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RANDOM ATTRACTOR FOR THE REACTION-DIFFUSION EQUATION PERTURBED BY A STOCHASTIC CÀDLÀG PROCESS

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Abstract. We study a stochastically perturbed reaction-diffusion equation by using the methods of the theory of stochastic attractors. It is proved that solutions of the equation form a multivalued random dynamic system for which there exists a random attractor in the phase space.

Introduction

The theory of random dynamic systems [1] provides convenient tools to study many random and stochastic equations. Evolution systems described by these equations have, as a rule, unbounded trajectories in ordinary time. The theory of random attractors (i.e., measurable invariant sets attracting trajectories in inverse time) is developed in the papers [5]–[7] to obtain a qualitative description of such systems and to study finite-dimensional systems of stochastically perturbed differential equations that have unique solutions.

A generalization of the theory of random attractors is proposed in the paper [8] for the case of infinite-dimensional systems that do not necessarily have unique solutions. However an extra assumption is made in [8] that the corresponding multivalued random dynamic system is dissipative with probability one.

In this paper, we develop the theory of random attractors for a system that is dissipative in probability. Based on this extension we study the qualitative behavior of solutions of the reaction-diffusion equation perturbed by a càdlàg process.

The setting of the problem

Consider the problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = a\triangle u(t,x) - f(u(t,x)) + h(x) + g(u(t,x))\xi(t,w), \\ u\big|_{\partial Q} = 0,\ u\big|_{t=0} = u_0(x), \end{cases} \tag{1}$$

where $a > 0$, $Q \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, $f, g \in C(\mathbb{R})$, $h \in L_2(Q)$, $uf(u) \geq \alpha|u|^p - C$, $|f(u)| \leq C(1 + |u|^{p-1})$, $p \geq 2$, $\alpha > 0$, and

$$|g(u)| \leq C_1|u| + C_2.$$

We assume that $\xi(t, w)$ is a stochastic càdlàg process; that is, its trajectories have no discontinuities of the second kind. In view of the canonical representation of such processes

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described in [1, 2], we assume in what follows that $\xi(t,w)\colon \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}$, where

$$\begin{aligned}\Omega &= D(\mathbb{R})\\ &= \Big\{ w(\cdot)\colon \mathbb{R} \mapsto \mathbb{R} \;\Big|\; \text{for all } t \in \mathbb{R} \text{ there exist } \lim_{s\to t-} w(s) =: w(t-) \\ &\qquad\qquad\qquad\qquad \text{and } \lim_{s\to t+} w(s) = w(t) \Big\}\end{aligned}$$

is the Skorokhod metric space [2] with the Borel σ-algebra Φ, the transformation

$$\theta_t w(\cdot) = w(t + \cdot),$$

and with a θ_t-invariant probabilistic measure P. In this case $\xi(t,w) = w(t) = \pi(\theta_t w)$, where $\pi\colon \Omega \mapsto \mathbb{R}$ and $\pi(w) = w(0)$. The metric ρ is defined by

$$\rho(w_1,w_2) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\rho_i(w_1,w_2)}{1+\rho_i(w_1,w_2)},$$

where

$$\rho_i\left(w_1,w_2\right) = \inf_{\lambda\in\Lambda} \left(\sup_{t\in[-i,i]} \left|w_1\left(t\right) - w_2\left(\lambda(t)\right)\right| + \sup_{t\in[-i,i]} \left|t - \lambda\left(t\right)\right| \right)$$

and Λ is the set of continuous increasing functions

$$\lambda(\cdot)\colon [-i,i] \to [-i,i]$$

such that $\lambda(-i) = -i$ and $\lambda(i) = i$.

Note that an arbitrary function $w(\cdot) \in \Omega$ is measurable, bounded on finite intervals, and has an at most countable set of points of discontinuity of the first kind. Moreover, if $w_n \to w_0$ in Ω, then

$$w_n(t) \to w_0(t)$$

almost everywhere on $[a,b]$ for arbitrary $[a,b] \subset \mathbb{R}$; and

$$\sup_{t\in[a,b]} |w_n(t)| \le A + B \sup_{t\in[a,b]} |w_0(t)|,$$

where the nonnegative constants A and B do not depend on n. Therefore $w_n \to w_0$ in $L^q(a,b)$ for all $1 \le q < \infty$ by the Lebesgue theorem.

It follows from [1] that $\{\theta_t\colon \Omega \mapsto \Omega\}_{t\in\mathbb{R}}$ is a metric dynamic system; that is, the mapping $(w,t) \mapsto \theta_t w$ is measurable, $\theta_0 w = w$, $\theta_{t+s} w = \theta_t\theta_s w$, and $\theta_t\,\mathsf{P} = \mathsf{P}$.

Our main aim is to prove that solutions of (1) form a (multivalued, generally speaking) random dynamic system whose phase space contains a minimal measurable attracting set called a random attractor.

Existence of solutions of problem (1) and prior estimates

Let $H = L_2(Q)$ be equipped with a norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. Applying Galerkin's approximations [3, 4] in a standard way we prove that, for all $w \in \Omega$, the class

$$W = L_p^{loc}(0,+\infty; L_p(Q)) \cap L_2^{loc}(0,+\infty; H_0^1(Q)) \cap C([0,+\infty); H)$$

contains at least one solution $u = u(t,w)u_0$ of problem (1) and, moreover,

$$\begin{aligned}&\frac{d}{dt}\|u(t)\|^2 + 2a\|u(t)\|^2_{H_0^1(Q)} + 2\alpha\|u(t)\|^p_{L_p(Q)} \\ &\qquad \le 2C + 2(h,u(t)) + C_\gamma|w(t)| + (2C_1+\gamma)|w(t)|\|u(t)\|^2\end{aligned} \tag{2}$$

for any solution $u \in W$ of problem (1) and almost all $t > 0$, where $\gamma > 0$ is an arbitrary number and $C_\gamma = C_2^2|Q|/\gamma$. It follows from (2) that

$$\|u(t)\|^2 \le \|u(s)\|^2 + \left(2C + \frac{1}{a\lambda_1}\|h\|^2\right)(t-s) + C_\gamma \int_s^t |w(p)|\,dp + \int_s^t \big((2C_1+\gamma)|w(p)| - \lambda_1 a\big)\|u(p)\|^2\,dp \tag{3}$$

for all $t \ge s \ge 0$, where $\lambda_1 > 0$ is the first eigenvalue of $-\triangle$ in $H_0^1(Q)$. Using the Gronwall lemma we obtain

$$\begin{aligned}\|u(t)\|^2 \le \|u_0\|^2 \exp\left(\int_0^t \big((2C_1+\gamma)|w(p)| - \lambda_1 a\big)\,dp\right) &\\ + \int_0^t \exp\left(\int_s^t \big((2C_1+\gamma)|w(p)| - a\lambda_1\big)\,dp\right) &\\ \times \left(2C + \frac{1}{a\lambda_1}\|h\|^2 + C_\gamma |w(s)|\right) ds &\end{aligned} \tag{4}$$

for all $t \ge 0$.

In what follows we need the following auxiliary result.

Lemma 1. *Let $\{u_n = u_n(t, w_n)u_n^0\} \subset W$ be an arbitrary sequence of solutions of problem (1), where $w_n \to w_0$ in Ω, $u_n^0 \to u_0$ weakly in H, and $t_n \to t_0 > 0$.*

Then

$$u_n(t_n, w_n)u_n^0 \to u(t_0, w_0)u_0$$

in H at least along some subsequence, where $u = u(t, w_0)u_0 \in W$ is a solution of problem (1).

Proof. Let $T > 0$. It follows from estimates (2) and (4) that the sequence $\{u_n\}$ is bounded in

$$L_p(0,T;L_p(Q)) \cap L_2(0,T;H_0^1(Q)) \cap L_\infty(0,T;H).$$

Since

$$H_0^s(Q) \subset L_p(Q) \cap H_0^1(Q)$$

for $s \ge \max\{1; (1/2 - 1/p)n\}$, we have $(H_0^1(Q))^* \subset (H_0^s(Q))^*$ and $L_q(Q) \subset (H_0^s(Q))^*$, where $q = p/(p-1)$. By the assumptions imposed on the function f,

$$u_{n_t} \in L_2\big(0,T;(H_0^1(Q))^*\big) + L_q(0,T;L_q(Q));$$

thus the sequence $\{u_{n_t}\}$ is bounded in $L_q(0,T;(H_0^s(Q))^*)$. The lemma on compactness [3] implies that (at least along a subsequence) $u_n(\cdot, w_n)u_n^0 \to u(\cdot)$ in $L_2(0,T;H)$, $u_n(t,w_n)u_n^0 \to u(t)$ strongly in H for almost all $t \in (0,T)$ and weakly in H uniformly in $t \in [0,T]$.

Passing to the limit in (1) we prove that $u = u(t,w_0)u_0 \in W$ is a solution of (1). Let $K := 2C + \|h\|^2/(a\lambda_1)$ and

$$L_n(p) := (2C_1+\gamma)|w_n(p)| - \lambda_1 a, \qquad L_0(p) := (2C_1+\gamma)|w_0(p)| - \lambda_1 a.$$

Then it follows from (3) that

$$J(t,w_0) \le J(s,w_0), \qquad J_n(t,w_n) \le J_n(s,w_n)$$

for all $t \geq s$, $t, s \in [0,T]$, and $J_n(t, w_n) \to J(t, w_0)$ for almost all $t \in (0,T)$ where the functions

$$J_n(t, w_n) := \|u_n(t)\|^2 - Kt - C_\gamma \int_0^t |w_n(p)|\, dp - \int_0^t L_n(p)\|u_n(p)\|^2\, dp,$$
$$J(t, w_0) := \|u(t)\|^2 - Kt - C_\gamma \int_0^t |w_0(p)|\, dp - \int_0^t L_0(p)\|u(p)\|^2\, dp$$

belong to $C([0,T])$. Now it is easy to show that $J_n(t, w_n) \to J(t, w_0)$ uniformly on an arbitrary interval $[a,b] \subset (0,T)$.

The convergence $w_n \to w_0$ implies that $L_n(\cdot) \to L_0(\cdot)$ in $L^1(0,t)$ and that $L_n(\cdot)$, $n \geq 1$, are bounded in $L^\infty(0,t)$. Then

$$\int_0^t L_n(p)\|u_n(p)\|^2\, dp \to \int_0^t L_0(p)\|u(p)\|^2\, dp.$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} J_n(t_n, w_n) &= J(t_0, w_0)\\ &\geq \liminf_{n\to\infty} \|u_n(t_n)\|^2 - Kt_0 - C_\gamma \int_0^{t_0} |w_0(p)|\, dp - \int_0^{t_0} L_0(p)\|u(p)\|^2\, dp\end{aligned}$$

for all $t_n \to t_0 > 0$.

Hence

$$\|u(t_0)\| \geq \liminf_{n\to\infty} \|u_n(t_n)\|.$$

Since $u_n(t_n) \to u(t_0)$ weakly in H, the converse inequality also holds and therefore

$$\liminf_{n\to\infty} \|u_n(t_n)\| \leq \|u(t_0)\| \leq \liminf_{n\to\infty} \|u_n(t_n)\|.$$

Moreover $u_n(t_n)$ converges (at least along a subsequence) strongly in H to $u(t_0)$. The lemma is proved. □

Consider the mapping $G\colon \mathbb{R}_+ \times \Omega \times H \mapsto 2^H$ defined by

$$G(t, w)u_0 := \big\{u(t, w)u_0 \;\big|\; u(t, w)u_0 \in W \text{ is a solution of (1), } u(0, w)u_0 = u_0\big\}. \tag{5}$$

Mapping (5) describes the dynamics of the solutions of problem (1), and our aim is to study the behavior of (5) as $t \to +\infty$. In doing so we apply some methods of the theory of random attractors of dynamic systems.

A general theory of multivalued stochastic dynamic systems

Let $(X, \|\cdot\|)$ be a separable Banach space equipped with the Borel σ-algebra $\sigma(X)$ and let $C(X)$ ($\beta(X)$) be the family of all nonempty closed (nonempty and bounded) subsets of X. For $A, B \subset X$ we put $\|A\| := \sup_{a\in A} \|a\|$,

$$\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|,$$

$B_\delta(A) = \{x \in X \mid \operatorname{dist}(x, A) < \delta\}$, and $B_r = \{x \in X \mid \|x\| \leq r\}$, where $\overline{A}$ is the closure of A in X. Let $(\Omega, \Phi, \mathsf{P})$ be a probability space, $\{\theta_t\colon \Omega \mapsto \Omega\}_{t\in\mathbb{R}}$ be a metric dynamic system, and let $\overline{\Phi}$ be the Lebesgue completion of Φ with respect to the measure P. The definitions and properties of multivalued mappings given below are taken from [9].

A mapping $F\colon \Omega \mapsto C(X)$ is called measurable if $\{w \in \Omega \mid F(w) \cap O \neq \varnothing\} \in \Phi$ for any open set $O \subset X$. The following conditions are equivalent:

(1) F is measurable;
(2) $w \mapsto \operatorname{dist}(x, F(w))$ is measurable for an arbitrary $x \in X$;
(3) there exist measurable functions $\{f_n(w)\}_{n=1}^\infty$ such that $F(w) = \overline{\bigcup_{n=1}^\infty f_n(w)}$.

It is also known that if F is measurable, then $w \mapsto \|F(w)\|$ is $\overline{\Phi}$-measurable, and if $G \in \overline{\Phi} \times \sigma(X)$, then $\pi_\Omega G := \{w \in \Omega \mid \text{there exists } x \in X, (w, x) \in G\} \in \overline{\Phi}$.

Definition 1. A multivalued mapping $G\colon \mathbb{R}_+ \times \Omega \times X \mapsto C(X)$ is called a multivalued random dynamic system if

1) the mapping $(t, w) \mapsto G(t, w)x$ is measurable for arbitrary $x \in X$;
2) $G(0, w)x = x$ and $G(t + s, w)x \subset G(t, \theta_s w)G(s, w)x$ for all $t, s \in \mathbb{R}_+$, $x \in X$, $w \in \Omega$.

Having in mind some specific applications, it is worth mentioning that an invariant set $\{\theta_t\}_{t\in\mathbb{R}}$ of full measure can replace the set Ω in the definition of a multivalued random dynamic system.

A measurable mapping $F\colon \Omega \mapsto C(X)$ is called a measurable set $F(w)$.

Definition 2. A measurable set $A(w)$ is called a random attractor for a multivalued random dynamic system G if for P-almost all $w \in \Omega$,

1) $A(\theta_t w) \subset G(t, w)A(w)$ for all $t \in \mathbb{R}_+$ (semi-invariance);
2) for all $B \in \beta(X)$,
$$\operatorname{dist}(G(t, \theta_{-t}w)B, A(w)) \to 0, \qquad t \to +\infty$$
(attraction);
3) $A(w)$ is a compact set in X.

Below are some assumptions concerning the multivalued random dynamic system G.

(G1) The mapping $(t, w) \mapsto \overline{G(t, w)B}$ is measurable for all $B \in \beta(X)$;

(G2) for all $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that for all $B \in \beta(X)$ there is $T = T(B, R, \varepsilon)$ for which
$$\mathsf{P}\left\{\sup_{t\ge T} \|G(t, \theta_{-t}w)B\| > R\right\} < \varepsilon.$$

Assumption (G1) implies the measurability of the mapping $(t, w) \mapsto \overline{G(t, \theta_{-t}w)B}$. Indeed, the set $K_1 = \{(t, \xi) | \operatorname{dist}(x, G(t, \xi)B) < a\}$ is measurable for all $a \in \mathbb{R}$. Since the mapping $v\colon (t, w) \mapsto (t, \theta_{-t}w)$ is measurable, the set
$$K_2 = v^{-1}(K_1) = \{(t, w) \mid \xi = \theta_{-t}w, (t, \xi) \in K_1\}$$
is measurable too. We also have $K_2 = \{(t, w) \mid \operatorname{dist}(x, G(t, \theta_{-t}w)B) < a\}$, whence the required statement follows.

On the other hand, the mapping $w \mapsto \operatorname{dist}\big(x, \bigcup_{t\ge T} G(t, \theta_{-t}w)B\big)$ is $\overline{\Phi}$-measurable, since
$$\operatorname{dist}\left(x, \bigcup_{t\ge T} G(t, \theta_{-t}w)B\right) = \inf_{t\ge T} \operatorname{dist}(x, G(t, \theta_{-t}w)B)$$
and
$$\left\{w \;\middle|\; \inf_{t\ge T} \operatorname{dist}(x, G(t, \theta_{-t}w)B) < a\right\} = \pi_\Omega\big\{(t, w) \bigm| \operatorname{dist}(x, G(t, \theta_{-t}w)B) < a,\ t \ge T\big\}.$$
This implies that the mapping $w \mapsto \sup_{t\ge T} \|G(t, \theta_{-t}w)B\|$ is $\overline{\Phi}$-measurable. Thus all terms in condition (G2) are well defined provided condition (G1) holds.

Condition (G2) can be checked for some classes of systems by using the following result.

Lemma 2. *If condition* (G1) *holds for a multivalued random dynamic system G and if there exists a bounded, measurable set $B(w)$ such that for P-almost all $w \in \Omega$ and all $B \in \beta(X)$ there is $T = T(B, w)$ for which $G(t, \theta_{-t}w)B \subset B(w)$ for all $t \ge T$, then condition* (G2) *holds.*

Proof. Suppose condition (G2) does not hold; that is, there exists $\varepsilon^* > 0$ such that for all R, there exists $B \in \beta(X)$ for which

$$\mathsf{P}\left\{\sup_{t\ge T} \|G(t,\theta_{-t}w)B\| \ge R\right\} \ge \varepsilon^* \quad \text{for all } T.$$

Since the mapping $w \mapsto \|B(w)\|$ is $\overline{\Phi}$-measurable and P-almost everywhere bounded,

$$\mathsf{P}\{\|B(w)\| > R\} \to 0, \qquad R \to \infty.$$

Then there exists R such that $\mathsf{P}\{\|B(w)\| > R\} < \varepsilon^*/2$. Since

$$K_N := \left\{w \;\Big|\; \sup_{t\ge N} \|G(t,\theta_{-t}w)B\| > R\right\},$$

$\mathsf{P}\{K_N\} \ge \varepsilon^*$, and $K_{N+1} \subset K_N$ for all $N \ge 1$, we have $K = \bigcap_{N=1}^\infty K_N$,

$$\mathsf{P}\{K\} = \lim_{N\to\infty} \mathsf{P}\{K_N\} \ge \varepsilon^*,$$

and $K \subset \{w \mid \|B(w)\| > R\}$. This implies that $\mathsf{P}\{K\} < \varepsilon^*/2$, leading to a contradiction. The lemma is proved. $\square$

For a given $B \in \beta(X)$, consider the sets

$$\Lambda_B(w) := \bigcap_{T>0} \overline{\bigcup_{t\ge T} G(t,\theta_{-t}w)B}, \qquad A(w) := \overline{\bigcup_{B\in\beta(X)} \Lambda_B(w)} = \overline{\bigcup_{M=1}^{\infty} \Lambda_{B_M}(w)}.$$

It follows from [8] that $\Lambda_B(w)$ consists of the limits of all convergent sequences $\{\xi_n\}$, where $\xi_n \in G(t_n,\theta_{-t_n}w)B$, $t_n \nearrow \infty$.

Lemma 3. *Let, for all $t \in \mathbb{R}_+$ and $w \in \Omega$, the mapping $x \mapsto G(t,w)x$ be upper semicontinuous and have compact values. Assume that conditions* (G1) *and* (G2) *hold for a multivalued random dynamic system G. Let, for all $w \in \Omega$, $t > 0$, and $R > 0$, the set $G(t,w)B_R$ be precompact in X (that is, let the closure of $G(t,w)B_R$ be compact). Then there exists a set $\Omega(B)$ such that $\mathsf{P}(\Omega(B)) = 1$ and, for all $w \in \Omega(B)$, the set $\Lambda_B(w) \ne \varnothing$ attracts B and is compact and semiinvariant; that is, $\Lambda_B(\theta_t w) \subset G(t,w)\Lambda_B(w)$ for all $t \in \mathbb{R}_+$.*

Remark 1. The assumption that the set $G(t,w)B_R$ is precompact follows from other assumptions of Lemma 3 if $\dim X < \infty$.

Proof. We have

$$G(t,\theta_{-t}w)B \subset G(1,\theta_{-1}w)G\big(t-1,\theta_{-(t-1)}\theta_{-1}w\big)B$$

for all $w \in \Omega$. Let $\Omega(N,R,B) := \{w \mid \sup_{t\ge N} \|G(t-1,\theta_{-(t-1)}\theta_{-1}w)B\| \le R\}$. Then for all $\varepsilon > 0$, there exist numbers $R = R(\varepsilon)$ and $N = N(B,R,\varepsilon)$ such that

$$\mathsf{P}\{\Omega(N,R,B)\} \ge 1 - \varepsilon$$

and $\Omega(N,R,B) \subset \Omega(N+1,R,B)$. Thus

$$G(t,\theta_{-t}w)B \subset G(1,\theta_{-1}w)B_R$$

for all $w \in \Omega(N,R,B)$ and $t \ge N$. Then $\Lambda_B(w)$ is a nonempty compact set. Hence $\Lambda_B(w)$ is a nonempty compact set for all $w \in \Omega(R,B) := \bigcup_{N\ge N(B,R,\varepsilon)}^{\infty} \Omega(N,R,B)$ and moreover $\mathsf{P}\{\Omega(R,B)\} \ge 1-\varepsilon$. Since $\Omega(R,B) \subset \Omega(R+1,B)$, one can choose $\varepsilon_i \to 0$ and $R_i = R(\varepsilon_i) \nearrow \infty$ such that the set $\Lambda_B(w)$ is nonempty and compact for all $w \in \Omega(B) := \bigcup_{i=1}^\infty \Omega(R_i,B)$ and moreover $\mathsf{P}\{\Omega(B)\} = 1$.

Now we prove the semiinvariance and attraction for $w \in \Omega(B)$. For all $w \in \Omega(B)$, there exist R and N such that $w \in \Omega(N, R, B)$. Thus

$$G\big(t_n + t, \theta_{-(t_n+t)}\theta_t w\big)B \subset G(t, w)G(t_n, \theta_{-t_n} w)B \subset G(t, w)\overline{G(1, \theta_{-1}w)B_R}$$

starting with some n, where $\{t_n\}$ is an arbitrary sequence such that $t_n \nearrow \infty$ and $t \in \mathbb{R}_+$. The latter set is compact, since $G(t, w)$ is upper semicontinuous and compact. Therefore $\Lambda_B(\theta_t w) \neq \varnothing$.

Further, for any $y \in \Lambda(\theta_t w)$ there exists a sequence $\{t_n\}$ such that $y = \lim y_n$ and $y_n \in G(t_n + t, \theta_{-(t_n+t)}\theta_t w)B$. Then there is $\eta_n \in G(t_n, \theta_{-t_n} w)B$ for which $y_n \in G(t, w)\eta_n$. Recall that the mapping $x \mapsto G(t, w)x$ is upper semicontinuous and has compact values. Thus the graph of the mapping $x \mapsto G(t, w)x$ is closed [9]. Since the set $\overline{G(1, \theta_{-1}w)B_R}$ is compact, the limit $\eta = \lim \eta_n \in \Lambda_B(w)$ exists, and $y \in G(t, w)\eta \subset G(t, w)\Lambda_B(w)$. This observation completes the proof of the semiinvariance.

Suppose $\Lambda_B(w)$ does not attract B for some $w \in \Omega(B)$. Then there exist a number $\delta > 0$ and a sequence $\{t_n\}$ such that $t_n \nearrow \infty$ and

$$\operatorname{dist}\big(G(t_n, \theta_{-t_n} w)B, \Lambda_B(w)\big) \geq \delta > 0.$$

Hence the limit $\xi_n \in G(t_n, \theta_{-t_n} w)B$ exists and is such that $\operatorname{dist}(\xi_n, \Lambda_B(w)) \geq \delta$. It follows from the above reasoning that the set $\{\xi_n\}$ is precompact, whence it follows that the limit $\xi = \lim \xi_n \in \Lambda_B(w)$ exists. This leads to a contradiction, and thus the lemma is proved. $\square$

Theorem 1. *Let a mapping $x \mapsto G(t, w)x$ be upper semicontinuous and compact-valued for all $t \in \mathbb{R}_+$ and $w \in \Omega$. Let conditions* (G1) *and* (G2) *hold for a multivalued random dynamic system G and let the set $G(t, w)B_R$ be precompact in X for all $w \in \Omega$, $t > 0$, and $R > 0$. Then the set*

$$A(w) = \overline{\bigcup_{n=1}^{\infty} \Lambda_{B_n}(w)}$$

is a random attractor for G. Therefore the attractor is unique, it is a minimal set among closed attracting sets, and it is a maximal set among compact, measurable, semiinvariant sets.

Proof. We prove that $A(w) = \overline{\bigcup_{n=1}^{\infty} \Lambda_{B_n}(w)}$ is a compact set for P-almost all $w \in \Omega$. Given B_n, consider the set $\Omega(B_n)$ defined in Lemma 3. Since $\Omega(B_{n+1}) \subset \Omega(B_n)$,

$$\Omega_0 := \bigcap_{n=1}^{\infty} \Omega(B_n)$$

is a set of the full measure and $\Lambda_{B_n}(w)$ satisfies the assumptions of Lemma 3 for all $w \in \Omega_0$ and all $n \geq 1$. Consider the set

$$\Omega_0(R, B_n) := \Omega_0 \cap \Omega(R, B_n),$$

where $\Omega(R, B_n)$ is defined in Lemma 3. Then for all $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that

$$\mathsf{P}\{\Omega_0(R, B_n)\} \geq 1 - \varepsilon,$$

$\Omega_0(R, B_{n+1}) \subset \Omega_0(R, B_n)$ for all $n \geq 1$, and

$$\Lambda_{B_n}(w) \subset \overline{G(1, \theta_{-1}w)B_R}$$

for all $w \in \Omega_0(R, B_n)$. Thus $\mathsf{P}\{\Omega_0(R)\} \geq 1 - \varepsilon$ for $\Omega_0(R) = \bigcup_{n=1}^{\infty} \Omega_0(R, B_n)$ and $A(w) \subset \overline{G(1, \theta_{-1}w)B_R}$ for all $w \in \Omega_0(R)$. Hence $A(w)$ is a compact set. Choose $\varepsilon_i \to 0$

and $R_i = R(\varepsilon_i) \nearrow \infty$. Since $\Omega_0(R_i) \subset \Omega_0(R_{i+1})$, the set $A(w)$ is compact and $\mathsf{P}\{\Omega^0\} = 1$ for all

$$w \in \Omega^0 := \bigcup_{i=1}^{\infty} \Omega_0(R_i).$$

In what follows we restrict the consideration to subsets of the set Ω^0. The attraction follows from Lemma 3. Furthermore,

$$A(\theta_t w) \subset \overline{\bigcup_{B\in\beta(X)} G(t,w)\Lambda_B(w)} \subset \overline{G(t,w)\overline{\bigcup_{B\in\beta(X)} \Lambda_B(w)}} = \overline{G(t,w)A(w)} = G(t,w)A(w),$$

since $A(w)$ is a compact set and the mapping $x \mapsto G(t,w)x$ is upper semicontinuous [9]. The measurability of $A(w)$ with respect to the σ-algebra Φ follows from the $\overline{\Phi}$-measurability of $A(w) = \overline{\bigcup_{n=1}^{\infty} \bigcap_{\tau_m \geq 0} \overline{\bigcup_{t\geq\tau_m} G(t,\theta_{-t}w)B_n}}$. In the proof of the measurability we use the following well-known result [6]: for an arbitrary compact $\overline{\Phi}$-measurable set $A(w)$, there exists a compact Φ-measurable set $\tilde{A}(w)$ that coincides with $A(w)$ almost everywhere with respect to P.

To prove the uniqueness we follow the method of [8]. Let $E(w)$ be a semiinvariant, compact, and measurable set. Let $F = \{w \mid E(w) \subset B\}$ for a nonrandom set $B \in \beta(X)$. Then the Poincaré recurrence theorem implies that $\mathsf{P}\{F_\infty\} \geq \mathsf{P}\{F\}$ for

$$F_\infty = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \theta_n F = \big\{w \mid \theta_{-n} w \in F \text{ infinitely often}\big\}.$$

If

$$w \in F_\infty,$$

then $E(\theta_{-n}w) \subset B$ for infinitely many $n \in \mathbb{N}$. Since $E(w)$ is semiinvariant, we have $E(w) \subset G(n,\theta_{-n}w)E(\theta_{-n}w) \subset G(n,\theta_{-n}w)B$; that is, $E(w) \subset \Lambda_B(w)$. Thus

$$\mathsf{P}\{E(w) \subset \Lambda_B(w)\} \geq \mathsf{P}\{E(w) \subset B\}.$$

On the other hand, it follows from [6] that, for an arbitrary compact measurable set $E(w)$ and for an arbitrary $\varepsilon > 0$, there exists a nonrandom compact set $K_\varepsilon \subset X$ such that

$$\mathsf{P}\{E(w) \subset K_\varepsilon\} > 1 - \varepsilon.$$

Hence $\mathsf{P}\{E(w) \subset A(w)\} \geq \mathsf{P}\{E(w) \subset \Lambda_{K_\varepsilon}(w)\} > 1 - \varepsilon$, whence

$$\mathsf{P}\{E(w) \subset A(w)\} = 1.$$

Therefore a random attractor is unique and is the maximal set among semiinvariant, compact, and measurable sets.

Now let $K(w)$ be a closed attracting set. We prove that

$$\Lambda_B(w) \subset K(w)$$

for all $B \in \beta(X)$ and $w \in \Omega^0$. Were this wrong we would find $w \in \Omega^0$, $y \in \Lambda_B(w)$, and $\delta > 0$ such that $\operatorname{dist}(y, K(w)) > \delta$. At the same time, there exist $t_n \nearrow \infty$ and $y_n \in G(t_n, \theta_{-t_n}w)B$ such that $y = \lim y_n$. Thus there exists a number $n(w)$ such that $\operatorname{dist}(y_n, K(w)) < \delta/2$ for all $n \geq n(w)$ and this leads to a contradiction. The theorem is proved. □

Remark 2. The assumptions (G1) and (G2) are necessary to check only for all $B = B_r$, $r > 0$.

The following result allows one to check assumption (G1).

Lemma 4. *Let Ω be a metric space and Φ be a Borel σ-algebra. Assume that a multivalued mapping $G\colon R_+ \times \Omega \times X \mapsto C(X)$ satisfies the following condition: if $x_n \to x_0$ weakly in X as $t_n \to t_0 > 0$, $w_n \to w_0$ in Ω, and $y_n \in G(t_n, w_n)x_n$, then $y_n \to y_0 \in G(t_0, w_0)x_0$ in X for some subsequence.*

Then assumption (G1) *holds for the mapping G.*

Proof. According to Remark 2, one only needs to prove that the mapping

$$(t, w) \mapsto \overline{G(t, w)B_r}$$

is measurable. For an arbitrary closed set $C \subset X$ and for $n \geq 1$, consider the set

$$L_n := \left\{ (t, w) \;\middle|\; t \geq \frac{1}{n},\, \overline{G(t, w)B_r} \cap C \neq \varnothing \right\}.$$

Then L_n is closed and

$$L = \begin{cases} \bigcup_{n\geq 1} L_n & \text{if } B_r \cap C = \varnothing, \\ \bigcup_{n\geq 1} L_n \cup \{0 \times \Omega\} & \text{if } B_r \cap C \neq \varnothing, \end{cases}$$

where $L := \{(t, w) | \overline{G(t, w)B_r} \cap C \neq \varnothing\}$. Thus L is a Borel set, and therefore the lemma is proved. ☐

Application to problem (1)

The results obtained above can be applied to the family of mappings (5). Indeed, we prove that (5) generates a multivalued random dynamic system for which a random attractor exists. The main result of the paper reads as follows.

Theorem 2. *Let $\xi(t, w) = w(t)$ be a process and $\delta > 0$ be some number. If for all $\varepsilon > 0$ there exists $T > 0$ such that*

$$\mathsf{P}\left\{ \sup_{t\geq T} \frac{1}{t} \int_{-t}^{0} |w(p)|\, dp \leq \frac{a\lambda_1}{2C_1 + \gamma} - \delta \right\} > 1 - \varepsilon, \tag{6}$$

and for all $\varepsilon > 0$ there exists D such that

$$\sup_{t\geq 0} \mathsf{P}\left\{ \int_{-t}^{0} |w(s)|\, e^{\delta(2C_1+\gamma)s}\, ds \leq D \right\} > 1 - \varepsilon, \tag{7}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and C_1 is a constant such that

$$|g(u)| \leq C_1|u| + C_2, \qquad C_1 > 0,$$

then the family of mappings (5) *forms a multivalued random dynamic system for which a random attractor exists.*

Proof. It easily follows from Lemmas 1 and 4 that the family of mappings (5) forms a multivalued random dynamic system (Definition 1 is checked analogously to the paper [8, Proposition 4]). Condition (G1) holds for this multivalued random dynamic system and the set $G(t, w)B_r$ is precompact. Moreover, Lemmas 1 and 4 also imply that the mapping $x \mapsto G(t, w)x$ is upper semicontinuous and has compact values. Thus it remains to check condition (G2), and then we are ready to apply Theorem 1.

Lemma 1 implies that, for all $T_1 > T_2 > 0$ and all $w \in \Omega$, there exist $t(w) \in [T_1, T_2]$ and $x_0(w) \in B_r$ such that

$$\sup_{t\in[T_1,T_2]} \|G(t, \theta_{-t}w)B_r\| = \left\| u\big(t(w), \theta_{-t(w)}w\big)x_0(w) \right\|.$$

Since $w \mapsto \sup_{t\in[T_1,T_2]} \|G(t,\theta_{-t}w)B_r\|$ is $\overline{\Phi}$-measurable,

$$\|u(t(w),\theta_{-t(w)}w)x_0(w)\|$$

is $\overline{\Phi}$-measurable too.

Fix $\varepsilon > 0$ and consider (4) for $t \in [T, T+N]$. Let

$$L^N := \left\{ w\colon \sup_{t\in[T,T+N]} \|G(t,\theta_{-t}w)B_r\|^2 > R^2 \right\} = \left\{ w\colon \|u(t(w),\theta_{-t(w)}w)x_0(w)\|^2 > R^2 \right\}.$$

Then estimate (4) yields

$$\begin{aligned} L^N \subset \Bigg\{ w\colon r^2 \exp\left(\frac{1}{t(w)} \int_{-t(w)}^0 |w(p)| dp - \frac{a\lambda_1}{2C_1+\gamma} \right) (2C_1+\gamma)\, t(w)) \\ + \int_{-t(w)}^0 \exp\left(\left(\frac{1}{s} \int_s^0 |w(p)|\, dp + \frac{a\lambda_1}{2C_1+\gamma} \right) (2C_1+\gamma)\, s \right) \\ \times \left(2C + \frac{1}{a\lambda_1} \|h\|^2 + C_\gamma \,|w(s)| \right) ds > R^2 \Bigg\}. \end{aligned}$$

Put

$$\begin{aligned} A_1 &:= \left\{ w\colon r^2 \exp\left(\frac{1}{t(w)} \int_{-t(w)}^0 |w(p)|\, dp - \frac{a\lambda_1}{2C_1+\gamma} \right) (2C_1+\gamma)\, t(w) \ge 1 \right\} \\ &\subset \left\{ w : \left(\frac{1}{t(w)} \int_{-t(w)}^0 |w(p)|\, dp - \frac{a\lambda_1}{2C_1+\gamma} \right) \ge \frac{1}{(2C_1+\gamma)\, t(w)} \ln\left(\frac{1}{r^2} \right) \right\}. \end{aligned}$$

Now we pick up a number $T = T(r)$ such that

$$\frac{1}{(2C_1+\gamma)T} \ln\left(\frac{1}{r^2} \right) > -\delta.$$

Then

$$\begin{aligned} A_1 &\subset \left\{ w\colon \frac{1}{t(w)} \int_{-t(w)}^0 |w(p)|\, dp - \frac{a\lambda_1}{2C_1+\gamma} > -\delta \right\} \\ &\subset \left\{ w\colon \sup_{t\ge T} \frac{1}{t} \int_{-t}^0 |w(p)|\, dp - \frac{a\lambda_1}{2C_1+\gamma} > -\delta \right\}. \end{aligned}$$

Condition (6) implies that there exist a number $T_1 = T_1(\varepsilon)$ and a random event $A_2 \subset \Omega$ such that $\mathsf{P}\{A_2\} < \frac{\varepsilon}{4}$ and

$$\sup_{t\ge T_1} \frac{1}{t} \int_{-t}^0 |w(p)|\, dp - \frac{a\lambda_1}{2C_1+\gamma} \le -\delta$$

for all $w \in \Omega \setminus A_2$. Thus there exists $T_2 = T_2(\varepsilon, r) \ge T_1 + T(r)$ such that

$$\mathsf{P}\{A_1\} < \frac{\varepsilon}{4}$$

for all $t(w) \in [T_2, T_2+N]$.

Then

$$L^N \subset A_1 \cup \Bigg\{ w\colon \int_{-t(w)}^{0} \exp\left(\left(\frac{1}{s}\int_s^0 |w(p)|\,dp + \frac{a\lambda_1}{2C_1+\gamma}\right)(2C_1+\gamma)s\right) \times \left(2C + \frac{1}{a\lambda_1}\|h\|^2 + C_\gamma |w(s)|\right) ds > R^2 - 1 \Bigg\}$$

$$\subset A_1 \cup A_2 \cup \Bigg\{ w\colon \int_{-T_1}^{0} \left(2C + \frac{1}{a\lambda_1}\|h\|^2 + C_\gamma |w(s)|\right) \times \exp\left\{\left(\frac{1}{s}\int_s^0 |w(p)|\,dp + \frac{a\lambda_1}{2C_1+\gamma}\right)(2C_1+\gamma)s\right\} ds + \int_{-T_2-N}^{0} \left(2C + \frac{1}{a\lambda_1}\|h\|^2 + C_\gamma |w(s)|\right) e^{\delta(2C_1+\gamma)s}\,ds > R^2 - 1 \Bigg\}$$

$$= A_1 \cup A_2 \cup \left\{ w\colon f_\varepsilon(w) + \int_{-T_2-N}^{0} |w(s)| e^{\delta(2C_1+\gamma)s}\,ds > \frac{R^2 - A}{B} \right\},$$

where $A > 0$ and $B > 0$ are some constants, and $f_\varepsilon : \Omega \mapsto \mathbb{R}$ is a measurable and P-almost everywhere bounded function. Hence there exist a number $R_1 = R_1(\varepsilon)$ and a random event $A_3 \subset \Omega$ such that $f_\varepsilon(w) > R_1$ and $\mathsf{P}\{A_3\} < \frac{\varepsilon}{4}$ for all $w \in A_3$. Thus

$$L^N \subset A_1 \cup A_2 \cup A_3 \cup \left\{ w\colon \int_{-T-N}^{0} |w(s)| e^{\delta(2C_1+\gamma)s}\,ds > \frac{R^2 - A}{B} - R_1(\varepsilon) \right\}.$$

Condition (7) implies that there exists $D = D(\varepsilon)$ such that

$$\mathsf{P}\left\{ w\colon \int_{-t}^{0} |w(s)| e^{\delta(2C_1+\gamma)s}\,ds > D \right\} < \frac{\varepsilon}{4}$$

for all $t > 0$.

Choose $R = R(\varepsilon)$ such that

$$\frac{R^2 - A}{B} - R_1(\varepsilon) > D.$$

Then, for all $\varepsilon > 0$ one can find $R = R(\varepsilon)$ for which, whatever B_r is, there exists $T = T(\varepsilon, R, r)$ such that

$$\mathsf{P}\{L^N\} = \mathsf{P}\left\{ w\colon \sup_{t \in [T, T+N]} \|G(t, \theta_{-t} w) B_r\|^2 > R^2 \right\} < \varepsilon$$

for all $N \geq 1$. Since $L^N \subset L^{N+1}$, we have $\mathsf{P}\{L\} < \varepsilon$ and

$$\left\{ w\colon \sup_{t \geq T} \|G(t, \theta_{-t} w) B_r\|^2 > R^2 \right\} = L,$$

where $L := \bigcup_{N=1}^{\infty} L^N$. Therefore condition (G2) holds and the theorem is proved. $\square$

Theorem 2 holds under weaker restrictions imposed on the function g if one assumes stronger conditions concerning the process $\xi(t, w)$. Namely, let the function g be such that

$$|g(u)| \leq C_1 |u|^r + C_2 \tag{8}$$

for $r < p-1$. In this case condition (7) must be replaced with the following: for all $\varepsilon > 0$ there exists $D > 0$ such that

$$\sup_{t\geq 0} \mathsf{P}\left\{ \int_{-t}^{0} |w(p)|^{p/(p-r-1)} e^{a\lambda_1 s}\, ds \leq D \right\} > 1-\varepsilon. \tag{9}$$

Theorem 3. *Suppose all the assumptions of Theorem 2 hold and additionally the function g satisfies condition* (8)*, while the process $\xi(t,w) = w(t)$ satisfies conditions* (6) *and* (9)*. Then the family of mappings* (5) *forms a multivalued random dynamic system for which a random attractor exists.*

Proof. Multiplying both sides of (1) by u we get

$$\begin{aligned}\frac{d\left\|u(t)\right\|^2}{dt} &\leq -\lambda_1 a \left\|u(t)\right\|^2 - \alpha \left\|u\left(t\right)\right\|_{L_p}^{p} + 2C_1 \left|w(t)\right| \left\|u(t)\right\|_{L_{r+1}}^{r+1} + 2C_2 \left|w(t)\right| \cdot \left\|u(t)\right\|_{L_1} \\ &\quad + 2C + \frac{\left\|h\right\|^2}{a\lambda_1} \\ &\leq -\lambda_1 a \left\|u(t)\right\|^2 + C_3 \left|w(t)\right|^{p/(p-r-1)} + C_4.\end{aligned}$$

Then inequalities (3) and (4) can be rewritten as follows:

$$\left\|u(t)\right\|^2 \leq \left\|u(s)\right\|^2 + C_3 \int_s^t \left|w(s)\right|^{p/(p-r-1)}\, ds + C_4\left(t-s\right), \tag{10}$$

$$\left\|u(t)\right\|^2 \leq e^{-\lambda_1 a t} \left\|u_0\right\|^2 + \int_0^t e^{-\lambda_1 a(t-s)} \left(C_3 \left|w(s)\right|^{p/(p-r-1)} + C_4\right) ds. \tag{11}$$

□

The proof of Lemma 1 is the same in this case, since

$$J(t,w) = \left\|u(t)\right\|^2 - C_4 t - C_3 \int_0^t \left|w(s)\right|^{p/(p-r-1)}\, ds.$$

Condition (G2) is checked similarly to the proof of the preceding theorem by observing that

$$\left\|y\right\|^2 \leq e^{-\lambda_1 a t} r^2 + \int_{-t}^{0} e^{\lambda_1 a p} \left(C_3 \left|w(p)\right|^{p/(p-r-1)} + C_4\right) dp$$

for all B_r and $y \in G(t, \theta_{-t} w) B_r$. □

Concluding remarks

In this paper, we proved that a random attractor exists and is unique for a multivalued random dynamic system that is dissipative in probability. Using this result, we proved that solutions of the reaction-diffusion equation perturbed by a stochastic càdlàg process form a multivalued random dynamic system for which a random attractor exists in the phase space.

Bibliography

1. L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998. MR1723992 (2000m:37087)
2. I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, "Nauka", Moscow, 1977; English transl. of the first edition, Scripta Technica, Inc. W. B. Saunders Co., Philadelphia, Pa.–London–Toronto, Ont., 1969. MR0488196 (58:7758)
3. A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, "Nauka", Moscow, 1989; English transl., North-Holland Publishing Co., Amsterdam, 1992. MR1007829 (92f:58101)
4. A. V. Kapustyan, *Global attractors of a nonautonomous reaction-diffusion equation*, Differ. Uravn. **38** (2002), no. 10, 1378–1382; English transl. in Differ. Equ. **38** (2002), no. 10, 1467–1471. MR1984456 (2004g:35028)
5. H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields **100** (1994), 365–393. MR1305587 (95k:58092)

6. H. Crauel, *Global random attractors are uniquely determined by attracting deterministic compact sets*, Ann. Mat. Pura Appl. **126** (1999), no. 4, 57–72. MR1746535 (2000m:37090)
7. K. R. Schenk-Hoppe, *Random attractors—general properties, existence and applications to stochastic bifurcation theory*, Discrete Contin. Dyn. Syst. **4** (1998), no. 1, 99–130. MR1485366 (98h:34103)
8. T. Caraballo, J. A. Langa, and J. Valero, *Global attractors for multivalued random dynamical systems*, Nonlinear Anal. **48** (2002), 805–829. MR1878338 (2003a:37068)
9. J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990. MR1048347 (91d:49001)

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