Теорія Ймовір. та Матем. Статист.
Вип. 73, 2005

Theor. Probability and Math. Statist.
No. 73, 2006, Pages 151–162
S 0094-9000(07)00689-8
Article electronically published on January 17, 2007

MULTIDIMENSIONAL WEAKLY STATIONARY RANDOM FUNCTIONS ON SEMIGROUPS

UDC 519.21

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Abstract. We consider some problems in the spectral analysis of weakly stationary Hilbert-valued random functions on involutive semigroups. We obtain spectral representations for such functions and for their correlation functions. These representations are extensions and improvements of the corresponding results proved earlier by the first author and by V. Girardin and R. Senoussi.

1. Weakly stationary vector random functions on semigroups

Denote by H a complex separable Hilbert space and by $L_2(\Omega)$ the Hilbert space of second order complex-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Then the set $\mathcal{M}(\Omega, H) = \mathcal{L}(H, L_2(\Omega))$ of all $L_2(\Omega)$-valued linear continuous random functionals on the space H can be viewed as the set of generalized second order random elements in the space H defined on $(\Omega, \mathcal{F}, \mathsf{P})$ (see [4]). Every element $\Xi \in \mathcal{M}(\Omega, H)$ is generated by an usual second order random element ξ defined up to P-equivalence and assuming values in some quasi-kernel extension H_- of the space H, $\Xi x \in (x|\xi)$, $x \in H$, where $(\cdot\,|\,\cdot)$ is the scalar product in H (see [5]). If the space H is finite dimensional, then $H_- = H$ and Ξ is identified with the corresponding H-valued random vector ξ.

The mathematical expectation $m = \mathsf{E}\,\Xi$ of the element $\Xi \in \mathcal{M}(\Omega, H)$ is an element of H uniquely defined by the equality

$$\mathsf{E}(\Xi x) = (x|m), \qquad x \in H.$$

The covariance operator $[\Xi, \Psi]$ for elements $\Xi, \Psi \in \mathcal{M}(\Omega, H)$ is an element of the complex algebra $B(H)$ of linear bounded operators in H which is uniquely defined by the equality

$$\mathsf{E}(\Xi x)(\overline{\Psi y}) = ([\Xi, \Psi]x|y), \qquad x, y \in H.$$

Note that $[\Xi, \Psi] = \Psi^*\Xi$, where Ψ^* is the dual operator for $\Psi\colon H \to L_2(\Omega)$ and $(\Psi^*\eta|x) = (\eta|\Psi x)_{L_2(\Omega)}$, $x \in H$, $\eta \in L_2(\Omega)$. Denote by $B_+(H)$ the convex cone of positive Hermitian operators in $B(H)$. Note that $[\Xi, \Psi]$, as a function of Ξ and Ψ, is a sesquilinear $B(H)$-valued form on $\mathcal{M}(\Omega, H)$ such that $[\Xi, \Xi] \in B_+(H)$.

In what follows we need some notions and results in the theory of semigroups (see [6]–[8]). Below we discuss briefly some of them.

Let $\mathbb{S}$ be an Abelian semigroup with a binary associative and commutative operation $\circ$. A semigroup $\mathbb{S}$ is called involutive (or a $*$-semigroup) if a unitary operation $*$, $s \to s^*$,

2000 *Mathematics Subject Classification.* Primary 60G10, 60G57, 60G15.

Key words and phrases. Random functions, spectral representations, involutive semigroup, weakly stationary random functions.

$s \in \mathbb{S}$, is defined on $\mathbb{S}$ and has the following properties:

$$(s \circ t)^* = t^* \circ s^*, \qquad (s^*)^* = s, \qquad s, t \in \mathbb{S}.$$

Thus the involution $*$ on $\mathbb{S}$ is an idempotent antiautomorphism acting from $\mathbb{S}$ to itself. An involutive Abelian semigroup $\mathbb{S}$ is denoted by $(\mathbb{S}, \circ, *)$ in what follows. If there exists a neutral element e in $\mathbb{S}$, that is, if $e \circ s = s \circ e = s$ for all $s \in \mathbb{S}$, then $e^* = e$.

Any Abelian semigroup $\mathbb{S}$ can be viewed as involutive for the identity involution $s^* = s$, $s \in \mathbb{S}$. This semigroup involution is denoted by Id. Any Abelian group $\mathbb{G}$ is an involutive semigroup with the involution $g^* = g^{-1}$, $g \in \mathbb{G}$.

One-dimensional representations (homomorphisms) of a semigroup $(\mathbb{S}, \circ, *)$ in the set of complex numbers $\mathbb{C}$ viewed as a $*$-semigroup $(\mathbb{C}, \cdot), ^-))$ with the operation of multiplication $\cdot)$ and involution $^-)$ (complex conjugation) are called semicharacters of $(\mathbb{S}, \circ, *)$ or its elementary harmonics. Thus every semicharacter χ of a semigroup $\mathbb{S}$ is a function $\chi\colon \mathbb{S} \to \mathbb{C}$ such that

$$\chi(s \circ t) = \chi(s)\chi(t), \qquad \chi(t^*) = \overline{\chi(t)}, \qquad s, t \in \mathbb{S}.$$

If there exists a neutral element e in $\mathbb{S}$, then $\chi(e) = 1$.

The set $\mathbb{S}^*$ of all semicharacters of a semigroup $\mathbb{S}$ is its maximal dual object. By $\widehat{\mathbb{S}}$ we denote the set of semicharacters such that $\sup_{t\in\mathbb{S}} |\chi(t)| \le 1$ and that are representations of $(\mathbb{S}, \circ, *)$ in the semigroup $(\mathbb{D}, \cdot),^-))$, where $\mathbb{D} = \{z \in \mathbb{C}\colon |z| \le 1\}$ is the unit disk in $\mathbb{C}$. The set of semicharacters of a semigroup $\mathbb{S}$ such that $|\chi(t)| = 1$, $t \in \mathbb{S}$, and that are representations of $(\mathbb{S}, \circ, *)$ in $(\mathbb{T}, \cdot),^-))$ is the one-dimensional torus in $\mathbb{C}$, where $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. This set is denoted by $\widetilde{\mathbb{S}}$. It is obvious that $\widetilde{\mathbb{S}} \subset \widehat{\mathbb{S}} \subset \mathbb{S}^*$. If $\mathbb{S}$ is a group, that is, $t^* = t^{-1}$, $t \in \mathbb{S}$, then the dual object $\mathbb{S}^*$ is the group of characters of the group $\mathbb{S}$, since

$$|\chi(t)|^2 = \chi(t)\overline{\chi(t)} = \chi(t)\chi(t^{-1}) = \chi(t \circ t^{-1}) = \chi(e) = 1, \qquad t \in \mathbb{S}.$$

Considering pointwise multiplication and complex conjugation as a binary operation and an involution in the dual objects $\widetilde{\mathbb{S}}$, $\widehat{\mathbb{S}}$, and $\mathbb{S}^*$, respectively, one can treat these dual objects as involutive semigroups with the unit element $1_{\mathbb{S}}(t) \equiv 1$, $t \in \mathbb{S}$. If $\mathbb{S}^*$ is equipped with the topology of pointwise convergence on $\mathbb{C}^{\mathbb{S}}$, then $\mathbb{S}^*$ becomes a completely regular Hausdorff space and its subsemigroup $\widehat{\mathbb{S}}$ is compact.

Note that the semicharacters of a semigroup $\mathbb{S}$ are positive definite functions on $\mathbb{S}$; that is,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_j\overline{c_k}\chi(t_j \circ t_k^*) = \left|\sum_{j=1}^{n} c_j\chi(t_j)\right|^2 \ge 0$$

for all positive integer numbers $n \in \mathbb{N}$, all complex numbers $c_j \in \mathbb{C}$, $j = 1, \dots, n$, and all elements $t_j \in \mathbb{S}$, $j = 1, \dots, n$.

A function $\alpha\colon \mathbb{S} \to \mathbf{R}_+ = [0, +\infty)$ is called the absolute value of a $*$-semigroup $\mathbb{S}$ if $\alpha(t^*) = \alpha(t)$ for all $t \in \mathbb{S}$, all $\alpha(s \circ t) \le \alpha(s)\alpha(t)$, and all $t, s \in \mathbb{S}$ (this property characterizes the so-called submultiplicative functions α). If there exists a neutral element e in $\mathbb{S}$, then we assume that $\alpha(e) = 1$.

A function $f\colon \mathbb{S} \to \mathbb{C}$ is called α-bounded if $|f(t)| \le c\alpha(t)$, $t \in \mathbb{S}$, for some constant $c > 0$. A function f is called exponentially bounded if it is bounded for at least one absolute value on $\mathbb{S}$. A function f is bounded on $\mathbb{S}$ if $|f(t)| \le c$ for all $t \in \mathbb{S}$.

Denote by $\mathbb{S}^*_\alpha$ the set of all α-bounded semicharacters χ on $\mathbb{S}$; that is,

$$|\chi(t)| \le \alpha(t), \qquad t \in \mathbb{S}.$$

Let S^*_α be equipped with the topology of pointwise convergence.

Definition 1.1. A family $\{\Xi_t, t \in T\}$ of generalized random elements $\Xi_t \in \mathcal{M}(\Omega, H)$ is called a generalized second order random function defined on the set T, $T \neq \varnothing$, and assuming values in H.

Definition 1.2. A generalized second order random function Ξ_t, $t \in \mathbb{S}$, on a $*$-semigroup S is called weakly stationary if its mean function is constant, that is, $\mathsf{E}\,\Xi_t \equiv m \in H$ for all $t \in \mathbb{S}$, and its correlation function $[\Xi_t, \Xi_s]$, $s, t \in \mathbb{S}$, depends on $t \circ s^*$ only; that is, there exists a $B(H)$-valued function $R\colon \mathbb{S} \to B(H)$ such that

$$[\Xi_t, \Xi_s] = R(t \circ s^*), \qquad t, s \in \mathbb{S}.$$

In what follows we assume without loss of generality that $m = 0$ (otherwise one can consider the stationary function $\widetilde{\Xi_t}$, $(\widetilde{\Xi_t}x) = (\Xi_t x) - (x|m)$, $t \in \mathbb{S}$, $x \in H$, for which $\mathsf{E}\,\Xi_t \equiv 0$).

Note that $R\colon \mathbb{S} \to B(H)$ is a correlation function for some weakly stationary random function $\{\Xi_t, t \in \mathbb{S}\}$ if and only if $R(t)$ is positive definite; that is,

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\bigl(R(t_j \circ t_k^*)x_j|x_k\bigr) \geq 0$$

for all $n \in N$ and all $t_j \in \mathbb{S}$, $x_j \in H$, $j = 1, \dots, n$ (see [1, 2]).

Below are a few simple examples of weakly stationary functions on $*$-semigroups.

Example 1. A random harmonic polynomial Ψ_t, $t \in \mathbb{S}$,

$$\Psi_t = \sum_{k=1}^{n} \chi_k(t) Z_k, \qquad t \in \mathbb{S},\ \chi_k \in \mathbb{S}^*, \tag{1}$$

is a stationary function on $\mathbb{S}$ with $\mathsf{E}\,\Psi_t = 0$, $t \in \mathbb{S}$, and the correlation function

$$[\Psi_t, \Psi_s] = \sum_{k=1}^{n} \chi_k(t \circ s^*) F_k, \qquad t, s \in \mathbb{S}, \tag{2}$$

if its amplitudes Z_k are uncorrelated (orthogonal) and belong to $\mathcal{M}(\Omega, H)$; that is, $[Z_k, Z_j] = \delta_{kj} F_k$ and $\mathsf{E}\,Z_k = 0$ for all $k, j = 1, \dots, n$ and some $F_k \in B_+(H)$, where δ_{kj} is the Kronecker delta.

Example 2. Let W_t, $t \in \mathbf{R}_+ = [0, \infty)$, be the generalized standard Brownian motion in H; that is, W is a Gaussian process in H with mean $\mathsf{E}\,W_t \equiv 0$ and correlation function

$$[W_t, W_s] = (t \wedge s) I, \qquad t, s \in \mathbf{R}_+,$$

where $t \wedge s = \min(t, s)$ and I is the unit operator in $B(H)$. If one treats $\mathbf{R}_+$ as an Abelian $*$-semigroup with the operation $\wedge$ and involution Id, $(\mathbf{R}_+, \wedge, \mathrm{Id})$, then W_t is a stationary function on this semigroup.

Example 3. Let W_t, $t \in \mathbf{R}_+^n$, be a Chentsov–Wiener random field in H (multiparameter Brownian motion); that is, W_t is a Gaussian field such that $W_t \in \mathcal{M}(\Omega, H)$, $t \in \mathbf{R}_+^n$, $\mathsf{E}\,W_t \equiv 0$, and

$$[W_t, W_s] = \prod_{j=1}^{n} (t_j \wedge s_j) Q$$

for $t = (t)_{j=1}^n \in \mathbf{R}_+^n$ and $s = (s_j)_{j=1}^n \in \mathbf{R}_+^n$, where $Q \in B_+(H)$ for $Q > 0$. Considering $\mathbf{R}_+^n$ as a $*$-semigroup $(\mathbf{R}_+^n, \circ, \mathrm{Id})$ with $t \circ s = \prod_{j=1}^n (t_j \wedge s_j)$, $t, s \in \mathbf{R}_+^n$, the function W_t is stationary (homogeneous) on this semigroup.

Example 4. Let $(U,\mathcal{A})$ be a measurable space, where $\mathcal{A}$ is the σ-algebra of subsets of U, and let $Z\colon \mathcal{A}\to\mathcal{M}(\Omega,H)$ be a generalized random measure in H defined on $\mathcal{A}$ (the σ-additivity of Z is understood in the strong topology of the space $\mathcal{L}(H,L_2(\Omega))$). Assume that the mean value of Z is zero, that is, $\mathsf{E}\,Z(\triangle)=0$, $\triangle\in\mathcal{A}$. We further assume that the values of Z are orthogonal in the following sense: there exists an operator measure $F\colon\mathcal{A}\to B_+(H)$ (the structure measure for Z) such that

$$[Z(\triangle_1),Z(\triangle_2)]=F(\triangle_1\cap\triangle_2),\qquad \triangle_1,\triangle_2\in\mathcal{A}.$$

The σ-additivity of F is understood in the weak topology of $B(H)$. Considering $\mathcal{A}$ as an involutive semigroup $(\mathcal{A},\cap,\mathrm{Id})$, we prove that the measure Z is a stationary function on $\mathcal{A}$.

Example 5. Let H and M be real Hilbert spaces, $\mathcal{L}(M,H)$ be the Banach space of linear continuous operators acting from M to H, and let Z be a random orthogonal measure in H on $(U,\mathcal{A})$ with the structure function F such that $F(\Delta)=\mu(\Delta)Q$, where $Q\in B_+(H)$ and μ is a nonnegative real-valued finite measure on $(U,\mathcal{A})$. The stochastic integrals

$$I_\triangle(A)=\int_\triangle Z(du)A(u),\qquad \triangle\in\mathcal{A},$$

and

$$[I_{\triangle_1}(A_1),I_{\triangle_2}(A_2)]=\int_{\triangle_1\cap\triangle_2}A_2^*(u)QA_1(u)\,\mu(du)\in B(M),$$

are well defined for strongly measurable functions $A\colon U\to\mathcal{L}(M,H)$ such that

$$\int_U \|A(u)\|^2\,\mu(du)<\infty$$

(see [9]). Note that $I_\triangle(A)\in\mathcal{M}(\Omega,M)$. Then $I_\triangle(A)$, $\triangle\in\mathcal{A}$, are stationary functions on $(\mathcal{A},\cap,\mathrm{Id})$ with $\mathsf{E}\,I_\triangle(A)=0$, $\triangle\in\mathcal{A}$.

Theorem 1. *(i) If a generalized random function Ξ_t, $t\in\mathbb{S}$, is weakly stationary where $\mathbb{S}$ is a $*$-semigroup, then its correlation kernel $K(t,s)=[\Xi_t,\Xi_s]$ is cross-involutive invariant, that is,*

$$K(t\circ a,s\circ b)=K(t\circ b^*,s\circ a^*),\qquad t,s,a,b\in\mathbb{S}. \tag{3}$$

(ii) If there exists a neutral element $e\in\mathbb{S}$, then property (3) *implies that Ξ_t, $t\in\mathbb{S}$, is weakly stationary for a random function Ξ_t, $t\in\mathbb{S}$, with $\mathsf{E}\,\Xi_t\equiv 0$.*

(iii) If $\mathbb{S}$ is a group, then property (3) *is equivalent to the classical condition that the kernel K is invariant with respect to translations, namely,*

$$K(t\circ a,s\circ a)=K(t,s),\qquad t,s,a\in\mathbb{S}. \tag{4}$$

Proof. Statement (i) follows from the equality

$$(t\circ a)\circ(s\circ b)^*=t\circ a\circ b^*\circ s^*=(t\circ b^*)\circ(s\circ a^*)^*.$$

Statement (ii) is a corollary of the relation

$$K(t,s)=K(t\circ e,e\circ s)=K(t\circ s^*,e\circ e)=R(t\circ s^*)$$

in view of property (3). Statement (iii) also is straightforward. Indeed, if $\mathbb{S}$ is a group and $s^*=s^{-1}$, then property (3) applied to the function Ξ_t, $t\in S$, with zero mean implies for $a=s^{-1}$ and $b=a$ that

$$K(t\circ a,s\circ a)=K(t\circ s^{-1},e\circ e)=K(t,s), \tag{5}$$

whence it follows that the function Ξ_t, $t\in\mathbb{S}$, is stationary.

Conversely, if a function Ξ_t, $t \in \mathbb{S}$, is stationary, then relation (5) holds. Substituting $t \circ a$ and $s \circ b$ for s and t in (5) we obtain

$$K(t \circ a, s \circ b) = K((t \circ a) \circ (b^* \circ s^*), e) = K((t \circ b^*) \circ (a \circ s^*), e) = K(t \circ b^*, s \circ a^*).$$

Thus (3) and (4) are equivalent if $\mathbb{S}$ is a group. □

2. Spectral representations of stationary functions

Let $\mathbb{S}$ be an Abelian $*$-semigroup with a neutral element e and let $\alpha\colon \mathbb{S} \to \mathbf{R}_+$ be some absolute value on $\mathbb{S}$.

Theorem 2. *If Ξ_t, $t \in \mathbb{S}$, is a generalized stationary random function in H whose correlation function $R\colon \mathbb{S} \to B(H)$ is α-bounded, that is,*

$$\|R(t)\| \le c\alpha(t), \qquad t \in \mathbb{S}, \tag{6}$$

then R admits the following spectral representation:

$$R(t) = \int_{\mathbb{S}^*} \chi(t)\, F(d\chi), \qquad t \in \mathbb{S}, \tag{7}$$

where F is a uniquely defined $B_+(H)$-valued Radon measure (the spectral measure of the stationary function Ξ_t) whose support is compact in $\mathbb{S}^*$. If the function R is bounded, that is, if $\|R(t)\| \le c$, $t \in \mathbb{S}^*$, then $\widehat{\mathbb{S}}$ can be substituted for $\mathbb{S}^*$ in representation (7). *In this case, the stationary function Ξ_t, $t \in \mathbb{S}$, itself admits the spectral representation*

$$\Xi_t = \int_{\mathbb{S}^*} \chi(t)\, \Phi(d\chi), \qquad t \in \mathbb{S}^*, \tag{8}$$

where Φ is a random $\mathcal{M}(\Omega, H)$-valued Radon measure on $\mathbb{S}^*$ (the random spectral measure of the function Ξ_t) whose support is compact and

$$[\Phi(\Delta_1), \Phi(\Delta_2)] = F(\Delta_1 \cap \Delta_2).$$

If R is bounded, then $\widehat{\mathbb{S}}$ can be substituted for $\mathbb{S}^*$ in representation (8).

Proof. Consider a sesquilinear form $(R(t)x|y)$, $x, y \in H$, in H that depends on the parameter $t \in \mathbb{S}$. This form is uniquely generated by the corresponding quadratic form $r_x(t) = (R(t)x|x)$, $x \in H$, with the help of the well-known polarization formula

$$(R(t)x|y) = \frac{1}{4}\big(r_{x+y}(t) - r_{x-y}(t) + i r_{x+iy}(t) - i r_{x-iy}(t)\big),$$

where i is the imaginary unit in $\mathbb{C}$. The complex-valued function $r_x(t)$, $t \in \mathbb{S}$, is positive definite and exponentially bounded for any $x \in H$ in view of condition (6). According to an analog of the Bochner–Khintchine theorem for the semigroup $\mathbb{S}$ (see [6, 7]), there exists a uniquely defined positive Radon measure μ_x on $\mathbb{S}^*$ whose support is compact and

$$r_x(t) = \int_{\mathbb{S}^*} \chi(t)\, \mu_x(d\chi), \qquad t \in \mathbb{S}. \tag{9}$$

Moreover, the set $\widehat{\mathbb{S}}$ can be substituted for $\mathbb{S}^*$ in representation (9) if R is bounded. Then

$$(R(t)x|y) = \int_{\mathbb{S}^*} \chi(t)\, \mu_{x,y}(d\chi), \qquad t \in \mathbb{S},$$

for vectors $x, y \in H$ where $\mu_{x,y}$ is a complex-valued Radon measure on $\mathbb{S}^*$ defined by

$$\mu_{x,y} = \frac{1}{4}\big(\mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}\big).$$

The above reasoning implies that, for any measurable set Δ in $\mathbb{S}^*$, $\mu_{x,y}(\Delta)$ as a function of arguments $(x, y) \in H \times H$ is a sesquilinear form that is uniformly bounded in Δ on $H \times H$:

$$|\mu_{x,y}(\Delta)| \le |\mu_{x,y}(\mathbb{S}^*)| = |(R(e)x|y)| \le \|R(e)\| \cdot \|x\| \cdot \|y\|.$$

Thus there exists a uniquely defined $B(H)$-valued Radon measure F on $\mathbb{S}^*$ such that

$$\mu_{x,y}(\Delta) = (F(\Delta)x|y), \qquad x, y \in H,$$

for all measurable sets Δ in $\mathbb{S}^*$. It is easy to check that the measure F is $B_+(H)$-valued and therefore representation (7) holds for it.

Representation (8) for the stationary function Ξ_t, $t \in \mathbb{S}^*$, follows from the representation

$$R(t \circ s^*) = \int_{\mathbb{S}^*} \chi(t)\overline{\chi(s)}\, F(d\chi)$$

and Theorem 3 of [10] on the spectral representations of generalized random functions in vector spaces. Theorem 2 is proved. □

Below we consider a somewhat different kind of spectral representation for multidimensional stationary functions on $*$-semigroups $\mathbb{S}$ for which a neutral element e not necessarily exists in $\mathbb{S}$ (that is, for the case where $\mathbb{S}$ is not necessarily a monoid).

Theorem 3. *If Ξ_t, $t \in \mathbb{S}$, is a stationary random function on a $*$-semigroup $(\mathbb{S}, \circ)$ in a complex Hilbert space H, α is an absolute value on $\mathbb{S}$, and if the correlation kernel K of the function Ξ_t is α-bounded on $\mathbb{S}$, that is,*

$$(K(t \circ s, t \circ s)x|x)_H \le \alpha^2(s)(K(t, t)x|x)_H \tag{10}$$

for all $t, s \in \mathbb{S}$ and $x \in H$, then there are a $B_+(H)$-valued Radon measure F and a $\mathcal{M}(\Omega, H)$-valued random Radon measure Φ on $\mathbb{S}^*_\alpha$ such that

$$[\Phi(\Delta_1), \Phi(\Delta_2)] = F(\Delta_1 \cap \Delta_2)$$

and the following spectral representations hold:

$$\begin{aligned} R(t \circ s^*) = [\Xi_t, \Xi_s] &= \int_{\mathbb{S}^*_\alpha} \chi(t \circ s^*)\, F(d\chi), \\ \Xi_t &= \int_{\mathbb{S}^*_\alpha} \chi(t)\, \Phi(d\chi), \qquad t, s \in \mathbb{S}. \end{aligned} \tag{11}$$

Proof. Representation (11) for the correlation function R of the random function Ξ_t follows from Theorem 15.7 in [8], since R is a positive definite $B(H)$-valued function on $\mathbb{S}$ that is α-bounded in the sense of [8]. Other statements of the theorem are corollaries of representation (11) for R that can be proved by the same method as the one used in the proof of Theorem 2. □

Corollary 2.1. *Let the dimension of the space H equal n, that is, $\dim H = n$, and let $\{\varphi\}_{i=1}^n$ be an orthonormal basis in H. Then a stationary function Ξ_t, $t \in \mathbb{S}$, in H can be identified with an n-measurable random function*

$$\xi(t) = \{\xi_j(t)\}_{j=1}^n, \quad t \in \mathbb{S}, \qquad \xi_j(t) = \Xi_t\varphi_j, \quad j = 1, \dots, n,$$

whose components are zero mean stationary functions such that

$$\mathsf{E}\, \xi_j(t)\overline{\xi_k(s)} = r_{jk}(t \circ s^*) = (R(t \circ s^*)\varphi_j|\varphi_k)_H, \qquad j, k = 1, \dots, n,$$

where $R(t)$ is the correlation function of Ξ_t, $t \in \mathbb{S}$, that can be identified with the matrix function $\{r_{jk}(t)\}_{j,k=1}^n$. If the correlation kernel K of the function Ξ_t, $t \in \mathbb{S}$, is α-bounded, then there are a uniquely defined matrix-valued Radon measure $F = \{F_{jk}\}_{j,k=1}^n$ and a *random vector-valued Radon measure* $Z = \{Z_j\}_{j=1}^n$ *on* $\mathbb{S}_\alpha^*$ *(the spectral and random spectral measures of the function* $\xi(t)$*) for which*

$$r_{jk}(t) = \int_{\mathbb{S}_\alpha^*} \chi(t)\, F_{jk}(d\chi), \qquad j,k = 1,\dots,n, \tag{12}$$

$$\xi_j(t) = \int_{\mathbb{S}_\alpha^*} \chi(t)\, Z_j(d\chi), \qquad j = 1,\dots,n, \tag{13}$$

where

$$\mathsf{E}\, Z_j(\Delta_1)\overline{Z_k(\Delta_2)} = F_{jk}(\Delta_1 \cap \Delta_2). \tag{14}$$

Relations (12)–(14) follow from Theorem 3 with $F_{jk}(\Delta) = (F(\Delta)\varphi_j|\varphi_k)_H$ and

$$Z_j(\Delta) = \Phi(\Delta)\varphi_j, \qquad k,j = 1,\dots,n,$$

where Δ are measurable sets in $\mathbb{S}_\alpha^*$.

When general results on the spectral representations for stationary random functions and those for their correlation functions are applied to particular $*$-semigroups, it is a common approach to associate the semicharacters of these semigroups with a certain parameter $\lambda \in \Lambda$, where the set Λ is isomorphic to $\mathbb{S}^*$ so that $\mathbb{S}^* = \{\chi_\lambda(t),\, t \in \mathbb{S}\colon \lambda \in \Lambda\}$. Then the spectral representations take the form

$$R(t \circ s^*) = \int_\Lambda \chi_\lambda(t \circ s^*)\, F(d\lambda), \qquad \Xi_t = \int_\Lambda \chi_\lambda(t)\, \Phi(d\lambda). \tag{15}$$

It is obvious that representation (1) of a random harmonic polynomial on $\mathbb{S}$ and representation (2) of its correlation function are their spectral representations with a discrete spectrum (this means that the supports of the corresponding spectral measures belong to a finite set of characters involved in the representations).

Example 6. The multiparameter Chentsov motion W_t, $t \in \mathbf{R}_+^n$, in H is a stationary field on the semigroup $\mathbb{S} = (\mathbf{R}_+^n, \wedge^n, \mathrm{Id})$. Since the semicharacters of $\mathbb{S}$ are of the form

$$\chi_\lambda(t) = \prod_{j=1}^n 1_{[\lambda_j,+\infty)}(t_j), \qquad \lambda = (\lambda_j)_{j=1}^n \in \Lambda = \mathbf{R}_+^n,\ t = (t_j)_{j=1}^n \in \mathbf{R}_+^n,$$

where the symbol 1_A stands for the indicator of a set A, we obtain the following spectral representation of W_t:

$$W_t = \int_{\mathbf{R}_+^n} \chi_\lambda(t)\, dW_\lambda = \int_0^{t_1} \dots \int_0^{t_n} dW_\lambda, \qquad t \in \mathbf{R}_+^n.$$

Thus W_t is its own random spectral function.

Example 7. Let $f(s)$, $s \in \mathbf{R}_+^n$, be a measurable random function assuming values in H and defined on the same probability space as the Chentsov field W_s, $s \in \mathbf{R}_+^n$, discussed in the preceding example. Assume that $f(s)$ is anticipating with respect to W_s for all $s \in \mathbf{R}_+^n$ and the function $\mathsf{E}\,\|f(s)\|^2$ is locally integrable on $\mathbf{R}_+^n$ (see [5]). Then the Itô type stochastic integrals

$$I_t(f) = \int_0^{t_1} \dots \int_0^{t_n} dW_s f(s), \qquad t = (t_1,\dots,t_n) \in \mathbf{R}_+^n,$$

are well defined and

$$\mathsf{E}\, I_t(f) = 0, \qquad t \in \mathbf{R}_+^n,$$

$$\mathsf{E}\, I_s(f) I_t(g) = \int_0^{t_1 \wedge s_1} \dots \int_0^{t_n \wedge s_n} \mathsf{E}\big(f(\lambda)|g(\lambda)\big)_H\, d\lambda, \qquad s,t \in \mathbf{R}_+^n.$$

Thus $I_t(f)$, $t \in \mathbf{R}_+^n$, is a stationary random field on $S = (\mathbf{R}_+^n, \bigwedge^n, I_d)$. Using the preceding example, one can easily prove that the structure spectral measure of the field $I_t(f)$, $t \in \mathbf{R}_+^n$, is

$$F(d\lambda) = \mathsf{E}\,\|f(\lambda)\|^2\,d\lambda, \qquad \lambda \in \mathbf{R}_+^n,$$

while its random spectral measure is

$$\Phi(d\lambda) = dW_\lambda f(\lambda), \qquad \lambda \in \mathbf{R}_+^n.$$

Now we consider the reduction of a generalized second order random function in H defined on a certain set $\mathbb{V}$ to a stationary function on a $*$-semigroup by transforming the space of its arguments. We also obtain the corresponding spectral representations in this case. Similar questions are discussed in [3, 11] for real-valued processes and fields.

Theorem 4. *Let Ξ_υ, $\upsilon \in \mathbb{V}$, be a generalized random function in H defined on a set $\mathbb{V}$, $\Xi_\upsilon \in \mathcal{M}(\Omega, H)$, and let $(\mathbb{S}, \bullet, *)$ be an involutive semigroup with a neutral element. Assume that there is a bijection f between the sets $\mathbb{V}$ and $\mathbb{S}$ such that the correlation kernel $[\Xi_\upsilon, \Xi_u] = K(\upsilon, u)$ of the function Ξ_υ is of the form*

$$K(\upsilon, u) = R((f(\upsilon)) \bullet (f(u))^*), \qquad \upsilon, u \in \mathbb{V}, \tag{16}$$

where R is an exponentially bounded $B(H)$-valued positive definite function on $\mathbb{S}$. Then $K(\upsilon, u)$, $\upsilon, u \in \mathbb{V}$, and the function Ξ_υ, $\upsilon \in \mathbb{V}$, admit a kind of spectral representation that will be specified in the proof of the theorem.

Proof. The bijection $f\colon \mathbb{V} \to \mathbb{S}$ generates on $\mathbb{V}$ the structure of the involutive semigroup $(\mathbb{V}, \bigstar, \sharp)$, where $\upsilon \bigstar u = f^{-1}((f(\upsilon)) \bullet (f(u)))$ and $\upsilon^\sharp = f^{-1}((f(\upsilon))^*)$ for $\upsilon, u \in \mathbb{V}$. The mapping f is isomorphic for semigroups $\mathbb{V}$ and $\mathbb{S}$. In this case, the semicharacters of $\mathbb{V}$ are $\widetilde{\chi} = \chi \circ f$, $\chi \in \mathbb{S}^*$, while the positive definite (with respect to the structure of the involutive semigroup) $B(H)$-valued functions on $\mathbb{V}$ are of the form $\widetilde{R} = R \circ f$, where R are $B(H)$-valued positive definite functions on $\mathbb{S}$.

By Theorem 2, the function $\widetilde{R}$ admits the spectral representation

$$\widetilde{R}(\upsilon) = R(f(\upsilon)) = \int_{\mathbb{S}^*} \chi(f(\upsilon))\,F(d\chi) = \int_{\mathbb{V}^*} \widetilde{\chi}(\upsilon)\,F_f(d\widetilde{\chi}), \qquad \upsilon \in \mathbb{V}, \tag{17}$$

where F_f denotes the image of the $B_+(H)$-valued measure F for the mapping

$$\chi \to \widetilde{\chi} = \chi \circ f$$

acting from $\mathbb{S}^*$ to $\mathbb{V}^*$. By assumption (16), representation (17) implies that the correlation kernel K of the function Ξ_υ, $\upsilon \in \mathbb{V}$, has the spectral representation

$$K(\upsilon, u) = \int_{\mathbb{S}^*} \chi(f(\upsilon))\overline{\chi(f(u))}\,F(d\chi) = \int_{\mathbb{V}^*} \widetilde{\chi}(\upsilon)\overline{\widetilde{\chi}(u)}\,F_f(d\widetilde{\chi}), \qquad \upsilon, u \in \mathbb{V}, \tag{18}$$

where F and F_f are uniquely defined $B_+(H)$-valued measures on $\mathbb{S}^*$ and $\mathbb{V}^*$, respectively (the spectral measures of the function Ξ_υ).

Then Theorem 3 in [10] implies the following spectral representation for Ξ_υ, $\upsilon \in \mathbb{V}$:

$$\Xi_\upsilon = \int_{\mathbb{S}^*} \chi(f(\upsilon))\,\Phi(d\chi) = \int_{\mathbb{V}^*} \widetilde{\chi}(\upsilon)\,\Phi_f(d\widetilde{\chi}), \qquad \upsilon \in \mathbb{V}, \tag{19}$$

where Φ and Φ_f are $\mathcal{M}(\Omega, H)$-valued orthogonal random measures on $\mathbb{S}^*$ and $\mathbb{V}^*$ whose structure measures are F and F_f, respectively (the random spectral measures of the function Ξ_υ). □

Example 8. Let Ξ_t, $t = (t_1, t_2) \in \mathbf{R}^2_{++} = (0, +\infty)^2$, be a normalized Lévy sheet in H; that is, Ξ_t is an $\mathcal{M}(\Omega, H)$-valued Gaussian field with the correlation function

$$K(t,s) = \frac{\|t\| + \|s\| - \|t-s\|}{2\sqrt{\|t\| \cdot \|s\|}} I, \qquad t, s \in \mathbf{R}^2_{++},\ u = (u_1, u_2) \in \mathbf{R}^2,$$

where I is the identity operator in H. The transformation $f(t) = (\ln \|t\|, \arctan(t_2/t_1))$ reduces Ξ_t to a weakly homogeneous random field $\Psi_{f(t)}$ on $\mathbb{S} = \mathbf{R}^2$, since

$$K(t,s) = R(f(t) - f(s))$$

for

$$R(u) = \left[\cosh\left(\frac{u_1}{2}\right) - \sqrt{\frac{(\cosh(u_1/2) - \cos u_2)}{2}}\right] I, \qquad u = (u_1, u_2) \in \mathbf{R}^2.$$

Determining appropriate transformations of the space of values of stationary random functions is another approach to obtaining spectral representations for nonstationary random functions on semigroups.

Example 9. Let Ξ_t, $t \in \mathbb{S}$, be a stationary random function on a $*$-semigroup $\mathbb{S}$ in H that admits (perhaps, under some additional assumptions) spectral representations of the integral form (15) similar to those in Theorems 2 or 3. Let A be an operator of $B(H)$, and let Q be an operator in $B(L_2(\Omega))$. Then the function $\Psi_t = Q\Xi_t A$, $t \in \mathbb{S}$, is harmonizable on $\mathbb{S}$; that is, Ψ admits the spectral representation

$$\Psi_t = \int_\Lambda \chi_\lambda(t)\, Z(d\lambda), \qquad t \in \mathbb{S},$$

where Z is a random $\mathcal{M}(\Omega, H)$-valued measure on Λ of the form $Z(d\lambda) = Q\,\Phi(d\lambda)A$ with Φ being the spectral random measure of the function Ξ_t, $t \in \mathbb{S}$ (see [15]). Moreover, the correlation kernel $K(t,s) = [\Psi_t, \Psi_s]$, $t, s \in \mathbb{S}$, of the function Ψ_t admits the following spectral representation:

$$K(t,s) = \int_\Lambda \int_\Lambda \chi_{\lambda_1}(t)\overline{\chi_{\lambda_2}(s)}\, G(d\lambda_1, d\lambda_2), \qquad t, s \in \mathbb{S},$$

where G is a $B(H)$-valued positive definite bimeasure on $\Lambda \times \Lambda$:

$$G(d\lambda_1, d\lambda_2) = [Z(d\lambda_1), Z(d\lambda_2)]$$

(see [10]).

3. Stationary processes and fields on classical semigroups

Below we apply the general results of spectral analysis of stationary functions on $*$-semigroups for several classes of stationary processes and fields on classical additive or multiplicative semigroups.

First we consider the case of additive stationary (A-stationary) processes on additive one-dimensional $*$-semigroups.

The stationarity of a stochastic process Ξ_t coincides with the usual stationarity in the case of the group $(\mathbf{R}, +, ^-))$ of real numbers or of the group $(\mathbb{Z}, +, ^-))$ of integers with respect to the addition and where the involution transforms a real number to its opposite, since the semicharacters are $\chi_\lambda(t) = e^{i\lambda t}$, $\lambda \in \Lambda$, in this case (where $\Lambda = \mathbf{R}$ or $\Lambda = \Pi = [-\pi, \pi]$, respectively).

If the $*$-semigroup $\mathbb{S}$ coincides with one of the following classical sets of numbers

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}_+, \quad \mathbb{Q}, \quad \Pi, \quad \mathbf{R}_+, \quad \mathbf{R}$$

considered together with the addition $+$ and where the involution is the identical transformation Id, then the semicharacters of the $*$-semigroup $(\mathbb{S}, +, \mathrm{Id})$ are $\chi_\lambda(t) = e^{\lambda t}$,

$\lambda \in \Lambda \subset \mathbf{R}$, and the correlation function R of the stationary process Ξ_t, $t \in \mathbb{S}$, depends on the sum of arguments only:

$$[\Xi_t, \Xi_s] = R(t+s), \qquad t, s \in \mathbb{S}.$$

Bounded $(\mathbb{S}, +, \mathrm{Id})$-positive definite functions are usually called exponentially bounded according to the existing terminology for one-dimensional functions. The corresponding stationary processes for which the correlation functions are exponentially bounded are called symmetric processes (see, for example, [13, 14]).

In particular, a bounded symmetric process Ξ_t, $t \in (\mathbf{R}, +, \mathrm{Id})$, admits the following spectral representations:

$$\Xi_t = \int_0^\infty \lambda^t \, \Phi(d\lambda), \qquad R(t+s) = [\Xi_t, \Xi_s] = \int_0^\infty \lambda^{t+s} \, F(d\lambda),$$

since the semicharacters of $(\mathbf{R}, +, \mathrm{Id})$ can be written as $\chi_\lambda(t) = \lambda^t$, $\lambda \in \mathbf{R}_+$.

If Ξ_t, $t \in (\mathbb{N}, +, \mathrm{Id})$, is a bounded symmetric process, then it admits the following spectral representations:

$$\Xi_n = \int_{-1}^1 \lambda^n \, \Phi(d\lambda), \qquad R(n+m) = \int_{-1}^1 \lambda^{n+m} \, F(d\lambda), \qquad n, m \in \mathbb{N},$$

in view of the relationship between the spectral representation of its correlation function and the classical problem of moments.

Now we consider multiplicative stationary (M-stationary) processes on multiplicative one-dimensional $*$-semigroups.

The semicharacters of the semigroup $\mathbb{S} = (\mathbf{R}_0, \times, (\cdot)^{-1})$, $\mathbf{R}_0 = \mathbf{R} \setminus \{0\}$, are

$$\chi_\lambda(t) = |t|^{i\lambda}, \qquad \lambda \in \mathbf{R},$$

while those for $\mathbb{S} = (\mathbb{R}_{++}, \times, (\cdot)^{-1})$, $\mathbf{R}_{++} = (0, \infty)$, are $\chi_\lambda(t) = t^{i\lambda}$, $\lambda \in \mathbf{R}$. Thus the spectral representations of a bounded stationary process Ξ_t, $t \in (\mathbb{R}_{++}, \times, (\cdot)^{-1})$, are of the form

$$\Xi_t = \int_{-\infty}^\infty t^{i\lambda} \, \Phi(d\lambda), \qquad [\Xi_t, \Xi_s] = R\left(\frac{t}{s}\right) = \int_{-\infty}^\infty \left(\frac{t}{s}\right)^{i\lambda} \, F(d\lambda).$$

In particular, the Kolmogorov spiral (the normalized fractional Brownian motion) in H, that is, the centered Gaussian process Ξ_t, $t \in \mathbf{R}_{++}$, in H with the correlation function

$$K(t,s) = \frac{t^{2h} + s^{2h} - |t-s|^{2h}}{2(st)^h} I,$$

where I is the identity operator of $B(H)$ and h is the fractal dimension of Ξ_t, $h \in [0,1]$, also is an M-stationary process on $(\mathbf{R}_{++}, \times, (\cdot)^{-1})$ whose correlation function is given by

$$R(\upsilon) = \frac{1 + \upsilon^{2h} - |1-\upsilon|^{2h}}{2\upsilon^h} I.$$

The semicharacters of the $*$-semigroup $(\mathbb{S}, \times, \mathrm{Id})$ with $\mathbb{S} = \mathbb{N} \setminus \{0\}$, $\mathbb{Z} \setminus \{0\}$, $\mathbf{R}_{++}$, or $\mathbf{R}_0$ are $\chi_\lambda(t) = |t|^\lambda$, $\lambda \in \mathbf{R}$. In particular, the spectral representations of a bounded stationary process Ξ_t, $t \in (\mathbf{R}_0, \times, \mathrm{Id})$, are given by

$$\Xi_t = \int_{-\infty}^\infty |t|^\lambda \, \Phi(d\lambda), \qquad [\Xi_t, \Xi_s] = R(ts) = \int_{-\infty}^\infty |ts|^\lambda \, F(d\lambda).$$

Stationary random fields on classical $*$-semigroups are easy to study in the framework of spectral theory if their arguments are treated as elements of direct products of underlying $*$-semigroups. This approach is developed in [15] for the case of classical Abelian locally compact groups. Many examples of spectral representations for specific classes of homogeneous random fields on those groups are also obtained in [15].

The approach mentioned above is based on the following result.

Definition 3.1. A $*$-semigroup $(\mathbb{U},\star,*)$ is called the direct product $\otimes$ of two $*$-semigroups $(\mathbb{S},\circ,\flat)$ and $(\mathbb{T},\bullet,\sharp)$ if $\mathbb{U}=\mathbb{S}\times\mathbb{T}$; the operation in the product is defined by

$$(s_1,t_1)\star(s_2,t_2)=(s_1\circ s_2,t_1\bullet t_2),\qquad (s_i,t_i)\in\mathbb{S}\times\mathbb{T},\ i=1,2,$$

and the involution is given by $(s,t)^*=(s^\flat,t^\sharp)$, $(s,t)\in\mathbb{S}\times\mathbb{T}$. Then *the dual object* $\mathbb{U}^*$ *of the semigroup* $\mathbb{U}$ *has the following structure:* $\mathbb{U}^*=\{\chi=\chi_1\chi_2\colon \chi_1\in\mathbb{S}^*,\chi_2\in\mathbb{T}^*\}$ (see [7]).

For example, considering the $*$-semigroup $(\mathbf{R}^n,+,\mathrm{Id})$ as the n-th power (as the result of the n-tuple direct product) of the $*$-semigroup $(\mathbf{R},+,\mathrm{Id})$, we obtain its semicharacters: $\chi_\lambda(t)=e^{(\lambda|t)}$, $t\in\mathbf{R}^n$, $\lambda\in\mathbf{R}^n$, where $(\cdot\,|\,\cdot)$ is the scalar product in $\mathbf{R}^n$. Thus a bounded stationary field Ξ_t, $t\in(\mathbf{R}^n,+,\mathrm{Id})$, admits the following spectral representations:

$$\Xi_t=\int_{\mathbf{R}^n}e^{(\lambda|t)}\,\Phi(d\lambda),\qquad R(t+s)=[\Xi_t,\Xi_s]=\int_{\mathbf{R}^n}e^{(\lambda|t+s)}\,\Phi(d\lambda).$$

Other simple examples of stationary random fields can be given in the complex plane

$$(\mathbb{C},+)=(\mathbf{R},+)\otimes(\mathbf{R},+).$$

This is equivalent to the representation of a complex number z in the form

$$z=\operatorname{Re}z+i(\operatorname{Im}z).$$

By $\mathbb{C}_0$ we denote the set $\mathbb{C}\setminus\{0\}$. Various involutions $*$ can be considered on $(\mathbb{C},+)$ or on $(\mathbb{C}_0,+)$.

If the involution coincides with the inversion, then

$$\left(\mathbb{C}_0,+,(\cdot)^{-1}\right)^*=\left\{\chi_\lambda\colon \chi_\lambda(z)=e^{i(\lambda_1\operatorname{Re}z+\lambda_2\operatorname{Im}z)},\lambda=(\lambda_1,\lambda_2)\in\mathbf{R}^2\right\},$$

whence

$$\Xi_z=\int_{\mathbf{R}^2}e^{i(\lambda_1\operatorname{Re}z+\lambda_2\operatorname{Im}z)}\,\Phi(d\lambda_1,d\lambda_2).\tag{20}$$

If the involution coincides with the complex conjugation, then

$$(\mathbb{C},+,^-))^*=\left\{\chi_\lambda\colon \chi_\lambda(z)=e^{\lambda_1\operatorname{Re}z}e^{i\lambda_2\operatorname{Im}z}\right\},$$

while

$$(\mathbb{C},+,\mathrm{Id})^*=\left\{\chi_\lambda\colon \chi_\lambda(z)=e^{\lambda_1\operatorname{Re}z+\lambda_2\operatorname{Im}z}\right\}$$

if the involution is the identity operator. The above equalities allow one to also obtain spectral representations similar to (20) for these cases.

For the multiplicative semigroup of complex numbers $(\mathbb{C},\times)$ we have

$$(\mathbb{C},\times)=(\mathbf{R}_+,X)\otimes(\Pi,+)$$

(the addition on Π is modulo 2π), and this is equivalent to the trigonometric representation of a complex number z, namely to the representation $z=|z|e^{i\varphi}$, where $\varphi=\operatorname{Arg}z$. Then

$$(\mathbb{C}_0,\times,(\cdot)^{-1})=\left\{\chi_\lambda\colon \chi_\lambda(z)=|z|^{i\lambda_1}e^{i\lambda_2\varphi},(\lambda_1,\lambda_2)\in\mathbb{R}\times\mathbb{Z}\right\},$$

and we obtain the following spectral representation of a bounded stationary random field Ξ_z on $(\mathbb{C}_0,\times,(\cdot)^{-1})$:

$$\Xi_z=\Xi_{|z|\exp(i\varphi)}=\int_{\mathbf{R}}\sum_{\lambda_2\in\mathbb{Z}}|z|^{i\lambda_1}e^{i\lambda_2\varphi}\,\Phi_{\lambda_2}(d\lambda_1),$$

where $\Phi_{\lambda_2}(d\lambda_1)$ is a family of orthogonal $\mathcal{M}(\Omega, H)$-valued measures such that

$$[\Phi_k(\triangle_1), \Phi_j(\triangle_2)] = \delta_{kj} F_k(\triangle_1 \cap \triangle_2), \qquad \triangle_1, \triangle_2 \subset \mathbf{R},\ k, j \in \mathbb{Z},$$

and F_k are operator $B_+(H)$-valued measures on $\mathbf{R}$.

Bibliography

1. A. I. Ponomarenko, *Wide sense homogeneous random fields on semigroups and homogeneous spaces with values in a Banach space*, Teor. Veroyatn. Matem. Statist. **7** (1972), 110–121; English transl. in Theory Probab. Math. Statist. **7** (1973), 104–114. MR0322940 (48:1300)
2. ______, *Infinite-dimensional random fields on semigroups*, Teor. Veroyatn. Matem. Statist. **30** (1984), 136–142; English transl. in Theory Probab. Math. Statist. **30** (1984), 153–158. MR0800839 (87b:60077)
3. V. Girardin and R. Senoussi, *Semigroup stationary processes and spectral representation*, Bernoulli **9** (2003), 857–876. MR2047689 (2005d:60054)
4. O. I. Ponomarenko, *Second-order random linear functionals. I*, Teor. Ĭmovīr. Mat. Stat. **54** (1996), 137–146; English transl. in Theory Probab. Math. Statist. **54** (1997), 143–151. MR1644594 (99h:60011)
5. L. L. Ponomarenko, *Infinite dimensional stochastic partial differential equations with multiparameter Brownian motion*, Kibernetika **4** (1976), 98–106. MR0488294 (58:7849)
6. R. Lindahl and P. Maserick, *Positive-definite functions on involution semigroups*, Duke Math. J. **38** (1971), 771–782. MR0291826 (45:916)
7. C. Berg, Y. P. Gristensen, and P. Ressel, *Harmonic Analysis on Semigroups*, Springer-Verlag, New York, 1984. MR747302 (86b:43001)
8. H. Glökner, *Positive Definite Functions on Infinite-Dimensional Convex Cones*, Mem. Amer. Math. Soc. **166** (2003). MR2008256 (2004m:43008)
9. A. I. Ponomarenko, *Stochastic integrals with respect to generalized random orthogonal measures in Banach spaces*, Teor. Veroyatn. Matem. Statist. **33** (1985), 92–99; English transl. in Theory Probab. Math. Statist. **33** (1986), 103–110. MR898202 (88h:60009)
10. O. I. Ponomarenko, *Integral representation of random functions with values in locally convex spaces*, Teor. Ĭmovīr. Mat. Stat. **46** (1992), 132–141; English transl. in Theory Probab. Math. Statist. **46** (1993), 129–136. MR1196216 (93k:60123)
11. O. Perjin and R. Senoussi, *Reducing non-stationary stochastic processes to stationary by a time deformation*, Statist. Probab. Lett. **43** (1999), 393–397. MR1707949 (2000d:60066)
12. O. I. Ponomarenko, *Operator bimeasures and stochastic integrals in normed spaces*, Teor. Ĭmovīr. Mat. Stat. **47** (1992), 129–139; English transl. in Theory Probab. Math. Statist. **47** (1993), 129–137. MR1272232 (95b:28010)
13. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Fizmatgiz, Moscow, 1961; English transl., Hafner Publishing Co., New York, 1965. MR0184042 (32:1518)
14. Y. Michalek, *Locally Stationary Covariances. Information Theory. Statistical Decision Functions*, Reidel, Dordrecht, 1988. MR1136266 (92i:60068)
15. A. I. Ponomarenko, *Stochastic Problems of Optimization*, Kiev University, Kiev, 1980. (Russian)

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Received 3/DEC/2004

Translated by V. V. SEMENOV