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INTERPOLATION OF MULTIDIMENSIONAL STATIONARY SEQUENCES

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This paper is dedicated to our teacher Mykhaĭlo Ĭosypovych Yadrenko.

Abstract. The problem of optimal estimation is considered for the linear functional $A_N\vec{\xi} = \sum_{j=0}^{N} \vec{a}(j)\vec{\xi}(j)$, where $\{\vec{\xi}(j)\}$ and $\{\vec{\eta}(j)\}$ are multidimensional stationary stochastic sequences. The estimation is based on observations of the sequence $\vec{\xi}(j)+\vec{\eta}(j)$ for $j \in Z\setminus\{0,\dots,N\}$. We obtain formulas for calculating the mean-square error and spectral characteristic of the optimal estimate of the functional. The least favorable spectral densities and minimax spectral characteristics of the optimal estimates of the functional are found for some classes of spectral densities.

1. Introduction

The classical methods of solution of linear extrapolation, interpolation and filtering problems for stationary stochastic processes are based on the assumption that the spectral densities of processes are known exactly (see, for example, selected works of Kolmogorov [6], the survey article of Kailath [4], and the books of Wiener [14], Rozanov [12], Yaglom [15, 16], and Hannan [3]). In practice, however, complete information on the spectral densities is not available in most cases. To minimize this complication one finds parametric or nonparametric estimates of unknown spectral densities or selects these densities by using other reasoning. Then one can apply the traditional estimation method assuming that the estimated or selected densities are the true ones. This procedure may result in a significant increase in errors, as Vastola and Poor [13] have demonstrated with concrete examples. This is a reason to find estimates which are optimal for all densities belonging to a certain class of admissible spectral densities. These estimates are called minimax, since they minimize the maximal value of the error. A survey of results of minimax (robust) methods of estimation can be found in the paper by Kassam and Poor [5]. Grenander [2] is the first to use the minimax approach to the extrapolation problem for stationary processes. Franke [1] investigated the extrapolation problem for stationary sequences with the help of methods of convex optimization. Pourahmadi [11] studied the extrapolation problem for vector-valued stationary sequences. In the papers by Moklyachuk [7]–[10] the extrapolation, interpolation, and filtering problems are studied for stationary processes and sequences.

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In this paper, the optimal linear estimation problem is considered for the functional

$$A_N\vec{\xi}=\sum_{j=0}^{N}\vec{a}(j)\vec{\xi}(j),$$

where the sequence $\vec{\xi}(j)+\vec{\eta}(j)$ is observed for $j\in Z\setminus\{0,\dots,N\}$, $\vec{\xi}(j)=\{\xi_k(j)\}_{k=1}^{T}$ are unknown values of a multidimensional stationary stochastic sequence, and $\vec{\eta}(j)=\{\eta_k(j)\}_{k=1}^{T}$ is an uncorrelated with $\vec{\xi}(j)$ multidimensional stationary stochastic sequence.

2. The classical projection method of linear interpolation

Let $\vec{\xi}(j)=\{\xi_k(j)\}_{k=1}^{T}$ and $\vec{\eta}(j)=\{\eta_k(j)\}_{k=1}^{T}$ be uncorrelated multidimensional stationary stochastic sequences with the spectral density matrices $F(\lambda)=\{f_{kl}(\lambda)\}_{k,l=1}^{T}$ and $G(\lambda)=\{g_{kl}(\lambda)\}_{k,l=1}^{T}$, respectively. Assume that the matrices $F(\lambda)$ and $G(\lambda)$ satisfy the minimality condition:

$$\int_{-\pi}^{\pi}\operatorname{Tr}\left[(F(\lambda)+G(\lambda))^{-1}\right]\,d\lambda<\infty. \tag{1}$$

This condition is necessary and sufficient in order that the error-free interpolation of unknown values of the sequence $\vec{\xi}(j)+\vec{\eta}(j)$ is impossible [12]. We also denote by $L_2\left(F\right)$ the Hilbert space of vector complex-valued functions $\varphi(\lambda)=\{\varphi_k(\lambda)\}_{k=1}^{T}$ that are second order integrable with respect to the measure whose density is $F(\lambda)=\{f_{kl}(\lambda)\}_{k,l=1}^{T}$:

$$\int_{-\pi}^{\pi}\varphi(\lambda)F(\lambda)\varphi^{*}(\lambda)\,d\lambda=\int_{-\pi}^{\pi}\sum_{k,l=1}^{T}\varphi_k(\lambda)\overline{\varphi_l(\lambda)}f_{kl}(\lambda)\,d\lambda<\infty.$$

We also denote by $L_2^{N-}(F)$ the subspace in $L_2(F)$ generated by the functions $e^{in\lambda}\delta_k$, $k=1,\dots,T$, $n\in Z\setminus\{0,\dots,N\}$, where $\delta_k=\{\delta_{kl}\}_{l=1}^{T}$ is such that $\delta_{kk}=1$ and $\delta_{kl}=0$ for $k\neq l$. Every linear estimate $\widehat{A}_N\vec{\xi}$ of the functional $A_N\vec{\xi}$ constructed from observations of the sequence $\vec{\xi}(j)+\vec{\eta}(j)$, $j\in Z\setminus\{0,\dots,N\}$, is of the form

$$\widehat{A}_N\vec{\xi}=\int_{-\pi}^{\pi}h(e^{i\lambda})\big(Z^{\xi}(d\lambda)+Z^{\eta}(d\lambda)\big)=\int_{-\pi}^{\pi}\sum_{k=1}^{T}h_k\left(e^{i\lambda}\right)\left(Z_k^{\xi}(d\lambda)+Z_k^{\eta}(d\lambda)\right),$$

where $Z^{\xi}(\Delta)=\{Z_k^{\xi}(\Delta)\}_{k=1}^{T}$ and $Z^{\eta}(\Delta)=\{Z_k^{\eta}(\Delta)\}_{k=1}^{T}$ are orthogonal random measures of the sequences $\vec{\xi}(j)$ and $\vec{\eta}(j)$, respectively, and $h(e^{i\lambda})=\{h_k(e^{i\lambda})\}_{k=1}^{T}$ is the spectral characteristic of the estimate $\widehat{A}_N\vec{\xi}$,

$$h(e^{i\lambda})\in L_2^{N-}(F+G).$$

The mean-square error $\Delta(h;F,G)$ of a linear estimate $\widehat{A}_N\vec{\xi}$ is given by

$$\begin{aligned}\Delta(h;F,G)&=\mathsf{E}\left|A_N\vec{\xi}-\widehat{A}_N\vec{\xi}\right|^2\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A_N\left(e^{i\lambda}\right)-h(e^{i\lambda})\right)F(\lambda)\left(A_N\left(e^{i\lambda}\right)-h\left(e^{i\lambda}\right)\right)^{*}d\lambda\\&\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}h\left(e^{i\lambda}\right)G(\lambda)h^{*}\left(e^{i\lambda}\right)\,d\lambda,\end{aligned}$$

$$A_N(e^{i\lambda})=\sum_{j=0}^{N}\vec{a}(j)e^{ij\lambda}.$$

The spectral characteristic $h(F,G)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ minimizes the mean-square error,

$$\Delta(F,G)=\Delta(h(F,G);F,G)=\min_{h\in L_2^{N-}(F+G)}\Delta(h;F,G)=\min_{\widehat{A}_N\vec{\xi}}\mathsf{E}\left|A_N\vec{\xi}-\widehat{A}_N\vec{\xi}\right|^2. \tag{2}$$

The optimal linear estimate $\widehat{A}_N\vec{\xi}$ is a solution to the optimization problem (2). This estimate is determined by two conditions [6, 12]:

$$\begin{aligned}\widehat{A}_N\vec{\xi}&\in H\big[\xi_k(n)+\eta_k(n),k=1,\dots,T,n\in Z\setminus\{0,\dots,N\}\big],\\ A_N\vec{\xi}-\widehat{A}_N\vec{\xi}&\perp H\big[\xi_k(n)+\eta_k(n),k=1,\dots,T,n\in Z\setminus\{0,\dots,N\}\big],\end{aligned}\tag{3}$$

where $H[\xi_k(n)+\eta_k(n),k=1,\dots,T,n\in Z\setminus\{0,\dots,N\}]$ is the subspace generated by the random variables $[\xi_k(n)+\eta_k(n),k=1,\dots,T,n\in Z\setminus\{0,\dots,N\}]$ in the Hilbert space $H=L_2(\Omega)$ of random variables with finite second moments and zero mathematical expectations. These conditions allow one to find the spectral characteristic $h(F,G)$ and the mean-square error $\Delta(F,G)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ in the case where the spectral density matrices $F(\lambda)$ and $G(\lambda)$ are known and condition (1) holds. In this case,

$$\begin{aligned}h(F,G)&=\big(A_N(e^{i\lambda})F(\lambda)-C_N(e^{i\lambda})\big)\big(F(\lambda)+G(\lambda)\big)^{-1}\\ &=A_N(e^{i\lambda})-\big(A_N(e^{i\lambda})G(\lambda)+C_N(e^{i\lambda})\big)\big(F(\lambda)+G(\lambda)\big)^{-1},\end{aligned}\tag{4}$$

$$\begin{aligned}\Delta(F,G)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(A_N(e^{i\lambda})G(\lambda)+C_N(e^{i\lambda})\big)\big(F(\lambda)+G(\lambda)\big)^{-1}\\ &\qquad\times F(\lambda)\big(F(\lambda)+G(\lambda)\big)^{-1}\big(A_N(e^{i\lambda})G(\lambda)+C_N(e^{i\lambda})\big)^{*}\,d\lambda\\ &\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(A_N(e^{i\lambda})F(\lambda)-C_N(e^{i\lambda})\big)\big(F(\lambda)+G(\lambda)\big)^{-1}\\ &\qquad\times G(\lambda)\big(F(\lambda)+G(\lambda)\big)^{-1}\big(A_N(e^{i\lambda})G(\lambda)+C_N(e^{i\lambda})\big)^{*}\,d\lambda\\ &=\langle\vec{c}_N,B_N\vec{c}_N\rangle+\langle\vec{a}_N,R_N\vec{a}_N\rangle,\end{aligned}\tag{5}$$

where

$$C_N(e^{i\lambda})=\sum_{j=0}^{N}\vec{c}(j)e^{ij\lambda},\qquad \vec{c}_N=\{\vec{c}(k)\}_{k=0}^{N}=B_N^{-1}D_N\vec{a}_N,\qquad \vec{a}_N=\{\vec{a}(k)\}_{k=0}^{N},$$

$\langle a,b\rangle$ denotes the scalar product, and B_N, D_N, and R_N are block matrices formed from the following $T\times T$ matrices:

$$\begin{aligned}B_N(k,j)&=\frac{1}{(2\pi)^T}\int_{-\pi}^{\pi}\big[(F(\lambda)+G(\lambda))^{-1}\big]^{T}e^{i(j-k)\lambda}\,d\lambda,\\ D_N(k,j)&=\frac{1}{(2\pi)^T}\int_{-\pi}^{\pi}\big[F(\lambda)(F(\lambda)+G(\lambda))^{-1}\big]^{T}e^{i(j-k)\lambda}\,d\lambda,\\ R_N(k,j)&=\frac{1}{(2\pi)^T}\int_{-\pi}^{\pi}\big[F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)\big]^{T}e^{i(j-k)\lambda}\,d\lambda,\\ &\qquad k,j=0,1,\dots,N.\end{aligned}$$

Therefore the following result holds.

Theorem 2.1. *Let $\vec{\xi}(j)=\{\xi_k(j)\}_{k=1}^{T}$ and $\vec{\eta}(j)=\{\eta_k(j)\}_{k=1}^{T}$ be uncorrelated multidimensional stationary stochastic sequences with the spectral density matrices*

$$F(\lambda)=\{f_{kl}(\lambda)\}_{k,l=1}^{T}$$

and $G(\lambda)=\{g_{kl}(\lambda)\}_{k,l=1}^{T}$*, respectively. Assume that the matrices* $F(\lambda)$ *and* $G(\lambda)$ *satisfy the minimality condition* (1) *and that the functional* $A_N\vec{\xi}$ *depends on unknown values of the sequence* $\vec{\xi}(j)$*. Let* $\Delta(F,G)$ *and* $h(F,G)$ *be the mean-square error and spectral characteristic of the optimal linear estimate constructed from observations of the sequence* $\vec{\xi}(j)+\vec{\eta}(j)$ *for* $j\in Z\setminus\{0,\dots,N\}$*. Then* $\Delta(F,G)$ *and* $h(F,G)$ *are given by equalities* (4) *and* (5)*.*

Corollary 2.1. *Let* $\vec{\xi}(j)=\{\xi_k(j)\}_{k=1}^{T}$ *be a multidimensional stationary stochastic sequence with the spectral density matrix* $F(\lambda)=\{f_{kl}(\lambda)\}_{k,l=1}^{T}$*. Assume that the matrix* $F(\lambda)$ *satisfies the minimality condition:*

$$\int_{-\pi}^{\pi}\operatorname{Tr}\left[(F(\lambda))^{-1}\right]d\lambda<\infty.$$

Let the functional $A_N\vec{\xi}=\sum_{j=0}^{N}\vec{a}(j)\vec{\xi}(j)$ *depend on unknown values of the sequence* $\vec{\xi}(j)$ *and let the optimal linear estimate be constructed from observations of the sequence* $\vec{\xi}(j)$ *for* $j\in Z\setminus\{0,\dots,N\}$*. Then the mean-square error* $\Delta(F)$ *and spectral characteristic* $h(F)$ *of the optimal linear estimate are given by*

$$h(F)=A_N(e^{i\lambda})-C_N(e^{i\lambda})[F(\lambda)]^{-1}, \tag{6}$$

$$\Delta(F)=\frac{1}{2\pi}\int_{-\pi}^{\pi}C_N(e^{i\lambda})[F(\lambda)]^{-1}C_N^{*}(e^{i\lambda})d\lambda=\left\langle B_N^{-1}\vec{a}_N,\vec{a}_N\right\rangle, \tag{7}$$

where $C_N(e^{i\lambda})=\sum_{j=0}^{N}\vec{c}(j)e^{ij\lambda}$*,* $\vec{c}_N=B_N^{-1}\vec{a}_N$*, and* B_N *is the block matrix formed from the* $T\times T$ *matrices:*

$$B_N(k,j)=\frac{1}{(2\pi)^T}\int_{-\pi}^{\pi}\left[(F(\lambda))^{-1}\right]^{T}e^{i(j-k)\lambda}\,d\lambda,\qquad k,j=0,1,\dots,N.$$

3. Minimax-robust approach to the linear interpolation problem

In order to use equalities (4)–(7) for calculating the mean-square error and the spectral characteristic of the optimal linear estimate of the functional $A_N\vec{\xi}$ it is necessary to know the spectral density matrices $F(\lambda)$ and $G(\lambda)$. The minimax-robust approach to the estimation of unknown values of the functional is appropriate in the case where exact values of the spectral density matrices are unknown, but a set $D=D_F\times D_G$ of admissible spectral density matrices is given.

When following the minimax-robust approach, we are searching an estimate which minimizes the mean-square error for all spectral density matrices $F(\lambda)$ and $G(\lambda)$ belonging to the class $D=D_F\times D_G$ instead of searching an estimate that is optimal for given spectral density matrices.

Definition 3.1. For a given class of spectral density matrices $D=D_F\times D_G$, the spectral density matrices $F^0(\lambda)\in D_F$ and $G^0(\lambda)\in D_G$ are called least favorable for the optimal linear interpolation of the functional $A_N\vec{\xi}$ if

$$\Delta\left(F^0,G^0\right)=\Delta\left(h\left(F^0,G^0\right);F^0,G^0\right)=\max_{(F,G)\in D}\Delta\left(h\left(F,G\right);F,G\right).$$

Definition 3.2. For a given class of spectral density matrices $D=D_F\times D_G$, the spectral characteristic $h^0(\lambda)$ of the optimal linear interpolation of the functional $A_N\vec{\xi}$ is called minimax-robust if

$$h^0(\lambda)\in H_D=\bigcap_{(F,G)\in D}L_2^{N-}(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta(h;F,G)=\max_{(F,G)\in D}\Delta\big(h^0;F,G\big).$$

Lemma 3.1. *Spectral density matrices $F^0(\lambda)$ and $G^0(\lambda)$ are least favorable in the class $D = D_F \times D_G$ for the optimal linear interpolation of the functional $A_N\vec{\xi}$ if the Fourier coefficients of the matrix functions*

$$\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}, \qquad F^0(\lambda)\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}, \qquad F^0(\lambda)\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}G^0(\lambda)$$

form matrices B_N^0, D_N^0, and R_N^0 that determine solutions to the conditional extremum problem:

$$\begin{aligned}\max_{(F,G)\in D}&\left\langle B_N^{-1}D_N\vec{a}_N, D_N\vec{a}_N\right\rangle+\langle\vec{a}_N, R_N\vec{a}_N\rangle\\ &=\left\langle (B_N^0)^{-1}D_N^0\vec{a}_N, D_N^0\vec{a}_N\right\rangle+\left\langle\vec{a}_N, R_N^0\vec{a}_N\right\rangle.\end{aligned}\tag{8}$$

The minimax-robust spectral characteristic $h^0 = h(F^0, G^0)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ is given by (4) *if $h(F^0, G^0) \in H_D$.*

Lemma 3.2. *Let* $\operatorname{Tr}\left[(F^0(\lambda))^{-1}\right]$ *be an integrable function. Then the spectral density matrix $F^0(\lambda) \in D_F$ is least favorable in the class D_F for the optimal linear interpolation of the functional $A_N\vec{\xi}$ constructed from observations of the sequence $\vec{\xi}(j)$ for*

$$j \in Z \setminus \{0, \dots, N\}$$

provided the Fourier coefficients of the matrix function $[F^0(\lambda)]^{-1}$ form the matrix B_N^0 determining a solution to the conditional extremum problem:

$$\max_{F\in D_F}\left\langle (B_N)^{-1}\vec{a}_N, \vec{a}_N\right\rangle=\left\langle (B_N^0)^{-1}\vec{a}_N, \vec{a}_N\right\rangle.\tag{9}$$

The minimax-robust spectral characteristic $h^0 = h(F^0)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ is given by (6) *if $h(F^0) \in H_D$.*

The least favorable spectral density matrices $F^0(\lambda)$ and $G^0(\lambda)$ and minimax-robust spectral characteristic $h^0 = h(F^0, G^0)$ form a saddle point of the function $\Delta(h; F, G)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta\left(h^0; F, G\right) \le \Delta\left(h^0; F^0, G^0\right) \le \Delta\left(h; F^0, G^0\right) \quad \forall h \in H_D,\ \forall F \in D_F,\ \forall G \in D_G,$$

hold if $h^0 = h(F^0, G^0)$, $h(F^0, G^0) \in H_D$, and (F^0, G^0) give a solution to the conditional extremum problem

$$\sup_{(F,G)\in D}\Delta\left(h\left(F^0,G^0\right);F,G\right)=\Delta\left(h\left(F^0,G^0\right);F^0,G^0\right),\tag{10}$$

$$\begin{aligned}\Delta&\left(h\left(F^0,G^0\right);F,G\right)\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A_N(e^{i\lambda})G^0(\lambda)+C_N^0(e^{i\lambda})\right)\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}\\ &\qquad\times F(\lambda)\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}\left(A_N(e^{i\lambda})G^0(\lambda)+C_N^0(e^{i\lambda})\right)^{*}d\lambda\\ &\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A_N(e^{i\lambda})F^0(\lambda)-C_N^0(e^{i\lambda})\right)\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}\\ &\qquad\times G(\lambda)\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}\left(A_N(e^{i\lambda})G^0(\lambda)+C_N^0(e^{i\lambda})\right)^{*}d\lambda.\end{aligned}$$

Lemma 3.3. *Let (F^0, G^0) be a solution to the conditional extremum problem* (10)*. The spectral density matrices $F^0(\lambda)$ and $G^0(\lambda)$ are least favorable in the class $D = D_F \times D_G$ and the spectral characteristic $h^0 = h(F^0, G^0)$ is minimax-robust for the optimal linear interpolation of the functional $A_N\vec{\xi}$ if $h(F^0, G^0) \in H_D$.*

Lemma 3.4. *Let* $\mathrm{Tr}\left[(F^0(\lambda))^{-1}\right]$ *be an integrable function. Then the spectral density matrix* $F^0(\lambda) \in D_F$ *is least favorable in the class* D_F *for the optimal linear interpolation of the functional* $A_N\vec{\xi}$ *constructed from observations of the sequence* $\vec{\xi}(j)$ *for*

$$j \in Z \setminus \{0, \dots, N\}$$

if $F^0(\lambda)$ *is a solution to the conditional extremum problem*

$$\sup_{F \in D_F} \Delta\left(h\left(F^0\right); F\right) = \Delta\left(h\left(F^0\right); F^0\right), \tag{11}$$

$$\Delta\left(h(F^0); F\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C_N^0(e^{i\lambda})\left[F^0(\lambda)\right]^{-1}[F(\lambda)]\left[F^0(\lambda)\right]^{-1}\left(C_N^0(e^{i\lambda})\right)^* d\lambda.$$

The spectral characteristic $h^0 = h(F^0)$ *is minimax-robust for the optimal linear interpolation of the functional* $A_N\vec{\xi}$ *if* $h(F^0, G^0) \in H_D$.

4. Least favorable spectral densities in the class D_0^-

Consider the problem of the minimax estimation of the functional $A_N\vec{\xi}$ from observations $\vec{\xi}(j)$, $j \in Z \setminus \{0, \dots, N\}$, under the condition that the spectral density matrix $F(\lambda)$ of the multidimensional stationary sequence $\vec{\xi}(j)$ belongs to the class

$$D_0^- = \left\{ F(\lambda) \;\middle|\; \frac{1}{(2\pi)^T}\int_{-\pi}^{\pi}[F(\lambda)]^{-1}d\lambda = P \right\},$$

where P is a positive definite matrix with nonzero determinant. With the help of Lemma 3.4 and the Lagrange multipliers method we can prove that the Fourier coefficients of the matrix function $[F_0(\lambda)]^{-1}$ satisfy the following relation:

$$C_N(e^{i\lambda})[F_0(\lambda)]^{-1} = (\alpha_1, \dots, \alpha_T)\,[F_0(\lambda)]^{-1}, \tag{12}$$

where $\vec{\alpha}_0 = (\alpha_1, \dots, \alpha_T)^T$ are constants (indeterminate Lagrange multipliers),

$$C_N(e^{i\lambda}) = \sum_{j=0}^{N} \vec{c}(j)e^{ij\lambda},$$

the vector $\vec{c}_N = \{\vec{c}(k)\}_{k=0}^N$ satisfies the equation $B_N^0\vec{c}_N = \vec{a}_N$, the matrix B_N^0 is constructed from the Fourier coefficients of the matrix function $[F_0(\lambda)]^{-1}$:

$$B_N^0(k,j) = R^0(k-j) = \frac{1}{(2\pi)^T}\int_{-\pi}^{\pi}\left[(F_0(\lambda))^{-1}\right]^T e^{i(j-k)\lambda}\, d\lambda, \qquad k, j = 0, 1, \dots, N.$$

The Fourier coefficients $R(k) = R^*(-k)$, $k = 0, 1, \dots, N$, found from the equation

$$B\vec{\alpha}_N^0 = \vec{a}_N$$

for $\vec{\alpha}_N^0 = (\vec{\alpha}_0, 0, \dots, 0)$, or from the equation $B\vec{\alpha}_0^N = \vec{a}_N$ for $\vec{\alpha}_0^N = (0, \dots, 0, \vec{\alpha}_0)$ satisfy relations (12) and $B_N^0\vec{c}_N = \vec{a}_N$. We find that $R(k) = \vec{a}(k)\vec{a}(0)^{-1}P^T$, where

$$\vec{a}(0)^{-1} = \left(\overline{a}_0^1, \dots, \overline{a}_0^T\right) D^{-1}$$

and $D = |a_0^1|^2 + \dots + |a_0^T|^2$. The coefficients $R(k) = \vec{a}(N-k)\vec{a}(N)^{-1}P^T$ satisfy the second equation, where $\vec{a}(N)^{-1} = (\overline{a}_N^1, \dots, \overline{a}_N^T)D^{-1}$ and $D = |a_N^1|^2 + \dots + |a_N^T|^2$. The equality $R(0)^T = P$ follows from the condition for the extremum. Let the vector-valued sequence $\vec{a}(k)$, $k = 0, \dots, N$, be such that the matrix function $[F^0(\lambda)]^{-1} = \sum_{k=-N}^{N} R^T(k)e^{ik\lambda}$ is positive definite and has nonzero determinant. Then $[F^0(\lambda)]^{-1}$ can be represented in the following form [3]:

$$\left[F^0(\lambda)\right]^{-1} = \left(\sum_{k=0}^{N} A_k e^{-ik\lambda}\right)\left(\sum_{k=0}^{N} A_k e^{-ik\lambda}\right)^*.$$

Thus $[F^0(\lambda)]^{-1}$ is the spectral density of the multidimensional autoregressive stochastic sequence of order N generated by the equation

$$\sum_{k=0}^{N} A_k \vec{\xi}(n-k) = \vec{\varepsilon}(n), \tag{13}$$

where $\vec{\varepsilon}(n)$ is a multidimensional "white noise" sequence. The minimax spectral characteristic $h(F^0)$ of the optimal linear interpolation of the functional $A_N\vec{\xi}$ is given by

$$h(F^0) = -\sum_{k=1}^{N} R^*(k) \left[P^T\right]^{-1} \vec{a}(0) e^{-ik\lambda} \tag{14}$$

or, equivalently, by

$$h(F^0) = -\sum_{k=1}^{N} R(k) \left[P^T\right]^{-1} \vec{a}(N) e^{i(N+k)\lambda}. \tag{15}$$

Therefore the following result holds.

Theorem 4.1. *The least favorable in the class D_0^- spectral density for the optimal linear interpolation of the functional $A_N\vec{\xi}$ determined by a sequence $\vec{a}(k)$, $k = 0, \ldots, N$, such that the matrix function*

$$[F^0(\lambda)]^{-1} = \sum_{k=-N}^{N} R(k)^T e^{ik\lambda}$$

is positive definite and has nonzero determinant, is the spectral density of the multidimensional autoregressive stochastic sequence (13) *whose Fourier coefficients are*

$$R(k)^T = R^*(-k)^T,$$

where $R(k) = \vec{a}(k)\vec{a}(0)^{-1}P^T$ or $R(k) = \vec{a}(N-k)\vec{a}(N)^{-1}P^T$. The minimax spectral characteristic $h(F^0)$ is given by (14) *or, equivalently, by* (15).

5. Least favorable spectral densities in the class D_M^-

Consider the problem of the minimax estimation of the functional $A_N\vec{\xi}$ from observations $\vec{\xi}(j)$, $j \in Z \setminus \{0, \ldots, N\}$, under the condition that the spectral density matrix $F(\lambda)$ of the multidimensional stationary sequence $\vec{\xi}(j)$ belongs to the class of spectral density matrices

$$D_M^- = \left\{ F(\lambda) \;\middle|\; \frac{1}{(2\pi)^T} \int_{-\pi}^{\pi} [F(\lambda)]^{-1} e^{-im\lambda}\, d\lambda = P(m), m = 0, \ldots, M \right\},$$

where $P(m)$, $m = 0, \ldots, N$, are such that the matrix function $\sum_{m=-M}^{M} P(m)e^{im\lambda}$ is positive definite, its determinant is nonzero, and $P(-m) = P^*(m)$. With the help of the Lagrange multipliers method we find that

$$C_N\left(e^{i\lambda}\right) F_0(\lambda)^{-1} = \left(\sum_{k=0}^{M} \alpha_k^1 e^{ik\lambda}, \ldots, \sum_{k=0}^{M} \alpha_k^T e^{ik\lambda} \right) [F_0(\lambda)]^{-1},$$

whence we obtain

$$\sum_{k=0}^{N} \vec{c}_k e^{ik\lambda} = \sum_{k=0}^{M} \vec{\alpha}_k e^{ik\lambda}. \tag{16}$$

Let $M \geq N$. In this case the Fourier coefficients determine the matrix B_N^0 and the extremum problem is degenerated. Put $\vec{\alpha}_{N+1} = 0, \ldots, \vec{\alpha}_M = 0$ and find $\vec{\alpha}_0, \ldots, \vec{\alpha}_N$ from the equation $B_N^0\vec{\alpha}_N^0 = \vec{a}_N$, where $\vec{\alpha}_N^0 = (\vec{\alpha}_0,\dots,\vec{\alpha}_N)^T$. Hence every spectral density $F(\lambda)\in D_M^-$ is least favorable and, as a corollary, the spectral density

$$F^0(\lambda) = \left(\sum_{m=-M}^{M} P(m)e^{im\lambda}\right)^{-1} = \left(\left(\sum_{k=0}^{M} A_k e^{-ik\lambda}\right)\left(\sum_{k=0}^{M} A_k e^{-ik\lambda}\right)^*\right)^{-1} \tag{17}$$

of the multidimensional autoregressive stochastic sequence

$$\sum_{k=0}^{M} A_k\vec{\xi}(n-k) = \vec{\varepsilon}(n) \tag{18}$$

is least favorable, too.

Let $M < N$. In this case the matrix B_N is determined by the Fourier coefficients of the function $[F(\lambda)]^{-1}$. Among them, $P(m)$, $m = 0,\dots,M$, are known and $P(m)$, $m = M+1,\dots,N$, are unknown. The unknown numbers $\vec{\alpha}_k$, $k = 0,\dots,M$, and $P(m)$, $m = M+1,\dots,N$, can be found from the equation

$$B_N\vec{\alpha}_M^0 = \vec{a}_N, \qquad \vec{\alpha}_M^0 = \left(\vec{\alpha}_0,\dots,\vec{\alpha}_M,\vec{0},\dots,\vec{0}\right)^T.$$

If the sequence $P(m)$, $m = 0,\dots,N$, is such that the matrix function $\sum_{m=-N}^{N} P(m)e^{im\lambda}$ is positive definite and has nonzero determinant, $P(-m) = P^*(m)$, then the least favorable spectral density $F^0(\lambda)$ is determined by the Fourier coefficients $P(m)$, $m = 0,\dots,N$, of the function $[F^0(\lambda)]^{-1}$:

$$\left(F^0(\lambda)\right)^{-1} = \sum_{m=-N}^{N} P(m)e^{im\lambda} = \left(\sum_{k=0}^{N} A_k e^{-ik\lambda}\right)\left(\sum_{k=0}^{N} A_k e^{-ik\lambda}\right)^*. \tag{19}$$

Thus $F^0(\lambda)$ is the spectral density of the multidimensional autoregressive stochastic sequence of order N:

$$\sum_{k=0}^{N} A_k\vec{\xi}(n-k) = \vec{\varepsilon}(n). \tag{20}$$

The above reasoning proves the following result.

Theorem 5.1. *The spectral density* (17) *of the multidimensional autoregressive stochastic sequence* (18) *determined by the coefficients $P(m)$, $m = 0,\dots,M$, is least favorable in the class D_M^- for the optimal linear interpolation of the functional $A_N\vec{\xi}$ if $M \ge N$. Otherwise (that is, if $M < N$ and solutions $P(m)$, $m = M+1,\dots,N$, to the equation $B_N\vec{\alpha}_M^0 = \vec{a}_N$ together with the coefficients $P(m)$, $m = 0,\dots,M$, form a positive definite matrix function $\sum_{m=-N}^{N} P(m)e^{im\lambda}$), the density* (19) *of the multidimensional autoregressive stochastic sequence* (20) *is least favorable in the class D_M^-. The minimax spectral characteristic $h(F^0)$ is given by* (6).

In the case of observations with the noise $\vec{\eta}(j)$, the minimax approach to the interpolation problem allows one to find equations that determine least favorable spectral densities in concrete classes $D = D_F \times D_G$ of admissible spectral densities.

6. Concluding remarks

In Section 2, we propose formulas to calculate the mean-square error and spectral characteristic of the optimal linear estimate of the functional $A_N\vec{\xi} = \sum_{j=0}^{N}\vec{a}(j)\vec{\xi}(j)$. The functional depends on unknown values of a multidimensional stationary stochastic sequence $\vec{\xi}(j)$, and the estimate is constructed from observations of the sequence

$$\vec{\xi}(j) + \vec{\eta}(j)$$

for $Z \setminus \{0, \dots, N\}$. We consider the problem in the case of observations without noise and in the case of observations with an uncorrelated noise.

In the next three sections, we deal with the estimation problem under the condition that spectral density matrices are not known, but classes $D = D_F \times D_G$ of admissible spectral densities are given. For concrete classes D of spectral density matrices, the least favorable spectral densities and minimax-robust spectral characteristics of the optimal estimate of the functional are proposed.

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