Hence the whole theory of [3] can be easily translated into an analogous theory for functions slowly varying at 0. In particular the following result holds:

Lemma 2.3. *A function H (defined on $(0, A]$) is slowly varying at $x = 0$ if and only if it is of the form*

$$H(x) = \psi(x) \exp\biggl(\int_x^A \frac{\phi(z)}{z}\, dz\biggr),$$

where ψ and ϕ are two functions defined in $(0, A]$ such that

$$\lim_{x \to 0} \psi(x) = c \in (0, +\infty)$$

and

$$\lim_{z \to 0} \phi(z) = 0.$$

Using Lemma 2.3, we can write

$$\begin{aligned} \frac{H(x)}{H(y)} &= \frac{\psi(x)}{\psi(y)} \exp\biggl(\int_x^A \frac{\phi(z)}{z}\, dz - \int_y^A \frac{\phi(z)}{z}\, dz\biggr) \\ &= \frac{\psi(x)}{\psi(y)} \exp\biggl(\int_x^y \frac{\phi(z)}{z}\, dz\biggr). \end{aligned}$$

Since it is evident that

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\psi(x)}{\psi(y)} = 1,$$

we are only concerned with the term $\exp\bigl(\int_x^y \phi(z)/z\, dz\bigr)$. Assume that $x \le y$ in order to fix ideas. Let $\varepsilon > 0$ be fixed. Since $\phi(x)$ goes to 0 as x goes to 0, there exists a number $R < A$ such that, if $x < z < y < R$, we have

$$\int_x^y \frac{|\phi(z)|}{z}\, dz \le \varepsilon \log \frac{y}{x}.$$

Since $a \le y/x \le b$, we deduce

$$a^\varepsilon \le \exp\biggl(\int_x^y \frac{|\phi(z)|}{z}\, dz\biggr) \le b^\varepsilon,$$

and, by letting ε go to 0, we get

$$\lim_{\substack{x \to 0,\, y \to 0 \\ a \le y/x \le b}} \exp\biggl(\int_x^y \frac{\phi(z)}{z}\, dz\biggr) = 1,$$

which concludes the proof of the lemma. □

We go back to the proof of Theorem 2.1. From (5) and (6) we obtain

$$\limsup_{t \to 0} \frac{h(t) M_t}{\log\log(1/t)} \le (1 + \delta)\bigl(1 \vee \theta^\alpha\bigr),$$

hence the statement of the theorem by letting δ go to 0 and θ go to 1. □

We shall now consider a particular family of processes.

Suppose that ϕ is an Orlicz N-function. We refer to [1] and [4] for the definition and properties of ϕ-sub-Gaussian random variables. We shall denote by ϕ^* the Young–Fenchel transform of ϕ (see [1] and [4] again).

Definition 2.4. A process $M = (M_t)$ is *ϕ-sub-Gaussian* if the random variable M_t is ϕ-sub-Gaussian for every t. The function $\tau_\phi\colon t \mapsto \tau_\phi(M_t)$ is called the *ϕ-sub-Gaussian standard* of M.

Theorem 2.5. *Let (M_t) be a ϕ-sub-Gaussian martingale, with ϕ-subGaussian standard τ_ϕ. Assume that the Young–Fenchel transform ϕ^* of ϕ is ultimately monotone and that τ_ϕ is regularly varying at $t = 0$. Then*

$$\mathsf{P}\Bigg(\limsup_{t\to 0} \frac{M_t}{\tau_\phi(t)(\phi^*)^{-1}\big(\log\log(1/t)\big)} \le 1\Bigg) = 1.$$

Proof. The maximal inequality gives, for every $\lambda > 0$ and T,

$$\mathsf{P}\left(\sup_{s\le t} M_s > x\right) \le e^{-\lambda x}\,\mathsf{E}\big[e^{\lambda M_t}\big] \le \exp\big(-\lambda x + \phi(\lambda\tau_\phi(t))\big).$$

By taking the infimum with respect to λ, we get

$$\mathsf{P}\left(\sup_{s\le t} M_s > x\right) \le \exp\big(-\phi^*(x/\tau_\phi(t))\big).$$

Fix any $\theta \in (0,1)$. For $t = \theta^n$ we get, for any $\delta > 0$,

$$\begin{aligned}
&\mathsf{P}\left(\sup_{s\le \theta^n} M_s > (1+\delta)\tau_\phi(\theta^n)(\phi^*)^{-1}\big(\log\log(1/\theta^n)\big)\right)\\
&\quad\le \exp\Bigg(-\phi^*(1+\delta)(\phi^*)^{-1}\big(\log\log(1/\theta^n)\big)\Bigg)\\
&\quad\le \exp\Bigg(-(1+\delta)\log\log(1/\theta^n)\Bigg),
\end{aligned}$$

where the last inequality is due to the relation

$$\phi^*((1+\delta)y) \ge (1+\delta)\phi^*(y), \qquad y > 0$$

(see [1] for the proof).
The series

$$\sum_n \exp\Bigg(-(1+\delta)\log\log(1/\theta^n)\Bigg)$$

is convergent, and the rest of the proof follows the arguments used in the proof of Theorem 2.1. $\square$

Remark 2.6. Analogous results to Theorem 2.5 were proved in [3] for partial sums of sub-Gaussian variables and vectors.

3. Some applications

The following proposition gives a sufficient condition in order that condition (2.1) holds.

Proposition 3.1. *Assume that the martingale $M = (M_t)$ is ϕ-sub-Gaussian, with ϕ-sub-Gaussian standard τ_ϕ. Suppose moreover that there exist a number $A > 0$ and a positive function $h(t)$ defined on $(0, A]$ such that, for $0 < t \le A$, we have*

$$h(t)\tau_\phi(t) \le 1. \tag{7}$$

Then

$$\sup_{t\le A} \mathsf{E}\big[e^{h(t)|M_t|}\big] = B < +\infty.$$

Proof. The assertion is immediate from the relation

$$\mathsf{E}\big[e^{h(t)|M_t|}\big] \leq 2\exp(h(t)\tau_\phi(t)) \leq 2e^{\phi(1)}. \qquad \square$$

Remark 3.2. Assume that $\phi(x) = x^p/p$ and let M be a ϕ-sub-Gaussian martingale with ϕ-sub-Gaussian standard $\tau_\phi(M_t)$. Suppose moreover that there exists a positive function h defined on $(0, A]$, regularly varying at $t = 0$ such that, for every real number $t \in (0, A]$, inequality (7) holds. Then applying Theorem 2.1 we get

$$\mathsf{P}\bigg(\limsup_{t\to 0} \frac{h(t)M_t}{\log\log(1/t)} \leq 1\bigg) = 1.$$

On the other hand, if also τ_ϕ is regularly varying, Theorem 2.5 yields

$$\mathsf{P}\bigg(\limsup_{t\to 0} \frac{M_t}{\tau_\phi(t)(\phi^*)^{-1}\big(\log\log(1/t)\big)} \leq 1\bigg) = 1.$$

But in this case $(\phi^*)^{-1}(t) = (qt)^{1/q}$, so, by using inequality (7), we easily obtain, up to constants,

$$\tau_\phi(t)\big(\log\log(1/t)\big)^{1/q} \prec \frac{\log\log(1/t)}{h(t)},$$

which shows that Theorem 2.5 gives a sharper estimation than Theorem 2.1.

In what follows we shall discuss a relevant example of a ϕ-sub-Gaussian martingale.

On the probability space $(\Omega, \mathcal{A}, \mathsf{P})$ let $B = (B_t)_t$ be a Brownian motion, $(\mathcal{F}_t)_t$ its natural filtration and $Y = (Y_t)_t$ a predictable process.

Consider the stochastic integral $M = (M_t)_t$ defined as

$$M_t = \int_0^t Y_s\, dB_s. \tag{8}$$

In the sequel we shall repeatedly use the following result:

Theorem 3.3. *On the probability space $(\Omega, \mathcal{A}, \mathsf{P})$ let $B = (B_t)_t$ be a Brownian motion, $(\mathcal{F}_t)_t$ its natural filtration and $Y = (Y_t)_t$ a predictable process, independent of B. Define the stochastic integral M as in (8). Then, for every integer k the $2k$-th moment of M_t satisfies*

$$\mathsf{E}\big[M_t^{2k}\big] = \frac{(2k)!}{2^k k!}\int_{[0,t]^k} \mathsf{E}\big[(Y_{x_1}\cdots Y_{x_k})^2\big]\, dx_1\cdots dx_k.$$

Proof. Let Σ be the set of all finite partitions of the interval $[0, t]$ of the form

$$\sigma = \{0 = t_0 < \cdots < t_n = t\}$$

and for every $\sigma \in \Sigma$, define $|\sigma| = \min_{i=1\ldots n} |t_i - t_{i-1}|$.

For every integer k, the $2k$-th moment of an approximating sum (defined as usual by $\sum_{h=1}^n Y_{t_h}(B_{t_h} - B_{t_{h-1}})$) is

$$\mathsf{E}\big[S_n^{2k}\big] = \sum_{\substack{0\leq h_i\leq 2k\\ \sum h_i = 2k}} \binom{2k}{h_1,\ldots,h_n} \mathsf{E}\big[Y_{t_0}^{h_1}\cdots Y_{t_{n-1}}^{h_n}(B_{t_1} - B_{t_0})^{h_1}\cdots(B_{t_n} - B_{t_{n-1}})^{h_n}\big].$$

Now by the independence of all involved increments of the Brownian motion and the fact that the process Y does not depend on any of them, we can separate the expectations as follows:

$$\sum_{\substack{0\le h_i\le 2k\\ \sum h_i=2k}} \binom{2k}{h_1,\dots,h_n} \mathsf{E}\big[Y_{t_0}^{h_1}\cdots Y_{t_{n-1}}^{h_n}(B_{t_1}-B_{t_0})^{h_1}\cdots(B_{t_n}-B_{t_{n-1}})^{h_n}\big]$$
$$= \sum_{\substack{0\le h_i\le 2k\\ \sum h_i=2k}} \binom{2k}{h_1,\dots,h_n} \mathsf{E}\big[Y_{t_0}^{h_1}\cdots Y_{t_{n-1}}^{h_n}\big]\,\mathsf{E}\big[(B_{t_1}-B_{t_0})^{h_1}\big]\cdots\mathsf{E}\big[(B_{t_n}-B_{t_{n-1}})^{h_n}\big].$$

Since for every pair of integers i and j,

$$\mathsf{E}\left[\left(B_{t_i}-B_{t_{i-1}}\right)^{2j+1}\right]=0,$$

in the sum above we can consider only even powers.

Split the sum into two parts: in the first one we only put the indexes h_i equal to 2 or 0, in the second one all the remaining powers. In the first part, since the sum of all indexes h_i must be equal to $2k$, exactly k among them are equal to 2, so that the multinomial coefficient has the value $(2k)!/2^k$ in any case. Hence, as $|\sigma|$ goes to 0, the first part converges to

$$\frac{(2k)!}{2^k}\int_{\{0\le x_1<\dots<x_k\le t\}} \mathsf{E}\big[(Y_{x_1}\cdots Y_{x_k})^2\big]\,d_{x_1}\cdots d_{x_k}$$
$$= \frac{(2k)!}{2^k k!}\int_{[0,t]^k} \mathsf{E}\big[(Y_{x_1}\cdots Y_{x_k})^2\big]\,d_{x_1}\cdots d_{x_k}.$$

We now achieve the proof of our statement if we show that the second sum goes to 0. This is easy to obtain if we remark that if j is greater than 2, then the following equality holds:

$$\lim_{|\sigma|\to 0}\sum_{h=0}^{n}\mathsf{E}\left[\left(B_{t_h}-B_{t_{h-1}}\right)^j\right]=0. \qquad \square$$

We are interested in finding conditions for the process Y in order to assure that M is ϕ-sub-Gaussian (so that Theorem 2.5 applies).

First, we present the following result, which roughly says that, if the L^{2k} norms of the process Y are "well controlled", then the stochastic integral M is ϕ-sub-Gaussian.

Theorem 3.4. *Assume that ϕ is an increasing Orlicz N-function, such that, for every large enough integer k and every s, we have*

$$\|Y_s\|_{L^{2k}} \le C\frac{\sqrt{k}}{\phi^{-1}(2k)}s^{\alpha} \tag{9}$$

with $\alpha > -\frac{1}{2}$ for a suitable constant C.

Then M is a ϕ-sub-Gaussian martingale and, for every t we have

$$\tau_\phi(M_t) \le C_\phi t^{\alpha+1/2}. \tag{10}$$

Proof. By Fubini's theorem we have

$$\mathsf{E}\left[\int_0^t Y_s^2\,ds\right] \le \int_0^t \mathsf{E}\big[Y_s^2\big]\,ds \le Lt^{2\alpha+1},$$

so the process M is a martingale.

Applying Theorem 3.3 we get

$$
\begin{aligned}
\mathsf{E}[M_t^{2k}] &\le \frac{(2k)!}{2^k k!}\int_{[0,t^k]} \|X_{s_1}\|_{L^{2k}}^2 \cdots \|X_{s_k}\|_{L^{2k}}^2 \, ds_1 \cdots ds_k \\
&\le C^{2k}\frac{(2k)!}{2^k k!}\frac{k^k}{\left(\phi^{-1}(2k)\right)^{2k}}\left[\int_0^t s^{2\alpha}\, ds\right]^k \\
&= C^{2k}\frac{(2k)!}{2^k k!}\frac{k^k}{(\phi^{-1}(2k))^{2k}}\frac{t^{k(2\alpha+1)}}{(2\alpha+1)^k}.
\end{aligned}
$$

The above implies

$$
\nu_\phi(M_t) \doteq \sup_k \frac{\left(\mathsf{E}[M_t^{2k}]\right)^{1/2k}\left(\phi^{-1}(2k)\right)}{\left((2k)!\right)^{1/2k}} \le C_1 \frac{\sqrt{k}}{(k!)^{1/2k}} t^{\alpha+1/2} \simeq C_2 t^{\alpha+1/2},
$$

where the last equivalence follows from Stirling's formula. From Lemma 4.3 of [1] we know that there exists a constant S_ϕ such that

$$
\tau_\phi(M_t) \le S_\phi \nu_\phi(M_t),
$$

which concludes the proof. □

Example 3.5. For $\phi(x) = x^4/4$ we shall exhibit a process Y for which assumption (9) holds with $\alpha = 1$, i.e., such that, for every real number t and all large integers k we have

$$
\|Y_t\|_{L^{2k}} \le C\frac{\sqrt{k}}{\phi^{-1}(2k)}t = C\,\frac{k^{1/4}}{2^{3/4}}t
$$

for a suitable constant C.

Let $Y = (Y_t)_{t>0}$ be any process such that, for each t, the law of Y_t has a density f_t given by

$$
f_t \colon x \mapsto \frac{2\Gamma(3/4)}{\pi\sqrt{2}t}\exp\left(-\frac{x^4}{t^4}\right),
$$

where Γ denotes Euler's function. We show that such a process Y satisfies (9).

First, it is easily proved by induction that the following equalities hold:

$$
\mathsf{E}\big[Y_t^{4k+2}\big] = \frac{\Gamma^2(3/4)}{\sqrt{2}\pi}\prod_{i=1}^k (4k+3-4i)\frac{t^{4k+2}}{4^k},
$$

$$
\mathsf{E}\big[Y_t^{4k}\big] = \prod_{i=1}^k (4k+1-4i)\frac{t^{4k}}{4^k}.
$$

For any fixed integer k, we can bound the quantity $\prod_{i=1}^k(4k+1-4i)$ by the square root of the product of all odd numbers smaller than $4k$, so that we get

$$
\prod_{i=1}^k (4k+1-4i) \le \sqrt{\frac{(4k)!}{2^{2k}(2k)!}} \approx \frac{2^{2k}k^k}{e^k},
$$

where the last equivalence is due to Stirling's formula. Hence, for every integer k large enough we obtain

$$
\|Y_t\|_{L^{4k}} \le C\left(\frac{4^k k^k}{e^k}\frac{t^{4k}}{4^k}\right)^{1/4k} = C\frac{1}{\sqrt[4]{e}}k^{1/4}t,
$$

for a suitable constant C obtaining (9). By the same argument, a similar bound can be found for $\|Y_t\|_{L^{4k+2}}$, and the proof is complete.

In the case of sub-Gaussianity (i.e., when $\phi(x) = x^2/2$), condition (9) can be dropped, as the following result shows.

Theorem 3.6. *Assume that Y is a sub-Gaussian process, with sub-Gaussian standard τ. Put $\psi(t) = \sup_{s\le t} \tau(s)$. Then M, defined in (8), satisfies*

$$\sup_{t\le A} \mathsf{E}\big[e^{|M_t|/\psi(t)}\big] = B < +\infty$$

for a suitable constant $A > 0$.

Proof. Put $\lambda(t) = 1/\psi(t)$ for simplicity. We begin with a preliminary result.

Lemma 3.7. *For a suitable constant $C > 0$ we have the inequality*

$$\sum_{k\ge 0} \frac{|\lambda(t)|^k}{k!} \mathsf{E}\big[|M_t|^k\big] \le C \sum_{k\ge 1} \frac{1}{\sqrt{k}} t^k.$$

Proof of Lemma 3.7. We have

$$\sum_{k\ge 0} \frac{|\lambda(t)|^k}{k!} \mathsf{E}\big[|M_t^k|\big] = \sum_{k\ge 0} \frac{|\lambda(t)|^{2k}}{(2k)!} \mathsf{E}\big[|M_t|^{2k}\big] + \sum_{k\ge 0} \frac{|\lambda(t)|^{2k+1}}{(2k+1)!} \mathsf{E}\big[|M_t|^{2k+1}\big]. \tag{11}$$

We now treat the first series above. From Theorem 3.3 we get

$$\begin{aligned} \mathsf{E}\big[|M_t|^{2k}\big] &\le \frac{(2k)!}{2^k k!} \int_{[0,t]^k} \|Y_{x_1}\|^2_{L^{2k}} \times \cdots \times \|Y_{x_k}\|^2_{L^{2k}}\, dx_1 \cdots dx_k \\ &\le \frac{(2k)!}{2^k k!} t^k \left(\sup_{s\le t} \|Y_s\|_{L^{2k}}\right)^{2k}. \end{aligned} \tag{12}$$

Now, the following inequality is proved in [1]:

$$\|Y_s\|^h_{L^h} \le 2\left(\frac{h}{e}\right)^{h/2} \tau^h(s),$$

whence we deduce

$$\sup_{s\le t} \|Y_s\|^h_{L^h} \le 2\left(\frac{h}{e}\right)^{h/2} \psi^h(t). \tag{13}$$

From (12) and (13) we conclude that

$$\sum_{k\ge 0} \frac{|\lambda(t)|^{2k}}{(2k)!} \mathsf{E}\big[|M_t|^{2k}\big] \le 2 \sum_{k\ge 0} \frac{1}{2^k k!} \left(\frac{2k}{e}\right)^k t^k \sim C \sum_{k\ge 1} \frac{1}{\sqrt{k}} t^k,$$

where the last equivalence follows from Stirling's formula.

In order to do the calculations for the second series in (11), we simply start from

$$\mathsf{E}\big[|M_t|^{2k+1}\big] \le \big(\mathsf{E}\big[|M_t|^{2k}\big]\, \mathsf{E}\big[|M_t|^{2k+2}\big]\big)^{1/2},$$

and then argue as for the first series, using again Theorem 3.3. $\square$

We go back to the proof of Theorem 3.6. Lemma 3.7 implies that, for every $t < 1$, the series

$$\sum_k \frac{\lambda(t)^k}{k!} \mathsf{E}\big[|M_t|^k\big]$$

is absolutely convergent. Hence, for any fixed $\varepsilon \in (0,1)$ we can write the following relations:

$$\begin{aligned}\sup_{0<t<1-\varepsilon} \mathsf{E}\big[e^{\lambda(t)|M_t|}\big] &= \sup_{0<t<1-\varepsilon}\left|\mathsf{E}\left[\sum_k \frac{\lambda(t)^k}{k!}|M_t|^k\right]\right| \\ &\le \sup_{0<t<1-\varepsilon}\sum_k \frac{|\lambda(t)|^k}{k!}\,\mathsf{E}\big[|M_t|^k\big] \\ &\le C\sum_{k\ge 1}\frac{1}{\sqrt{k}}(1-\varepsilon)^k = B \\ &< +\infty. \qquad \square\end{aligned}$$

Consider now the following situation. Let $B^1 = (B^1_t)$ and $B^2 = (B^2_t)$ be two independent Brownian motions, $(\mathcal{F}_t)_t$ the natural filtration of $B = (B^1, B^2)$ and $X = (X_t)_t$ a predictable process, independent of B. Define the double stochastic integral M of X as

$$M_t = \int_0^t\int_0^s X_u\, dB^1_u\, dB^2_s.$$

Corollary 3.8. *Assume that X satisfies condition* (9) *with respect to $\phi(x) = x^2/2$ (with $\alpha = 0$). Then Theorem* 2.1 *holds for M with $h(t) = Ct^{-1/2}$ (C is a constant).*

Proof. Put

$$Y_t = \int_0^t X_s\, dB^1_s.$$

Then the process Y is strictly sub-Gaussian by Theorem 3.4. Moreover, relation (10) implies

$$\psi(t) = \sup_{s\le t}\tau(s) \le C\sqrt{t}.$$

From Theorem 3.6 we deduce, for a suitable constant A,

$$\sup_{t\le A}\mathsf{E}\big[e^{Ct^{-1/2}|M_t|}\big] \le \sup_{t\le A}\mathsf{E}\big[e^{|M_t|/\psi(t)}\big] = B < +\infty. \qquad \square$$

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