Теорія Ймовір. та Матем. Статист.
Вип. 73, 2005

Theor. Probability and Math. Statist.
No. 73, 2006, Pages 135–149
S 0094-9000(07)00688-6
Article electronically published on January 17, 2007

TAUBERIAN THEOREMS FOR RANDOM FIELDS WITH AN *OR* SPECTRUM. I

UDC 519.21

A. YA. OLENKO

Abstract. We obtain Abelian and Tauberian theorems describing a relationship between the asymptotic behavior at the origin of the spectrum of a random field and that at infinity of the integral of the random field over a sphere or a ball. We consider the case of homogeneous isotropic fields with singular spectra at the origin. The asymptotic behavior is given in terms of OR functions.

1. Introduction

Abelian and Tauberian theorems (in what follows we simply say "Tauberian theorems") are not only of their own interest but also have many applications in asymptotic problems for stochastic processes and random fields (see, for example, [2, 3, 5, 6, 7, 8, 9, 10, 11, 15]). Most of the known Tauberian theorems provide the relations between the behavior of spectral characteristics at infinity and correlation characteristics at the origin.

We obtain Tauberian theorems describing the relation between the behavior of the spectrum at the origin and integrals over a sphere or a ball at infinity for random fields. The asymptotic behavior is given in terms of OR functions. Similar results are obtained in [7, 8, 9, 12] for regularly varying functions. We extend the class of functions appropriate for the proof of Tauberian theorems and propose simpler proofs as compared to the papers mentioned above. Moreover, we obtain a better result concerning the asymptotic behavior of spectral densities.

Tauberian theorems for the Laplace transform of OR functions are considered in [2, 13, 14]. For the general case of integral transforms with nonnegative kernels that decrease to zero, results of this type are obtained in [4]. In contrast to the earlier papers, nonmonotone kernels are considered in the current paper.

Necessary definitions and properties of OR functions are presented in Section 2. In Section 3, we introduce the functionals of random fields whose asymptotic behavior is studied in Section 4. We also recall some technical results concerning the Bessel functions in Section 4. The main results are stated and proved in Sections 5 and 6.

In what follows we use the symbols C and R with sub- and superscripts to denote constants whose precise values do not matter for our reasoning. Moreover, the same symbol may be used for different constants appearing in the same proof.

2000 *Mathematics Subject Classification.* Primary 60G60, 62E20, 40E05; Secondary 60F05, 26A12, 44A15.

Key words and phrases. Tauberian theorem, Abelian theorem, slowly varying functions, OR functions, random fields, homogeneous fields, isotropic fields, functionals of a random field, spectral function, correlation function, asymptotic behavior, strong dependence.

Supported by the NATO grant # SA(PST.CLG.976361)5437.

2. OR functions and their properties

Below we provide basic definitions and results for OR functions needed later in the proof. A comprehensive study of slowly varying functions and their various applications can be found in [2].

An important role in the proofs of Tauberian theorems and in asymptotic problems is played by the function class R_ρ defined as follows.

Definition 1. A positive measurable function f varies regularly at infinity if there exists a number $\rho \in \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho$$

for all $\lambda > 0$.

We consider a wider class of O-regularly varying functions.

Let

$$f^*(\lambda) := \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)}, \qquad f_*(\lambda) := \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)}, \qquad \lambda > 0.$$

Definition 2. A positive measurable function f is O-regularly varying (belongs to the class OR) if

$$0 < f_*(\lambda) \le f^*(\lambda) < +\infty \quad \text{for all } \lambda \ge 1.$$

The following Feller theorem is useful when checking whether a function belongs to the class OR.

Theorem 1. *Let f be positive and nonincreasing. If $f_*(\lambda_0)$ is nonzero for some $\lambda_0 > 1$, then $f \in OR$.*

Definition 3. Let f be a positive measurable function. The infimum of those α for which there are constants $C = C(\alpha)$ such that

$$\frac{f(\lambda x)}{f(x)} \le C\{1 + o(1)\}\lambda^\alpha \quad \text{uniformly in } \lambda \in [1, \Lambda]$$

for all $\Lambda > 1$ as $x \to \infty$ is called the upper Matuszewska index and is denoted by $\alpha(f)$.

The supremum of those β for which there are constants $D = D(\beta) > 0$ such that

$$\frac{f(\lambda x)}{f(x)} \ge D\{1 + o(1)\}\lambda^\beta \quad \text{uniformly in } \lambda \in [1, \Lambda]$$

for all $\Lambda > 1$ as $x \to \infty$ is called the lower Matuszewska index and is denoted by $\beta(f)$.

Theorem 2. *$f \in OR$ if and only if both of its Matuszewska indices $\alpha(f)$ and $\beta(f)$ are finite. In this case,*

a) $f_*(\lambda) \le \lambda^{\beta(f)} \le \lambda^{\alpha(f)} \le f^*(\lambda)$ *for all* $\lambda \ge 1$,

b) *for any $\alpha > \alpha(f)$, there are positive constants C and X such that*

$$\frac{f(y)}{f(x)} \le C\left(\frac{y}{x}\right)^\alpha, \qquad y \ge x \ge X,$$

c) *for any $\beta < \beta(f)$, there are positive constants C' and X' such that*

$$\frac{f(y)}{f(x)} \ge C'\left(\frac{y}{x}\right)^\beta, \qquad y \ge x \ge X'.$$

Definition 4. By $OR(\beta, \alpha)$ we denote the subclass of OR functions whose Matuszewska indices satisfy $\alpha(f) \le \alpha$ and $\beta \le \beta(f)$.

The following result provides a useful representation of OR functions.

Theorem 3 (Karamata, Aljančić and Arandelović). *$f \in OR$ if and only if*

$$f(x) = \exp\left\{\eta(x) + \int_1^x \frac{\zeta(t)}{t}\,dt\right\}, \qquad x \ge 1, \tag{1}$$

where the functions $\eta(\cdot)$ and $\zeta(\cdot)$ are measurable and finite on some interval $[X, \infty)$.

Given arbitrary α and β such that $\beta < \beta(f) \le \alpha(f) < \alpha$, there is a representation of the form (1) *where $\zeta(\cdot)$ is integrable and assumes values in the interval $[\beta; \alpha]$.*

Remark 1. Suppose a function $f(\cdot)$ assumes finite values and does not vanish for $x \in [1, X]$. It is shown in the proof of Theorem 3 of [2, §2.2.3] that the function $\eta(\cdot)$ is bounded on the interval $[1, X]$, since $\zeta(\cdot)$ is bounded in $[1, \infty)$. Therefore we assume in what follows that both functions $\eta(\cdot)$ and $\zeta(\cdot)$ are bounded in the interval $[1, \infty)$.

3. Functionals of random fields

Let $\mathbb{R}^n$ be a Euclidean space of dimension $n \ge 2$, $s(r) = \{t \in \mathbb{R}^n : \|t\| = r\}$ be a sphere, $v(r) = \{t \in \mathbb{R}^n : \|t\| \le r\}$ be a ball in $\mathbb{R}^n$, $|s(1)| = 2\pi^{n/2}/\Gamma(n/2)$ be the area of the surface of the sphere $s(1)$ in $\mathbb{R}^{n-1}$. Furthermore, let $\xi(t)$, $t \in \mathbb{R}^n$, be a real-valued measurable mean-square continuous homogeneous random field, isotropic in the wide sense (see [6, 16]), with zero mean and the correlation function

$$\mathsf{B}_n(r) = \mathsf{B}_n(\|t\|) = \mathsf{E}\,\xi(0)\xi(t), \qquad t \in \mathbb{R}^n.$$

It is known (see, for example, [6, 16]) that there is a bounded nondecreasing function $\Phi(x)$, $x \ge 0$, for which

$$\mathsf{B}_n(r) = 2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right)\int_0^\infty \frac{J_{(n-2)/2}(rx)}{(rx)^{(n-2)/2}}\,d\Phi(x), \tag{2}$$

where $J_\nu(z)$ is the Bessel function of the first kind and of order $\nu > -\frac{1}{2}$. The function $\Phi(x)$ is called the spectral function of the field $\xi(t)$, $t \in \mathbb{R}^n$. If there is a function $f(x)$, $x \in [0, +\infty)$, such that

$$x^{n-1} f(x) \in L([0,\infty)), \qquad \Phi(x) = |s(1)| \int_0^x z^{n-1} f(z)\,dz,$$

then f is called the isotropic spectral density of the field $\xi(t)$.

It is shown in [6, 16] that

$$l_n(r) = \operatorname{Var}\left[\int_{s(r)} \xi(t)\,dm(t)\right] = \int_{s(r)}\int_{s(r)} \mathsf{B}_n(\|t-s\|)\,dt\,ds$$

$$= \frac{2^n \pi^{n-1}}{(n-2)!} r^{n-1} \int_0^{2r} z^{n-2}\left(1 - \left(\frac{z}{2r}\right)^2\right)^{(n-3)/2} \mathsf{B}_n(z)\,dz \tag{3}$$

$$= (2\pi)^n r^{2(n-1)} \int_0^\infty \frac{J^2_{(n-2)/2}(rx)}{(rx)^{n-2}}\,d\Phi(x), \qquad n \ge 2, \tag{4}$$

where $m(\cdot)$ is the Lebesgue measure on the sphere $s(r)$;

$$b_n(r) = \operatorname{Var}\left[\int_{v(r)} \xi(t)\,dt\right] = \int_{v(r)}\int_{v(r)} \mathsf{B}_n(\|t-s\|)\,dt\,ds$$

$$= \frac{4\pi^n}{n\Gamma^2(n/2)} r^n \int_0^{2r} z^{n-1} I_{1-(z/(2r))^2}\left(\frac{n+1}{2}, \frac{1}{2}\right) \mathsf{B}_n(z)\,dz \tag{5}$$

$$= (2\pi)^n r^{2n} \int_0^\infty \frac{J^2_{n/2}(rx)}{(rx)^n}\,d\Phi(x). \tag{6}$$

Here

$$I_\mu(p,q) = \frac{1}{B(p,q)} \int_0^\mu t^{p-1}(1-t)^{q-1}\,dt, \qquad p>0,\ q>0,\ \mu\in[0,1],$$

is the incomplete beta function.

Note that the right hand side of (2) is a Hankel type integral transform. If the correlation function $\mathsf{B}_n(r)$ is nonintegrable, then the field has a long range dependence structure. Long range dependent fields can alternatively be characterized by a singularity of their spectra at zero (say, by the unboundedness of the spectral density at zero or by the convergence to zero of the spectral density). We are going to establish a relationship between the behavior of the function $\Phi(x)$ as $x\to+0$ and the functions $l_n(r)$ and $b_n(r)$ as $r\to+\infty$ for random fields with long range dependence.

It is convenient to introduce the following notation:

$$\tilde{l}_n(r) := \frac{l_n(r)}{r^{2(n-1)}}, \qquad \tilde{b}_n(r) := \frac{b_n(r)}{r^{2n}};$$

$$G_n(x) := (2\pi)^n \frac{J^2_{(n-2)/2}(x)}{x^{n-2}}. \tag{7}$$

We omit the subscript n if this does not cause any misunderstanding.

4. Properties of the function $G_n(\cdot)$

In what follows we need some asymptotic properties of the function $G(x)$ and its derivative. All of the properties needed later in the discussion are listed in the following result; their proof is based on some properties of the Bessel functions (see [1]).

Lemma 1. *We have*

1) $G(\cdot)$ *is continuous in the interval* $(0,1]$*;*
2) *the limit* $\lim_{x\to0} G(x)\neq 0$ *exists and is finite;*
3) $G_n(\cdot)$ *does not vanish in the interval* $(0, j_{(n-2)/2,1})$ *where* $j_{\nu,1}$ *is the minimal positive root of the Bessel function of order* $\nu>-1$*. Moreover,* $j_{(n-2)/2,1}>1$*, since* $j_{-\frac12,1}=\frac{\pi}{2}$*,* $j_{0,1}\approx 2.4048$*, and* $j_{\nu,1}<j_{\nu+1,1}$*.*

Proof. Statements 1) and 3) follow from properties of the Bessel functions $J_\nu(x)$. Statement 2) follows from (7) in view of the following expansion:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k}, \tag{8}$$

where $\Gamma(\cdot)$ is the gamma function. □

Lemma 2. *Let* $g(\cdot) := -G_n'(\cdot)$*. Then*

$$g(u) := (2\pi)^n \frac{2J_{(n-2)/2}(u)J_{n/2}(u)}{u^{n-2}}. \tag{9}$$

The asymptotic behavior of the function $g(\cdot)$ *is such that*

1) $g(z)=O(z)$ *as* $z\to0$*;*
2) $g(z)=O(z^{1-n})$ *as* $z\to\infty$*.*

Proof. Since $J_\nu'(z)=\frac12\left(J_{\nu-1}(z)-J_{\nu+1}(z)\right)$ and $\frac{2\nu}{z}J_\nu(z)=J_{\nu-1}(z)+J_{\nu+1}(z)$, we obtain

$$G'(u) = -(2\pi)^n\left[\frac{(n-2)J^2_{(n-2)/2}(u)}{u^{n-1}} - \frac{J_{(n-2)/2}(u)}{u^{n-2}}\left(J_{(n-2)/2-1}(u)-J_{(n-2)/2+1}(u)\right)\right]$$

$$= -(2\pi)^n \frac{2J_{(n-2)/2}(u)J_{n/2}(u)}{u^{n-2}}.$$

Now statement 1) follows from (8). Furthermore,

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right), \qquad z \to \infty, \tag{10}$$

and thus

$$g(z) \sim \frac{2^{n+2}\pi^{n-1}}{z^{n-1}} \cos\left(z - \frac{\pi n}{4} - \frac{\pi}{4}\right) \cos\left(z - \frac{\pi n}{4} - \frac{\pi}{4} + \frac{\pi}{2}\right) \sim \frac{2^{n+1}\pi^{n-1} \sin\left(\frac{\pi n}{2} - 2z\right)}{z^{n-1}}.$$

This proves statement 2). □

5. Tauberian theorem for spectral functions

This section is devoted to Tauberian theorems describing a relationship between the asymptotic behavior of functionals of random fields at infinity and that of their spectra at zero.

We write $f \asymp g$ if $f = O(g)$ and $g = O(f)$.

Theorem 4. *Let $0 > \alpha \geq \beta > 2 - n$. Then the following relations are equivalent:*

(i) $\Phi(1/\,\cdot) \in OR(\beta, \alpha)$,

(ii) $\tilde{l}(\cdot) \in OR(\beta, \alpha)$,

(iii) $\tilde{l}(r) \asymp \Phi(1/r)$ *as $r \to \infty$ and there are constants C, C', and r_0 such that*

$$C' \left(\frac{r_1}{r}\right)^{\beta} \leq \frac{\Phi(1/r_1)}{\tilde{l}(r)} \leq C \left(\frac{r_1}{r}\right)^{\alpha}, \qquad r_1 \geq r \geq r_0.$$

Proof. (i) ⇒ (iii). It follows from representation (4) for $\tilde{l}(\cdot)$ that

$$\tilde{l}(r) = \int_0^\infty G(rx)\, d\Phi(x) \geq \int_0^{1/\lambda r} G(rx)\, d\Phi(x) \geq \Phi\left(\frac{1}{\lambda r}\right) \cdot \min_{u \in [0, 1/\lambda]} G(u). \tag{11}$$

Let $\lambda = 1$. Lemma 1 implies that

$$\frac{\tilde{l}(r)}{\Phi(1/r)} \geq \min_{u \in [0,1]} G(u) > 0,$$

that is, $\Phi(1/r) = O(\tilde{l}(r))$ as $r \to \infty$.

Now we show that $\tilde{l}(r) = O(\Phi(1/r))$ as $r \to \infty$. Integrating the right hand side of (4) by parts we get

$$\tilde{l}(r) = \int_0^\infty G(u)\, \Phi\left(\frac{du}{r}\right) = G(u)\Phi\left(\frac{u}{r}\right)\Big|_0^\infty - \int_0^\infty G'(u)\Phi\left(\frac{u}{r}\right) du = -\int_0^\infty G'(u)\Phi\left(\frac{u}{r}\right) du,$$

since (10) holds as $u \to \infty$, and

$$\lim_{u \to 0} \Phi(u) = 0$$

according to statement b) of Theorem 2 for $\alpha < 0$.

Using Lemma 2 we obtain

$$\frac{\tilde{l}(r)}{\Phi(1/r)} = \underbrace{\int_0^1 g(u) \frac{\Phi(u/r)}{\Phi(1/r)}\, du}_{I_1} + \underbrace{\int_1^r g(u) \frac{\Phi(u/r)}{\Phi(1/r)}\, du}_{I_2} + \underbrace{\int_r^\infty g(u) \frac{\Phi(u/r)}{\Phi(1/r)}\, du}_{I_3}.$$

Below we estimate every integral separately.

Since $\Phi(\cdot)$ is a nondecreasing function, we have

$$|I_1| = \left| \int_0^1 g(u) \frac{\Phi(u/r)}{\Phi(1/r)}\, du \right| \leq \int_0^1 |g(u)|\, du. \tag{12}$$

Next, $\int_0^1 |g(u)|\, du < \infty$ by statement 1) of Lemma 2.

Theorem 3 on the integral representation of OR functions implies that

$$|I_2| \le \int_1^r |g(u)| \frac{\Phi(u/r)}{\Phi(1/r)}\, du = \int_1^r |g(u)| e^{\eta(r/u)-\eta(r)} \exp\left\{ -\int_{r/u}^{r} \frac{\zeta(t)}{t}\, dt \right\} du$$

for $r \ge 1$. It follows from Theorem 3 and from the remark after it that

$$|I_2| \le C \int_1^r |g(u)| \exp\left\{ \max_{t\ge 1} |\zeta(t)| \ln(u) \right\} du \le C \int_1^r |g(u)| u^{-\beta'}\, du \tag{13}$$

for all $\beta' < \beta$.

Statement 2) of Lemma 2 implies for $\beta > \beta' > 2-n$ that the integral in (13) does not exceed $\int_1^\infty |g(u)| u^{-\beta'}\, du < \infty$ whatever $r \ge 1$ is.

Since the spectral function is monotone and bounded,

$$|I_3| \le \int_r^\infty |g(u)| \frac{\Phi(u/r)}{\Phi(1/r)}\, du \le \frac{\Phi(+\infty) \int_r^\infty |g(u)|\, du}{\Phi(1/r)} = \frac{\mathsf{B}_n(0) \int_0^{1/r} |g(1/z)|\, dz/z^2}{\Phi(1/r)}.$$

Applying statement c) of Theorem 2 to $\Phi(1/\cdot)$ we see that, for any $\beta' < \beta$, there exists $R > 0$ such that

$$\frac{\Phi(1/r)}{\Phi(1/R)} \ge C' \left(\frac{r}{R}\right)^{\beta'}$$

for all $r \ge R$. Thus the integral I_3 is estimated as follows:

$$|I_3| \le \frac{\mathsf{B}_n(0) R^{\beta'}}{C' \Phi(1/R)} \frac{\int_0^{1/r} |g(1/z)|\, dz/z^2}{r^{\beta'}}, \qquad r \ge R. \tag{14}$$

Statement 2) of Lemma 2 yields that

$$\frac{\int_0^x |g(1/z)|\, dz/z^2}{x^{-\beta'}} \sim -\frac{|g(1/x)|/x^2}{\beta' x^{-\beta'-1}} = O\left(x^{n-2+\beta'}\right) \tag{15}$$

as $x \to 0$.

The latter expression is bounded in a neighborhood of zero if $\beta > 2-n$. By inequality (14), $|I_3|$ is bounded for $r \ge R$ if $\beta > 2-n$.

If r is sufficiently large, the sum $|I_1| + |I_2| + |I_3|$ is bounded and thus $\tilde{l}(r) = O(\Phi(1/r))$ as $r \to \infty$, whence $\tilde{l}(r) \asymp \Phi(1/r)$ as $r \to \infty$.

Therefore there are constants C_1, C_2, and R such that

$$0 < C_2 \le \frac{\Phi(1/r)}{\tilde{l}(r)} \le C_1 < +\infty \quad \text{for all } r \ge R. \tag{16}$$

Rewriting $\Phi(1/r_1)/\tilde{l}(r)$ as

$$\frac{\Phi(1/r_1)}{\Phi(1/r)} \cdot \frac{\Phi(1/r)}{\tilde{l}(r)}$$

and using bound (16) and Theorem 2, we obtain

$$C_2 C' \left(\frac{r_1}{r}\right)^{\beta} \le \frac{\Phi(1/r_1)}{\tilde{l}(r)} \le C_1 C \left(\frac{r_1}{r}\right)^{\alpha}$$

for sufficiently large r_1 and r, $r_1 \ge r$. This completes the proof of the implication (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). We use bound (11) and obtain that

$$\frac{\Phi(1/r)}{\Phi(1/(\lambda r))} \ge \min_{u\in[0,1/\lambda]} G(u) \cdot \frac{\Phi(1/r)}{\tilde{l}(r)}.$$

Let

$$\frac{1}{j_{(n-2)/2,1}} < \lambda < 1.$$

Statement 3) of Lemma 1 yields $\min_{u\in[0,1/\lambda]} G(u) > 0$. Thus it follows from (iii) that

$$\liminf_{r\to\infty} \frac{\Phi(1/r)}{\Phi(1/(\lambda r))} > 0$$

for this λ. Now Theorem 1 implies that $\Phi(1/\,\cdot) \in OR$.

Rewriting $\Phi(1/r_1)/\Phi(1/r)$ as

$$\frac{\Phi(1/r_1)}{\Phi(1/r)} = \frac{\Phi(1/r_1)}{\tilde{l}(r)} \cdot \frac{\tilde{l}(r)}{\Phi(1/r)}$$

and using (iii), we obtain

$$\frac{C'}{C_1}\left(\frac{r_1}{r}\right)^{\beta} \le \frac{\Phi(1/r_1)}{\Phi(1/r)} \le \frac{C}{C_2}\left(\frac{r_1}{r}\right)^{\alpha}$$

for sufficiently large r_1 and r, $r_1 \ge r$, where C_1 and C_2 are defined in (16). According to Definition 3, the Matuszewska indices of the function $\Phi(1/r)$ do not exceed β and α, respectively. This completes the proof of the implication (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (ii). Since (i) and (iii) are equivalent, the inclusion

$$\Phi(1/\,\cdot) \in OR(\beta,\alpha)$$

and relation $\tilde{l}(r) \asymp \Phi(1/r)$ imply (ii).

(ii) $\Rightarrow$ (i). We represent $\tilde{l}(r)$ as the sum of the following three terms:

$$\tilde{l}(r) = \underbrace{\int_0^c g(u)\Phi(u/r)\,du}_{I_1'} + \underbrace{\int_c^{r/K} g(u)\Phi(u/r)\,du}_{I_2'} + \underbrace{\int_{r/K}^{\infty} g(u)\Phi(u/r)\,du}_{I_3'},$$

where $c, K \ge 1$. Now we estimate every integral separately.

Integrating by parts, we get

$$|I_1'| = \left| G(u)\Phi(u/r)\big|_0^c - \int_0^c G(u)\,\Phi(du/r) \right| \le \Phi(c/r)\left(G(c) + \max_{[0,c]} G(u)\right).$$

Relation (11) implies that

$$|I_2'| \le \int_{K/r}^{1/c} |g(1/z)|\Phi\left(\frac{1}{rz}\right)\frac{dz}{z^2} \le \frac{1}{\min_{[0,1]} G(u)} \int_{K/r}^{1/c} \tilde{l}(zr)|g(1/z)|\frac{dz}{z^2}.$$

Applying Theorem 3 to $\tilde{l}(zr)/\tilde{l}(r)$ we obtain

$$|I_2'| \le C\tilde{l}(r) \int_{K/r}^{1/c} e^{\eta(zr)-\eta(r)} \exp\left\{-\int_{zr}^{r} \frac{\zeta(t)}{t}\,dt\right\} |g(1/z)|\frac{dz}{z^2}.$$

Choosing K such that η is bounded on $[K,+\infty)$ we get for an arbitrary $\beta' < \beta$ that

$$\begin{aligned} |I_2'| &\le C\tilde{l}(r) \int_{K/r}^{1/c} |g(1/z)| \exp\left\{-\ln(z)\max_{t\ge1}|\zeta(t)|\right\}\frac{dz}{z^2} \\ &\le C\tilde{l}(r)\int_c^{r/K} |g(u)|u^{-\beta'}\,du \end{aligned} \tag{17}$$

by Theorem 3. Statement 2) of Lemma 2 implies that the integral in (17) is bounded from above by $\int_c^{+\infty} |g(u)|u^{-\beta'}\,du < \infty$ for any $r \ge cK$ if $\beta > \beta' > 2-n$.

It follows from (11) that

$$|I_3'| \le \int_0^{K/r} |g(1/z)|\Phi\left(\frac{1}{rz}\right)\frac{dz}{z^2} \le \frac{1}{\min_{[0,1]} G(u)}\tilde{l}(r)\int_0^{K/r} \frac{\tilde{l}(zr)}{\tilde{l}(r)}|g(1/z)|\frac{dz}{z^2}.$$

Representation (3) yields

$$\tilde{l}(r) = C\int_0^1 z^{n-2}\left(1-z^2\right)^{(n-3)/2} \mathsf{B}_n(zr)\,dz \le C < +\infty.$$

Thus

$$|I_3'| \le C\tilde{l}(r)\frac{\int_0^{K/r} |g(1/z)|\frac{dz}{z^2}}{\tilde{l}(r)}.$$

Applying statement c) of Theorem 2 to $\tilde{l}(\cdot)$ we prove for any $\beta' < \beta$ that there exists $R > 0$ such that

$$\frac{\tilde{l}(r)}{\tilde{l}(R)} \ge C'\left(\frac{r}{R}\right)^{\beta'}$$

for all $r \ge R$. This allows one to estimate the integral I_3' as follows:

$$|I_3'| \le C\tilde{l}(r)\frac{\int_0^{K/r} \left|g\left(1/z\right)\right|\,dz/z^2}{(r/K)^{\beta'}}. \tag{18}$$

Thus relation (15) implies that $|I_3'|$ in (18) is bounded for $r \ge R$ if $\beta > \beta' > 2-n$.

Therefore the sum $|I_1'| + |I_2'| + |I_3'|$ is bounded and

$$\tilde{l}(r) \le \Phi(c/r)\left(G(c) + \max_{[0,c]} G(u)\right) + C_1\tilde{l}(r)\int_c^{r/K} |g(u)|u^{-\beta'}\,du$$
$$+ C_2\tilde{l}(r)\frac{\int_0^{K/r} \left|g\left(1/z\right)\right|\,dz/z^2}{r^{\beta'}}$$

for sufficiently large r. Note that the constants C_1 and C_2 do not depend on r and c. Thus there exists a sufficiently large number c such that

$$C_1\int_c^{r/K} |g(u)|u^{-\beta'}\,du + C_2\frac{\int_0^{K/r} \left|g\left(1/z\right)\right|\,dz/z^2}{r^{\beta'}} < \frac{1}{2} \quad \text{for all } r \ge cK,$$

whence

$$\tilde{l}(r) \le 2\Phi(c/r)\left(G(c) + \max_{[0,c]} G(u)\right). \tag{19}$$

Using bounds (11) and (19) we get

$$\frac{\Phi(1/(\lambda r))}{\Phi(1/r)} = \frac{\Phi(c/(\lambda rc))}{\Phi(1/r)} \ge \frac{\min_{u\in[0,1]} G(u)}{2\left(G(c) + \max_{[0,c]} G(u)\right)}\cdot\frac{\tilde{l}(\lambda rc)}{\tilde{l}(r)}, \qquad \lambda r > K.$$

Assumption (ii) and Definition 3 imply that

$$\frac{\Phi(1/(\lambda r))}{\Phi(1/r)} \ge C(1+o(1))\lambda^{\beta} \quad \text{uniformly in } \lambda \in [1,\Lambda]$$

for an arbitrary $\Lambda > 1$ as $r \to \infty$.

Using bounds (11) and (19) we obtain

$$\frac{\Phi(1/(\lambda r))}{\Phi(1/r)} \le \frac{2\left(G(c) + \max_{[0,c]} G(u)\right)}{\min_{u\in[0,1]} G(u)}\cdot\frac{\tilde{l}(\lambda r)}{\tilde{l}(cr)}, \qquad r > K.$$

Consider the following two cases: a) $\lambda \in [c,\Lambda]$, and b) $\lambda \in [1,c]$.

a) Assumption (ii) and Definition 3 imply

$$\frac{\Phi(1/(\lambda r))}{\Phi(1/r)} \le C(1+o(1))\lambda^{\alpha} \quad \text{uniformly in } \lambda \in [c,\Lambda]$$

for all $\Lambda > 1$ as $r \to \infty$.

b) If $1 \le \lambda \le c$, then

$$\frac{\tilde{l}(\lambda r)}{\tilde{l}(cr)} = \frac{1}{\tilde{l}(cr)/\tilde{l}(\lambda r)} \le \frac{1}{D(1+o(1))(c/\lambda)^{\beta}} \le C(1+o(1))\lambda^{\alpha}$$

as $r \to \infty$, since $\beta < \alpha$.

Therefore

$$\frac{\Phi(1/(\lambda r))}{\Phi(1/r)} \le C(1+o(1))\lambda^{\alpha} \quad \text{uniformly in } \lambda \in [1, \Lambda].$$

Now Definition 3 implies that $\Phi(1/\,\cdot) \in OR(\beta, \alpha)$.

This completes the proof of the implication (ii) $\Rightarrow$ (i) as well as the proof of the theorem itself. □

Theorem 5. *Let $0 > \alpha \ge \beta > -n$. Then the following statements are equivalent:*

(i) $\Phi(1/\,\cdot) \in OR(\beta, \alpha)$,
(ii) $\tilde{b}(\cdot) \in OR(\beta, \alpha)$,
(iii) $\tilde{b}(r) \asymp \Phi(1/r)$ *as* $r \to \infty$ *and there exist positive numbers* C, C', *and* r_0 *such that*

$$C'\left(\frac{r_1}{r}\right)^{\beta} \le \frac{\Phi(1/r_1)}{\tilde{b}(r)} \le C\left(\frac{r_1}{r}\right)^{\alpha}, \qquad r_1 \ge r \ge r_0.$$

Proof. The proof follows from Theorem 4 by noting that

$$(2\pi)^2\tilde{b}_n(r) = \tilde{l}_{n+2}(r)$$

according to (4) and (6). □

Remark 2. Conditions $\tilde{l}(r) \asymp \Phi(1/r)$ and $\tilde{b}(r) \asymp \Phi(1/r)$ in the statement (iii) of Theorems 4 and 5 can be omitted, since they follow from bounds for $\Phi(1/r_1)/\tilde{l}(r)$ and $\Phi(1/r_1)/\tilde{b}(r)$, respectively, if $r_1 = r$.

Remark 3. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are called the Abelian and Tauberian theorems, respectively.

6. Tauberian theorem for spectral densities

Below we discuss some results, similar to those in the preceding section but expressed in terms of isotropic spectral densities instead of in terms of spectral functions.

It turns out that analogs of the Tauberian part in Theorems 4 and 5 do not hold if one does not use an extra assumption. We show this by an example. In Example 1, we construct a spectral function $\Phi(\cdot)$ whose density $f(\cdot)$ is such that $\Phi(1/\,\cdot) \in OR$ but $f(1/\,\cdot) \notin OR$.

Example 1. Consider the following spectral density:

$$f(x) = \frac{1}{|s(1)|x^{n-1}} \cdot \begin{cases} 2^{2k}x, & \text{for } x \in (1/4^k, 2/4^k],\ k \ge 1, \\ 2^{3k}x^2, & \text{for } x \in (2/4^k, 1/4^{k-1}],\ k \ge 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then the spectral function $\Phi(x)$ is

$$\begin{cases} \int_{1/4^k}^{x} 2^{2k} z\,dz + \sum_{m=k+1}^{\infty} \int_{1/4^m}^{2/4^m} 2^{2m} z\,dz \\ \qquad + \sum_{m=k+1}^{\infty} \int_{2/4^m}^{1/4^{m-1}} 2^{3m} z^2\,dz, & x \in \left(1/4^k, 2/4^k\right],\ k \ge 1, \\ \int_{2/4^k}^{x} 2^{3k} z^2\,dz + \sum_{m=k+1}^{\infty} \int_{2/4^m}^{1/4^{m-1}} 2^{3m} z^2\,dz \\ \qquad + \sum_{m=k}^{\infty} \int_{1/4^m}^{2/4^m} 2^{2m} z\,dz, & x \in \left(2/4^k, 1/4^{k-1}\right],\ k \ge 2. \end{cases}$$

Introducing a new variable $2^m z$ and joining the intervals of integration we get

$$\Phi(x) = \begin{cases} \int_0^{2^k x} z\,dz + \int_0^{2/2^k} z^2\,dz, & \text{for } x \in \left(1/4^k, 2/4^k\right],\ k \geq 1, \\ \int_0^{2^k x} z^2\,dz + \int_0^{2/2^k} z\,dz, & \text{for } x \in \left(2/4^k, 1/4^{k-1}\right],\ k \geq 2, \end{cases}$$
$$(20) \qquad = \begin{cases} 2^{2k-1}x^2 + \frac{8}{3\cdot 2^{3k}}, & \text{for } x \in \left(1/4^k, 2/4^k\right],\ k \geq 1, \\ \frac{2^{3k}x^3}{3} + \frac{1}{2^{2k-1}}, & \text{for } x \in \left(2/4^k, 1/4^{k-1}\right],\ k \geq 2. \end{cases}$$

Consider the case of $\lambda = 2$. We have

$$\frac{f(x/\lambda)}{\lambda^{n-1} f(x)} = \begin{cases} \frac{2^{3k+1}x^2}{2^{2k}x}, & x \in \left(1/4^k, 2/4^k\right],\ k \geq 1, \\ \frac{2^{2k-1}x}{2^{3k}x^2}, & x \in \left(2/4^k, 1/4^{k-1}\right],\ k \geq 2, \end{cases}$$
$$= \begin{cases} 2^{k+1}x, & x \in \left(1/4^k, 2/4^k\right],\ k \geq 1, \\ \frac{1}{2^{k+1}x}, & x \in \left(2/4^k, 1/4^{k-1}\right],\ k \geq 2. \end{cases}$$

Thus

$$\liminf_{x\to\infty} \frac{f(1/(\lambda x))}{f(1/x)} = 0 \quad \text{and} \quad \limsup_{x\to\infty} \frac{f(1/(\lambda x))}{f(1/x)} = +\infty,$$

which means that $f(1/\cdot) \notin OR$.

On the other hand, it follows from (20) that

$$\Phi(x) \in \left[\frac{x}{2}\left(1 + \frac{4}{3}\sqrt{2x}\right), x\left(1 + \frac{8}{3}\sqrt{x}\right)\right]$$

for $x \in (0, \frac{1}{2}]$. Thus $x/2 \leq \Phi(x) \leq 3x$ in the interval $(0, \frac{1}{2}]$ and

$$\liminf_{x\to\infty} \frac{\Phi(1/(\lambda x))}{\Phi(1/x)} \geq \frac{1}{6\lambda} > 0, \qquad \limsup_{x\to\infty} \frac{\Phi(1/(\lambda x))}{\Phi(1/x)} \leq \frac{6}{\lambda} < +\infty.$$

Therefore $\Phi(1/\cdot) \in OR$. Moreover, the lower and upper Matuszewska indices coincide and equal -1.

The above example shows that analogs of the Tauberian part in Theorems 4 and 5 for spectral densities require some additional assumptions. Below we present two results of such a kind. First we provide the Abelian theorem for isotropic spectral densities and spectral functions that hold without any additional assumption.

Theorem 6. *Let $\alpha < n$. If $f(1/\cdot) \in OR(\beta, \alpha)$, then*

$$\Phi(1/\cdot) \in OR(\beta - n, \alpha - n)$$

and $\Phi(1/r) \asymp f(1/r)/r^n$ as $r \to \infty$.

Proof. Using the definition of the spectral density we obtain

$$\frac{\Phi(1/r)}{f(1/r)} = \frac{|s(1)| \int_0^{1/r} x^{n-1} f(x)\,dx}{f(1/r)} = \frac{|s(1)|}{r^n} \int_1^\infty \frac{f(1/(zr))}{f(1/r)} \frac{dz}{z^{n+1}}.$$

Applying Theorem 2 to the function $f(1/\cdot)$, we prove that there exists a number $r_0 > 0$ such that

$$C' \int_1^\infty \frac{dz}{z^{n+1-\beta}} \leq \frac{r^n \Phi(1/r)}{f(1/r)} \leq C \int_1^\infty \frac{dz}{z^{n+1-\alpha}}, \qquad r \geq r_0.$$

Since $\alpha < n$, $\Phi(1/r) \asymp f(1/r)/r^n$ as $r \to \infty$, and the inclusion $\Phi(1/\cdot) \in OR(\beta - n, \alpha - n)$ follows. □

In the case of the Tauberian theorem, we consider spectral densities that do not vanish in a neighborhood of the origin, since the function $f(1/\cdot)$ does not belong to the class OR otherwise.

Definition 5. A nonnegative function $g(\cdot)$ is called essentially positive (in a neighborhood of the origin) if there exists a number $\varepsilon > 0$ such that $g(x) > 0$ for all $x \in (0, \varepsilon]$.

Theorem 7. *Let $f(1/\cdot) \in OR$ be an essentially positive isotropic spectral density and let $\alpha < 0$. If $\Phi(1/\cdot) \in OR(\beta, \alpha)$, then $f(1/\cdot) \in OR(\beta + n, \alpha + n)$ and $\Phi(1/r) \asymp f(1/r)/r^n$ as $r \to \infty$.*

Proof. Since $\Phi(1/\cdot) \in OR(\beta, \alpha)$, Theorem 2 implies that there exists a number $r_0 > 0$ such that

$$C'\lambda^\beta \Phi\left(\frac{1}{r}\right) \le \Phi\left(\frac{1}{\lambda r}\right) \le C\lambda^\alpha \Phi\left(\frac{1}{r}\right), \qquad r \ge r_0,\ \lambda \ge 1. \tag{21}$$

According to Theorem 2, if $f(1/\cdot) \in OR$, then there are constants $\tilde\alpha, \tilde\beta \in \mathbb{R}$, C_1, C_2, and $A > 0$ such that

$$C_1 \left(\frac{y}{x}\right)^{\tilde\beta} \le \frac{f(1/y)}{f(1/x)} \le C_2 \left(\frac{y}{x}\right)^{\tilde\alpha}, \qquad y \ge x \ge A.$$

Thus

$$C_2 f(x) \left(\frac{x}{y}\right)^{\tilde\alpha} \ge f(y), \qquad \frac{1}{y} \ge \frac{1}{x} \ge A,$$

$$C_2 x^{n-1} f(x) \int_{x/\lambda}^{x} \left(\frac{x}{y}\right)^{\tilde\alpha} \left(\frac{y}{x}\right)^{n-1} dy \ge \int_{x/\lambda}^{x} y^{n-1} f(y)\, dy = \frac{1}{|s(1)|} \left(\Phi(x) - \Phi(x/\lambda)\right).$$

In view of (21),

$$C_2 x^n f(x) \int_{1/\lambda}^{1} y^{n-1-\tilde\alpha}\, dy \ge \frac{1}{|s(1)|} \Phi(x) \left(1 - C\lambda^\alpha\right)$$

for sufficiently small x. Thus

$$\frac{x^n f(x)}{\Phi(x)} \ge \frac{(1 - C\lambda^\alpha)}{C_2 |s(1)| \int_{1/\lambda}^{1} y^{n-1-\tilde\alpha}\, dy} > 0 \tag{22}$$

for sufficiently large λ. Analogously

$$C_1 f(x) \left(\frac{x}{y}\right)^{\tilde\beta} \le f(y), \qquad \frac{1}{y} \ge \frac{1}{x} \ge A,$$

$$C_1 x^{n-1} f(x) \int_{x/\lambda}^{x} \left(\frac{x}{y}\right)^{\tilde\beta} \left(\frac{y}{x}\right)^{n-1} dy \le \int_{x/\lambda}^{x} y^{n-1} f(y)\, dy = \frac{1}{|s(1)|} \left(\Phi(x) - \Phi(x/\lambda)\right).$$

The latter result and (21) imply that

$$C_1 x^n f(x) \int_{1/\lambda}^{1} y^{n-1-\tilde\beta}\, dy \le \frac{1}{|s(1)|} \Phi(x) \left(1 - C'\lambda^\beta\right)$$

for sufficiently small x. Thus

$$\frac{x^n f(x)}{\Phi(x)} \le \frac{\left(1 - C'\lambda^\beta\right)}{C_1 |s(1)| \int_{1/\lambda}^{1} y^{n-1-\tilde\beta}\, dy} < +\infty. \tag{23}$$

Finally, we obtain from (22) and (23) that $\Phi(1/r) \asymp f(1/r)/r^n$ as $r \to \infty$. Therefore

$$f(1/\cdot) \in OR(\beta + n, \alpha + n). \qquad \square$$

The following result contains a kind of extra assumption mentioned above. The assumption is given in terms of the monotonicity of spectral densities in a neighborhood of the origin.

Theorem 8. *Let $f(1/\cdot)$ be an essentially positive isotropic spectral density such that $x^{n-1}f(x)$ is monotone in a neighborhood of the origin and let $\alpha < 0$. If*

$$\Phi(1/\cdot) \in OR(\beta, \alpha),$$

then $f(1/\cdot) \in OR(\beta + n, \alpha + n)$ and $\Phi(1/r) \asymp f(1/r)/r^n$ as $r \to \infty$.

Proof. First we consider the case where $x^{n-1}f(x)$ is nondecreasing. We show that

$$\Phi(x) \asymp x^n f(x) \quad \text{as } x \to 0.$$

It follows from the definition of the spectral density that

$$\Phi(x) = |s(1)| \int_0^x z^{n-1} f(z)\, dz \leq |s(1)| x^n f(x), \tag{24}$$

$$\Phi(\lambda x) \geq |s(1)| \int_x^{\lambda x} z^{n-1} f(z)\, dz \geq |s(1)|(\lambda - 1) x^n f(x), \qquad \lambda \geq 1, \tag{25}$$

in a neighborhood of the origin where $x^{n-1}f(x)$ is nondecreasing.

According to (25),

$$\frac{\Phi(x)}{x^n f(x)} = \frac{\Phi(\lambda x)}{x^n f(x)} \frac{\Phi(x)}{\Phi(\lambda x)} \geq |s(1)|(\lambda - 1) C' \lambda^\beta > 0, \qquad \lambda > 1, \tag{26}$$

since $\Phi(1/\cdot) \in OR(\beta, \alpha)$.

Now it follows from (24) and (26) that $\Phi(1/r) \asymp f(1/r)/r^n$ as $r \to \infty$, that is,

$$f(1/\cdot) \in OR(\beta + n, \alpha + n).$$

Now we consider the case where $x^{n-1}f(x)$ is nonincreasing. We show again that $\Phi(x) \asymp x^n f(x)$ as $x \to 0$. It follows from the definition of the spectral density that

$$\Phi(x) = |s(1)| \int_0^x z^{n-1} f(z)\, dz \geq |s(1)| x^n f(x), \tag{27}$$

$$\frac{\Phi(\lambda x) - \Phi(x)}{\Phi(x)} = \frac{|s(1)|}{\Phi(x)} \int_x^{\lambda x} z^{n-1} f(z)\, dz \leq \frac{|s(1)|}{\Phi(x)} (\lambda - 1) x^n f(x), \qquad \lambda \geq 1, \tag{28}$$

in a neighborhood of the origin where $x^{n-1}f(x)$ is nonincreasing. The inclusion

$$\Phi(1/\cdot) \in OR(\beta, \alpha)$$

implies that

$$\frac{\Phi(\lambda x) - \Phi(x)}{\Phi(x)} \geq c\lambda^{-\alpha} - 1, \qquad \lambda \geq 1.$$

It follows from (28) that

$$\frac{x^n f(x)}{\Phi(x)} \geq \frac{c\lambda^{-\alpha} - 1}{|s(1)|(\lambda - 1)} > 0 \tag{29}$$

for some sufficiently large λ.

Finally, inequalities (29) and (27) imply that $\Phi(1/r) \asymp f(1/r)/r^n$ as $r \to \infty$, so that

$$f(1/\cdot) \in OR(\beta + n, \alpha + n). \qquad \square$$

Remark 4. The result of the preceding theorem remains true if the monotonicity is required for $f(x)$ instead of $x^{n-1}f(x)$. Indeed, if $f(x)$ is nondecreasing, then $x^{n-1}f(x)$ is nondecreasing, too. However, if $f(x)$ is nonincreasing, then $x^{n-1}f(x)$ is not necessarily a nonincreasing function. Nevertheless the proof is true in this case, too, since

$$\Phi(x) \geq \frac{|s(1)|}{n} x^n f(x)$$

in a neighborhood of the origin if $f(x)$ is nonincreasing. This inequality replaces (27) in the proof, while (28) is replaces by

$$\frac{\Phi(\lambda x)-\Phi(x)}{\Phi(x)} \leq \frac{|s(1)|(\lambda^n-1)}{n\Phi(x)} x^n f(x).$$

As a corollary of the latter three results we obtain the following Tauberian theorem.

Theorem 9. *Let $f(1/\cdot)$ be an essentially positive isotropic spectral density. If*

(i) $f(1/\cdot) \in OR(\beta+n, \alpha+n)$,

then for $0>\alpha\geq\beta>2-n$,

(ii) $\tilde{l}(\cdot) \in OR(\beta, \alpha)$;
(iii) $\tilde{l}(r) \asymp f(1/r)/r^n$ *as* $r\to\infty$ *and there are positive constants* C, C', *and* r_0 *such that*

$$C'\left(\frac{r_1}{r}\right)^{\beta} \leq \frac{f(1/r_1)}{r_1^n \tilde{l}(r)} \leq C\left(\frac{r_1}{r}\right)^{\alpha}, \qquad r_1\geq r\geq r_0;$$

and for $0>\alpha\geq\beta>-n$,

(iv) $\tilde{b}(\cdot) \in OR(\beta, \alpha)$;
(v) $\tilde{b}(r) \asymp f(1/r)/r^n$ *as* $r\to\infty$ *and there are positive constants* C, C', *and* r_0 *such that*

$$C'\left(\frac{r_1}{r}\right)^{\beta} \leq \frac{f(1/r_1)}{r_1^n \tilde{b}(r)} \leq C\left(\frac{r_1}{r}\right)^{\alpha}, \qquad r_1\geq r\geq r_0.$$

Moreover, if at least one of the following conditions holds, namely,

(1) $f(1/\cdot) \in OR$;
(2) $x^{n-1}f(x)$ *is monotone in a neighborhood of the origin;*
(3) $f(x)$ *is monotone in a neighborhood of the origin;*

then (i) *follows from any of the statements* (ii), (iii), (iv), *or* (v).

Proof. (i) $\Rightarrow$ (ii), (iii), (iv), (v). Theorem 6 and (i) imply that $\Phi(1/\cdot) \in OR(\beta,\alpha)$. Now Theorems 4 and 5 prove (ii) and (iv). Theorem 6 and statement (iii) of Theorems 4 and 5 imply statements (iii) and (v) of Theorem 9.

(ii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (i). Theorem 4 and (ii) (or, Theorem 5 and (iv)) imply that $\Phi(1/\cdot) \in OR(\beta,\alpha)$. Now Theorems 7 and 8 or Remark 4 prove (i).

(iii) $\Rightarrow$ (i), (v) $\Rightarrow$ (i). It follows from (iii) that

$$\frac{C'}{z}\left(\frac{1}{zr}\right)^{\beta} \leq \frac{z^{n-1}f(z)}{\tilde{l}(r)} \leq \frac{C}{z}\left(\frac{1}{zr}\right)^{\alpha}, \qquad 1/z\geq r\geq r_0.$$

Integrating with respect to z in the interval $[0,x]$ we obtain

$$C'\left(\frac{1}{xr}\right)^{\beta} \leq \frac{\Phi(x)}{\tilde{l}(r)} \leq C\left(\frac{1}{xr}\right)^{\alpha}, \qquad 1/x\geq r\geq r_0,$$

whence statement (iii) of Theorem 4 follows. Similarly, (v) implies statement (iii) of Theorem 5. The further proof of (iii) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i) follows the lines of the proof of (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i), respectively. ∎

Remark 5. The assumption imposed on the isotropic spectral density in Theorem 9 implies that $x^{n-1}f(x)$ has the following singularity properties:

(1) $x^{n-1}f(x)$ is unbounded at zero if $\beta > -1$;
(2) $x^{n-1}f(x)$ approaches 0 if $\alpha < -1$.

7. Concluding remarks

Abelian and Tauberian theorems are proved in the paper for a class of OR functions wider than the class of regularly varying functions. The proofs for OR functions are simpler than those for R_ρ functions in most cases. This is explained by the following observation. Despite the fact that the class OR is wider than R_ρ, the asymptotics does not require the exact value of the constants C and D involved in Definition 3 in the case of OR functions. On the other hand, if one wants to get a result for R_ρ functions as a corollary of the corresponding result for OR functions, then one needs not only choose $\alpha = \beta = \rho$ but also one should prove that $C = D$.

Comparing the results of Sections 5 and 6 with the analogous results for R_ρ functions in [6, 9, 12, 16] the following question appears. Is it possible to decrease the lower bound for indices α and β? This question and the asymptotic behavior of spectral and correlation characteristics of random fields with limit Matuszewska indices will be investigated elsewhere. We also plan to exhibit some applications of the above Tauberian theorems and study the asymptotic behavior of functionals of random fields.

Bibliography

1. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Reprint of the second (1944) edition, Cambridge University Press, Cambridge, 1995. MR1349110 (96i:33010)
2. N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1989. MR1015093 (90i:26003)
3. M. Broniatowski and A. Fuchs, *Tauberian theorems, Chernoff inequality, and the tail behavior of finite convolutions of distribution functions*, Adv. Math. **116** (1995), 12–33. MR1361477 (96k:60034)
4. J. L. Geluk, *Abelian and Tauberian theorems for* 0*-regularly varying functions*, Proc. Amer. Math. Soc. **93(2)** (1985), 235–241. MR770528 (86d:44001)
5. G. Laue, *Tauberian and Abelian theorems for characteristic functions*, Teor. Veroyatn. Mat. Stat. **37** (1987), 78–92; English transl. in Theory Probab. Math. Statist. **37** (1988), 89–103. MR913912 (89a:60044)
6. A. V. Ivanov and N. N. Leonenko, *Statistical Analysis of Random Fields*, "Vyshcha shkola", Kiev, 1986; English transl., Kluwer Academic Publishers, Dordrecht, 1989. MR917486 (89e:62125)
7. N. N. Leonenko and A. Ya. Olenko, *Tauberian and Abelian theorems for the correlation function of a homogeneous isotropic random field*, Ukrain. Mat. Zh. **43** (1991), no. 12, 1652–1664; English transl. in Ukrainian Math. J. **43** (1992), no. 12, 1539–1548. MR1172306 (93f:60069)
8. N. N. Leonenko and A. Ya. Olenko, *Tauberian theorems for correlation functions and limit theorems for spherical averages of random fields*, Random Oper. Stochastic Equations **1** (1993), no. 1, 57–67. MR1254176 (95a:60068)
9. N. N. Leonenko, *Limit Theorems for Random Fields with Singular Spectrum*, Kluwer Academic Publishers, Dordrecht–Boston–London, 1999. MR1687092 (2000k:60102)
10. C. Ma, *Long-memory continuous-time correlation models*, J. Appl. Prob. **40** (2003), 1133–1146. MR2012691 (2005a:62200)
11. A. A. Malyarenko, *Abelian and Tauberian theorems for random fields on two-point homogeneous spaces*, Teor. Ĭmovīr. Mat. Stat. **69** (2003), 106–118; English transl. in Theory Probab. Math. Statist. **69** (2004), 115–127. MR2110910 (2006e:60067)
12. A. Ya. Olenko, *Tauberian and Abelian theorems for strongly dependent random fields*, Ukrainian Math. J. **48** (1996), no. 3, 368–383. MR1408658 (97k:60143)
13. B. A. Rogozin, *A Tauberian theorem for increasing functions of dominated variation*, Sibirsk. Mat. Zh. **43** (2002), no. 2, 442–445; English transl in Siberian Math. J. **43** (2003), no. 2, 353–356. MR1902831 (2003f:40010)

14. B. A. Rogozin, *Tauberian theorems for decreasing functions of dominated variation*, Theory Probab. Appl. **47** (2002), no. 2, 351–357. MR2001841 (2004f:26003)
15. A. L. Rukhin, *Statistical estimation of exponential-type functions: Admissibility and Tauberian theorems*, J. Math. Anal. Appl. **191** (1995), 346–359. MR1324018 (96b:62046)
16. M. I. Yadrenko, *Spectral Theory of Random Fields*, Optimization Software, New York, 1983. MR697386 (84f:60003)

Department of Probability Theory and Mathematical Statistics, Faculty for Mathematics and Mechanics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

E-mail address: olenk@univ.kiev.ua

Received 1/FEB/2005

Translated by OLEG KLESOV