Теорія Ймовір. та Матем. Статист.
Вип. 73, 2005

Theor. Probability and Math. Statist.
No. 73, 2006, Pages 71–79
S 0094-9000(07)00682-5
Article electronically published on January 17, 2007

LUNDBERG APPROXIMATION FOR THE RISK FUNCTION IN AN ALMOST HOMOGENEOUS ENVIRONMENT

UDC 519.21

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Abstract. A generalization of the classical risk process is considered where the premium rate depends on the current reserve of an insurance company. We assume that the corresponding function converges to a limit with the exponential rate and prove that the limit of the exponentially weighted ruin function exists as the initial reserve increases. Two-sided estimates for the limit are found; the estimates show that the limit is positive under certain assumptions on the stability.

1. Introduction

Risk processes in a nonhomogeneous environment are studied in [2]–[14].

We consider a generalization of the classical risk process; namely, we assume that the premium rate depends on the current reserve. Using the results of the theory of stability [1], we obtain a generalization of the Lundberg theorem on the exponential behavior of the ruin probability for the case where premium rate functions are close to constants. Note that our results for constant premium rate functions coincide with the classical results.

2. Main notation

1. Let $c(x)$, $x \in \mathbb{R}_+$, be a measurable bounded strictly positive function such that $1/c(x)$ is locally integrable and let $Z(t) = \sum_{k=1}^{\nu(t)} \xi_k$ be a compound Poisson process, where $(\xi_n, n \geq 1)$ are independent identically distributed nonnegative random variables treated as premiums and the Poisson process $\nu(t)$ does not depend on premiums.

Consider a right continuous Markov process

$$dX_t = c(X_t)\,dt - dZ_t, \qquad t \geq 0,\ X_0 = x. \tag{1}$$

Then the ruin probability is given by

$$q(x) = \mathsf{P}\left(\bigcup_{t\geq 0}\{X_t < 0\} \;/\; X_0 = x\right), \qquad x \geq 0. \tag{2}$$

2. In what follows we use the following notation:

$$\lambda = \mathsf{E}\,\nu(1), \qquad m = \mathsf{E}\,\xi_1, \qquad F(x) = \mathsf{P}(\xi_1 < x), \qquad G(x) = 1 - F(x). \tag{3}$$

3. Assume that the Cramér condition holds, that is,

$$\gamma \equiv \sup\big(s \geq 0\colon\ \mathsf{E}\exp(s\xi_1) < \infty\big) > 0. \tag{4}$$

2000 *Mathematics Subject Classification.* Primary 60J45; Secondary 60A05.

Key words and phrases. Risk function, Lundberg index, Poisson process.

Condition (4) implies that the moment generating functions

$$\widehat{f}(s) = \int_0^\infty \exp(sx)\, dF(x), \tag{5}$$

$$\widehat{g}(s) = \int_0^\infty \exp(sx) G(x)\, dx = s^{-1}\big(\widehat{f}(s) - 1\big) \tag{6}$$

are well defined and are analytic for $\operatorname{Re} s < \gamma$.

4. Let the premium rate function be such that

$$\text{there exists } \beta > 0\colon \quad c(x) - c = O(\exp(-\beta x)),\ x \to \infty, \tag{7}$$

for some constant $c > 0$ and moreover let the following balance condition hold:

$$c > \lambda m. \tag{8}$$

Consider the *Lundberg index* for the risk problem with the constant premium rate c:

$$\alpha \equiv \sup\big(s < \gamma\colon \lambda \widehat{g}(s) < c\big) \in (0, \infty). \tag{9}$$

The positivity of the index α follows from (8) and (6), while its boundedness holds, since $\widehat{g}(s) \to \infty$ as $s \to \infty$. Using the continuity on $[0, \gamma)$ we obtain from the *Lundberg condition* $\alpha < \gamma$ that α is a unique positive root of the equation

$$c\alpha - \lambda \widehat{f}(\alpha) + \lambda = 0. \tag{10}$$

3. Main results

The following two results generalize the Lundberg theorem for the ruin probability. Both these results coincide with the Lundberg theorem if $c(x) \equiv c$.

Theorem 1. *Let a Markov process $(X_t, t \geq 0)$ be defined by equalities* (1). *Assume that the Cramér condition* (4) *holds. Let, for some $c > 0$, representation* (7) *and balance condition* (8) *hold. Define the indices $\alpha \leq \gamma$ by equalities* (9) *and* (4), *respectively.*

(a) *If $\alpha < \gamma$, then there exists $\rho \in (\alpha, \gamma)$ and a constant C_α such that*

$$q(x) - C_\alpha \exp(-\alpha x) = O(\exp(-\rho x)), \qquad x \to \infty, \tag{11}$$

where the constant C_α is positive if deviations $c(x) - c$ are sufficiently small and the limits

$$\lim_{x\to\infty} x^{-1} \ln q(x) = -\alpha, \qquad C_\alpha = \lim_{x\to\infty} \exp(\alpha x) q(x) > 0 \tag{12}$$

exist.

(b) *If $\alpha = \gamma$, then*

$$q(x) = O(\exp(-\beta x)), \qquad x \to \infty, \tag{13}$$

for all $\beta < \gamma$.

The following numbers can serve as deviation indices of the premium rate function from a constant:

$$\begin{aligned} \rho_s^{\pm} &= \sup_{x\geq 0} (\exp(sx) - 1)(c(x) - c)^{\pm}, \\ \sigma_s^{\pm} &= \int_0^\infty (\exp(sx) - 1)(c(x) - c)^{\pm}\, dx, \end{aligned} \tag{14}$$

where $y^{\pm}$ stand for the positive and negative parts of y.

The following result allows one to evaluate explicitly the constant C_α for the classical case $c(x) \equiv x$, since $\rho_s^{\pm} = \sigma_s^{\pm} = 0$ in this case.

Theorem 2. *Suppose the assumptions of Theorem 1 and the Lundberg condition $\alpha < \gamma$ hold.*

(a) *If*

$$\text{there exists } \beta \geq \alpha \text{ such that } c(x) - c = O(\exp(-\beta x)),\ x \to \infty, \tag{15}$$

then

$$\begin{aligned} C_\alpha &\leq \left(c - \lambda m + \rho_\alpha^-\right) \left(\lambda\alpha\widehat{g}'(\alpha)\right)^{-1}, \\ C_\alpha &\geq \left(c - \lambda m - \rho_\alpha^+\right) \left(\lambda\alpha\widehat{g}'(\alpha)\right)^{-1}, \end{aligned} \tag{16}$$

where C_α is the constant defined in Theorem 1 (see (12)*).*

(b) *If*

$$\inf_{x\geq 0} \left(c(x) - \lambda \int_0^x G(y)\, dy \right) > 0, \tag{17}$$

then

$$\delta_\beta \equiv \inf_{x \geq 0} \left(c(x) - \lambda \int_0^x \exp(\beta y) G(y)\, dy \right) > 0 \tag{18}$$

for all sufficiently small $\beta \geq 0$. Moreover

$$\begin{aligned} C_\alpha &\leq \left(c - \lambda m + D_\beta \sigma_{\alpha-\beta}^-\right) \left(\lambda\alpha\widehat{g}'(\alpha)\right)^{-1}, \\ C_\alpha &\geq \left(c - \lambda m - D_\beta \sigma_{\alpha-\beta}^+\right)\Big) \left(\lambda\alpha\widehat{g}'(\alpha)\right)^{-1}, \end{aligned} \tag{19}$$

where

$$D_\beta = \lambda \delta_\beta^{-1} \sup_{x \geq 0} \exp(\beta x) G(x) c(x) \leq \lambda \delta_\beta^{-1} \widehat{f}(\beta) \sup_{x\geq 0} c(x). \tag{20}$$

Condition (15) holds if $c(x) = c$ for all sufficiently large x, while condition (17) holds if, for example, $c(x) \geq c$ for all x. The infimum in (17) can be evaluated on a finite interval only, since the corresponding difference approaches $c - \lambda m > 0$ as $x \to \infty$ provided conditions (8) and (7) hold.

Example. Consider the simplest case where the evaluations can be done in a closed form. Let $G(x) = \exp(-\mu x)$, $x \geq 0$, $m = \mu^{-1}$, and let $c > \lambda m$ be the constant premium rate. Then the Lundberg coefficient is $\alpha = \mu(1 - \lambda m / c)$. Let

$$K_\alpha = C_\alpha \left(\lambda\alpha\widehat{g}'(\alpha)\right) = C_\alpha \alpha c^2 / \lambda, \tag{21}$$

where C_α is the constant defined in Theorem 1 (see (11)) and the factors are the same as in (16) and (19).

Consider the exponential perturbation of the premium rate

$$c(x) = c \pm \varepsilon \exp(-\theta x), \qquad x \geq 0. \tag{22}$$

(a) Assume that $\theta \geq \alpha$. Then assumption (15) of Theorem 2 holds. Moreover

$$c - \lambda m - \varepsilon \leq K_\alpha \leq c - \lambda m + \varepsilon \tag{23}$$

irrespective of the sign of ε.

(b) Let $c(x) = c + \varepsilon \exp(-\theta x)$ where $\varepsilon > 0$. Assume that $\theta < \alpha$. Then condition (18) holds for $\delta_\beta = c - \lambda m$, and the constant in (20) does not exceed $D_\beta = \lambda(c+\varepsilon)(c - \lambda m)^{-1}$ for $\beta \leq \mu$. If $\beta \in (\alpha - \theta, \mu]$ and $\theta > 0$ is arbitrary, then (19) implies that

$$c - \lambda m - \varepsilon\alpha D_\beta / (\theta + \beta - \alpha)(\theta + \beta) \leq K_\alpha \leq c - \lambda m. \tag{24}$$

(b′) Finally consider the perturbation $c(x) = c - \varepsilon \exp(-\theta x)$ where $\varepsilon > 0$. Then condition (18) holds for $\delta_\beta = c - \varepsilon - \lambda m(1-\beta m)^{-1} > 0$. The latter constant is positive if and only if $\beta < \alpha - \lambda\varepsilon/c(c-\varepsilon)$. The right hand side of (20) does not exceed

$$D_\beta = \lambda c^2(\mu - \beta)(c(c-\varepsilon)(\alpha-\beta) - \lambda\varepsilon)^{-1}.$$

If $\theta > 0$ is arbitrary and $\varepsilon < \theta c^2(\lambda + \theta c)$, then (19) implies that

$$c - \lambda m \le K_\alpha \le c - \lambda m + \varepsilon\alpha D_\beta/(\theta + \beta - \alpha)(\theta + \beta) \tag{25}$$

for $\beta \in (\alpha - \theta, \alpha - \lambda\varepsilon/c(c-\varepsilon))$.

Note that the two sided bounds for the limit constant C_α are stable as $\varepsilon \to 0$ if $\theta > 0$ is arbitrary and ε is sufficiently small. The balance condition (8) implies that the constant C_α is asymptotically positive as $\varepsilon \to 0$.

4. Proofs

Denote by τ_n, $n \ge 1$, the sequential times of jumps of the Poisson process $\nu(t)$ and put $\theta_n = \tau_n - \tau_{n-1}$, $n \ge 1$, $\tau_0 = 0$.

Consider the following increasing and mutually inverse functions:

$$D(x) = \int_0^x \frac{dy}{c(y)} \quad \text{and} \quad C(x) = D^{(-1)}(x), \tag{26}$$

which are bijections in $\mathbb{R}_+$.

Lemma 1. *The process X_t between the jumps is given by*

$$\begin{aligned} X_{\tau_n + t} &= C(D(X_{\tau_n}) + t), \qquad t \in [0, \theta_{n+1}), \\ X_{\tau_{n+1}} &= C(D(X_{\tau_n}) + \theta_{n+1}) - \xi_{n+1}. \end{aligned} \tag{27}$$

Proof. Without loss of generality we assume that $n = 0$.

Using (1) we have $dX_t = c(X_t)\,dt$ or $dD(X_t) = dt$ in the interval $t \in [0, \theta_1)$ up to the first jump. This implies that $D(X_t) = D(X_0) + t$, whence $X_t = C(D(X_0) + t)$ by the definition of the inverse function in (26). This proves the first equality in (27). The second equality follows from the first one, since the process X_t is right continuous. □

Lemma 2. *The ruin function $q(x)$ is nondecreasing and $q(x) \to 0$ as $x \to \infty$.*

Proof. Using (2) we rewrite the non-ruin function as follows:

$$1 - q(x) = \mathsf{P}\left(\inf_{n \ge 0} X_{\tau_n} \ge 0 \bigm/ X_0 = x\right).$$

Since $C(D(x) + t)$ is monotone in x, equation (27) implies that the sequence

$$(X_{\tau_n}, n \ge 0)$$

is stochastically monotone with respect to the initial state $X_{\tau_0} = x$. Thus the function $1 - q(x)$ is nondecreasing.

Let $c_1 \in (\lambda m, c)$. Using definition (26) and conditions (7) and (8) we prove that there exists x_1 such that the inequality $C(D(x) + t) \ge x + c_1 t$ holds for all $x \ge x_1$ and $t \ge 0$. Thus, up to the moment of the first up-crossing of the level x_1, the sequence X_{τ_n} is stochastically bounded from below by the underlying sequence constructed for the constant premium rate c_1. Changing x, the up-crossing time can be made infinite with the probability approximating 1. □

Note that X_t is a strong Markov process [17, Chapter I, §4], since its trajectories are right continuous and its distribution is a weakly continuous function of the initial state x in view of (27).

Lemma 3. *For almost all $x \geq 0$, the function $q(x)$ is differentiable and satisfies the equation*

$$-c(x)q'(x) = -\lambda q(x) + \lambda \int_0^x q(x-y)\,dF(y) + \lambda G(x). \tag{28}$$

Proof. Consider the Markov time $\upsilon = \min(\theta_1, \varepsilon)$ for $\varepsilon > 0$.

According to the strong Markov property, we obtain from Lemma 1 that

$$\begin{aligned} q(x) &= \mathsf{E}_x\, q(X_\upsilon) = \mathsf{E}_x(1_{\theta_1>\varepsilon} q(X_\varepsilon) + 1_{\theta_1\leq\varepsilon} q(X_{\theta_1})) \\ &= \mathsf{E}_x(1_{\theta_1>\varepsilon} q(C(D(x)+\varepsilon)) + 1_{\theta_1\leq\varepsilon} q(C(D(x)+\theta_1) - \xi_1)), \end{aligned}$$

where the random variables θ_1 and ξ_1 are independent, $\mathsf{P}(\theta_1 > t) = \exp(-\lambda t)$, and $\mathsf{P}(\xi_1 < x) = F(x)$.

Putting $\Delta_\varepsilon = C(D(x)+\varepsilon) - x$, we obtain from the latter equality that

$$q(x) - q(x+\Delta_\varepsilon) = \int_0^\varepsilon \lambda \exp(-\lambda s) \left(\mathsf{E}\, q(C(D(x)+s) - \xi_1) - q(x+\Delta_\varepsilon)\right) ds \tag{29}$$

(by definition, $q(y) = 1$ for $y < 0$).

In particular, (29) implies that $q(x)$ is continuous. Moreover (29) yields the existence of the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \left(q(x) - q(x+\Delta_\varepsilon)\right) = \lambda\, \mathsf{E}\, q(x-\xi_1) - \lambda q(x)$$

by the mean value theorem. Relation (28) follows from the latter equality, since

$$\varepsilon^{-1}\Delta_\varepsilon \to C'(D(x)) = 1/D'(x) = c(x) \quad \text{as } \varepsilon \to 0. \qquad \square$$

The function

$$r(x) = -q'(x) \geq 0 \tag{30}$$

is nonnegative by Lemma 2 and integrable by definition. Moreover

$$\int_0^\infty r(x)\,dx = q(0) - q(\infty) = q(0). \tag{31}$$

Lemma 4. *The function $r(x)$ is a solution of the equation*

$$c(x)r(x) = \lambda \int_0^x r(y)G(x-y)\,dy + \lambda(1-q(0))G(x), \qquad x \geq 0. \tag{32}$$

Moreover

$$\int_0^\infty c(x)r(x)\,dx = \lambda m. \tag{33}$$

Proof. To prove (32) we integrate by parts

$$\int_0^x q(x-y)\,dF(y) = -q(x-y)G(y)t\Big|_0^x + \int_0^x r(x-y)G(y)\,dy$$

and substitute the result in equation (28).

Integrating (32) and using (31) we derive the identity (33):

$$\int_0^\infty c(x)r(x)\,dx = \lambda \int_0^\infty r(y)\,dy\, m + \lambda(1-q(0))m = \lambda m. \qquad \square$$

Let

$$h(x) = (c(x) - c)r(x), \qquad x \geq 0, \tag{34}$$

and define the moment generating functions

$$\widehat{r}(s) = \int_0^\infty \exp(sx)r(x)\,dx, \qquad \widehat{h}(s) = \int_0^\infty \exp(sx)h(x)\,dx \tag{35}$$

for $\operatorname{Re} s \le 0$. The function h is integrable, since r is integrable and $c(x)r(x)$ is continuous (this follows from (32)).

Lemma 5. *We have*

$$\widehat{r}(s) = k(s)(c - \lambda\widehat{g}(s))^{-1} \tag{36}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s \le 0$ where

$$k(s) = \lambda(1 - q(0))\widehat{g}(s) - \widehat{h}(s). \tag{37}$$

Proof. We multiply both sides of (32) by $\exp(sx)$, then substitute $cr(x) + h(x)$ in the left hand side according to definition (34), and integrate the result on $[0, \infty)$. This completes the proof of the lemma. □

Lemma 6. *Let the assumptions of Theorem 1 hold.*

(a) *If $\alpha < \gamma$, then there exist $\rho \in (\alpha, \gamma)$ and a constant D_α such that the function*

$$v(s) \equiv (c - \lambda\widehat{g}(s))^{-1} - D_\alpha(\alpha - s)^{-1} \tag{38}$$

is analytic in the region $\operatorname{Re} s \le \rho$. The constant can be evaluated as follows:

$$D_\alpha = (\lambda\widehat{g}'(\alpha))^{-1}. \tag{39}$$

(b) *If $\alpha = \gamma$, then the function $(c - \lambda\widehat{g}(s))^{-1}$ is analytic in the region $\operatorname{Re} s < \gamma$.*

Proof. (a) The function $\widehat{g}(s)$ is analytic in the region $\operatorname{Re} s < \gamma$ in view of the Cramér condition (4). According to (9), the function $c - \lambda\widehat{g}(s)$ has no roots if $\operatorname{Re} s < \alpha$ and has a root if $s = \alpha$. Since $c^{-1}\lambda\widehat{g}(\alpha + it)$ is the characteristic function of an absolutely continuous distribution function, it does not equal 1 for any $t \ne 0$ and for $\operatorname{Im} t > -\varepsilon$ where $\varepsilon > 0$ is arbitrary. Thus the function $c - \lambda\widehat{g}(s)$ has no roots in the region $\operatorname{Re} s < \rho \equiv \alpha + \varepsilon$ except a simple root $s = \alpha$.

If D_α is defined by (39), then the function defined by (38) can be written as

$$v(s) = D_\alpha\big(\widehat{g}(s) - \widehat{g}(\alpha) - (s - \alpha)\widehat{g}'(\alpha)\big)\big(\widehat{g}(\alpha) - \widehat{g}(s)\big)^{-1}(\alpha - s)^{-1}.$$

Note that v is continuous at the point $s = \alpha$ in this case. Moreover v is analytic in the region $\operatorname{Re} s \le \rho$ by the principle of analytic continuation.

(b) The function $c - \lambda\widehat{g}(s)$ is analytic and has no roots in the region $\operatorname{Re} s < \gamma$ in view of (9) if $\alpha = \gamma$. □

Lemma 7. *Suppose the assumptions of Theorem 1 hold and $\alpha < \gamma$.*

Then there are $\rho \in (\alpha, \gamma)$ and a constant C_α such that the function

$$u(s) \equiv \widehat{r}(s) - \alpha C_\alpha(\alpha - s)^{-1} \tag{40}$$

is analytic in the region $\operatorname{Re} s \le \rho$. The constant is such that

$$C_\alpha = k(\alpha)(\lambda\alpha\widehat{g}'(\alpha))^{-1}. \tag{41}$$

Proof. The functions $u(s)$ and $v(s)$ defined by relations (40) and (38), respectively, are such that

$$u(s) = k(s)v(s) + D_\alpha(k(s) - k(\alpha))(\alpha - s)^{-1}. \tag{42}$$

Moreover the function $\widehat{g}(s)$ in (37) is analytic in the region $\operatorname{Re} s < \gamma$ by the Cramér condition (4).

Put

$$\delta = \sup\left(s < \gamma\colon \int_0^\infty \exp(sx)\,|h(x)|\,dx < \infty\right). \tag{43}$$

Since the function $r(x)$ defined by (34) is integrable, condition (7) implies that $\delta \ge \beta > 0$. Thus the function $\widehat{h}(s)$ defined in (35) is analytic in the region $\operatorname{Re} s < \delta$. By (37), $k(s)$

is also analytic in the same region. Using statement (a) of Lemma 6 and (42) we prove that the function $u(s)$ is analytic for $\operatorname{Re} s < \min(\delta, \rho)$.

Assume that $\delta < \alpha$. We have $\widehat{r}(\delta - \varepsilon) < \infty$ for all $\varepsilon > 0$ by (40). Moreover (34) and (7) imply that

$$h(x) = O(\exp(-\beta t - \delta t + \varepsilon t)), \qquad t \to \infty,$$

for all $\varepsilon > 0$. This contradicts (43) for $\varepsilon < \beta$. Hence $\delta \geq \alpha$. Then condition (7) implies that $\widehat{r}(\alpha - \beta/2) < \infty$ for $\beta > 0$. Thus

$$\delta \geq \alpha - \beta/2 + \beta = \alpha + \beta/2$$

by (34). Therefore the function $\widehat{h}(s)$, as well as $k(s)$ and $u(s)$, is analytic in the region $\operatorname{Re} s < \min(\alpha + \beta/2, \rho)$. □

Proof of Theorem 1. (a) According to definition (40), $u(it)$ is the characteristic function of an absolutely continuous function of bounded variation:

$$u(s) = \int_0^\infty \exp(sx)\big(r(x) - \alpha C_\alpha \exp(-\alpha x)\big)\, dx.$$

Thus one can invert the moment generating function

$$\begin{aligned} r(x) - \alpha C_\alpha \exp(-\alpha x) &= (2\pi i)^{-1} \int_{\operatorname{Re} s = 0} \exp(-sx)\widehat{u}(s)\, ds \\ &= (2\pi i)^{-1} \int_{\operatorname{Re} s = \theta} \exp(-sx)\widehat{u}(s)\, ds, \end{aligned} \tag{44}$$

where $\theta \in (\alpha, \rho)$. We used Lemma 7 in (44) to prove that the function $\widehat{u}(s)$ is analytic in the region $\operatorname{Re} s < \rho$ for $\rho > \alpha$. The second equality in (44) is a consequence of the analyticity and the Riemann lemma.

It follows from (44) that

$$\begin{aligned} |r(x) - \alpha C_\alpha \exp(-\alpha x)| &\leq (2\pi)^{-1} \exp(-\theta x) \sup_{\operatorname{Re} s = \theta} |\widehat{u}(s)| \\ &= O(\exp(-\theta x)), \qquad x \to \infty. \end{aligned} \tag{45}$$

Now we derive from (30) and (45) that

$$\begin{aligned} |q(x) - C_\alpha \exp(-\alpha x)| &= \left| \int_x^\infty \big(r(y) - \alpha C_\alpha \exp(-\alpha y)\big)\, dy \right| \\ &= O(\exp(-\theta x)), \qquad x \to \infty, \end{aligned} \tag{46}$$

where $\theta > \alpha$ by definition.

(b) It remains to apply (44) to the function $\widehat{r}(s)$ and to take into account statement (b) of Lemma 6. □

Proof of Theorem 2. According to relation (46), it is sufficient to prove bounds for the constant C_α defined in (41). The denominators in (41) are the same for both cases of the theorem; thus we obtain estimates for the numerators $k(\alpha)$ defined by (37):

$$\begin{aligned} k(\alpha) &= \lambda(1 - q(0))\widehat{g}(\alpha) - \widehat{h}(\alpha) = c(1 - q(0)) - \widehat{h}(\alpha) = c - c\int_0^\infty r(x)\, dx - \widehat{h}(\alpha) \\ &= c - \lambda m + \int_0^\infty (c(x) - c) r(x)\, dx - \widehat{h}(\alpha) = c - \lambda m - h_1(\alpha) \end{aligned} \tag{47}$$

in view of (33) and (31), where

$$h_1(\alpha) = \int_0^\infty (\exp(\alpha x) - 1)(c(x) - c) r(x)\, dx. \tag{48}$$

(a) It follows from (34) and (14) that

$$\begin{aligned} h_1(s) &\le \rho_s^+ \int_0^\infty r(x)\,dx \le \rho_s^+, \\ h_1(s) &\ge -\rho_s^- \int_0^\infty r(x)\,dx \ge -\rho_s^- \end{aligned} \tag{49}$$

for $s \ge 0$. Thus (14) and (34) imply that

$$\begin{aligned} k(\alpha) &\le c - \lambda m + \rho_\alpha^-, \\ k(\alpha) &\ge c - \lambda m - \rho_\alpha^+, \end{aligned}$$

whence (16) follows.

(b) The left hand side of (18) is positive for sufficiently small $\beta > 0$ in view of condition (17), since the expression in (18) approaches the corresponding function in (17) uniformly in x as $\beta \to 0$.

Equality (32) can be rewritten as follows:

$$r(x) = Kr(x) + v(x), \tag{50}$$

where $v(x) = \lambda(1 - q(0))G(x)$ and the operator K is given by

$$Kr(x) = (c(x))^{-1} \int_0^x r(x-y)G(y)\,dy.$$

Using (17) we get $\inf c(x) \equiv \delta > 0$. Thus

$$Kr(x) \le \lambda\delta^{-1} \int_0^x r(y)G(x-y)\,dy \equiv K_1 r(x). \tag{51}$$

It follows from (51) that $K^n r(x) \le K_1^n r(x) \to 0$ as $n \to \infty$ for all x, since K_1 is a Volterra operator. Now we conclude from (50) that $r(x) = \sum_{n\ge 0} K^n v(x)$ for an arbitrary integrable function v.

In particular, $r \ge 0$ for nonnegative v.

Now we obtain from (18) that

$$\exp(-\beta x)\left(c(x) - \int_0^x \exp(\beta y)G(y)\,dy\right) \ge \exp(-\beta x)\delta_\beta$$

for all $x \ge 0$, whence

$$u(x) \ge u(x)\delta_\beta(c(x))^{-1} + Ku(x)$$

for the function $u(x) = D\exp(-\beta x)$. The inequality $u(x)\delta_\beta(c(x))^{-1} \ge v(x)$ implies that $u(x) \ge r(x)$ for all x, since $r \ge 0$ for $v \ge 0$. The first of these inequalities is

$$D\exp(-\beta x)\delta_\beta(c(x))^{-1} \ge \lambda(1-q(0))G(x) \quad \text{for all } x \ge 0,$$

which holds for $D = D_\beta$ as seen from (20).

Thus the function $r(x)$ satisfies the inequality

$$r(x) \le D_\beta \exp(-\beta x) \quad \text{for all } x \ge 0. \tag{52}$$

Now we use (14), (34), (35), (47), and (48) and obtain

$$\begin{aligned} h_1(s) &\le D_\beta \int_0^\infty \exp((s-\beta)x)(c(x)-c)^+\,dx = D_\beta \sigma_{s-\beta}^+, \\ h_1(s) &\ge -D_\beta \int_0^\infty \exp((s-\beta)x)(c(x)-c)^-\,dx = -D_\beta \sigma_{s-\beta}^- \end{aligned} \tag{53}$$

for $s \ge 0$.

Therefore (14) and (34) imply that

$$k(\alpha) \le c - \lambda m + D_\beta \sigma^-_{\alpha-\beta}, \\ k(\alpha) \ge c - \lambda m - D_\beta \sigma^+_{\alpha-\beta}, \tag{54}$$

whence (19) follows. □

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Received 25/NOV/2004

Translated by OLEG KLESOV