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ASYMPTOTIC ANALYSIS OF A MEASURE OF VARIATION

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Abstract. Let X_i, $i = 1, \dots, n$, be a sequence of positive independent identically distributed random variables and define

$$T_n := \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{(X_1 + X_2 + \cdots + X_n)^2}.$$

Utilizing Karamata's theory of functions of regular variation, we determine the asymptotic behaviour of arbitrary moments $\mathsf{E}(T_n^k)$, $k \in \mathbb{N}$, for large n, given that X_1 satisfies a tail condition, akin to the domain of attraction condition from extreme value theory. As a by-product, the paper offers a new method for estimating the extreme value index of Pareto-type tails.

1. Introduction

Let X_i, $i = 1, \dots, n$, be a sequence of positive independent identically distributed (i.i.d.) random variables with distribution function F and define

$$T_n := \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{(X_1 + X_2 + \cdots + X_n)^2}. \tag{1}$$

The asymptotic behaviour of $\mathsf{E}(T_n)$ was investigated in [5], simplifying and generalizing earlier results in [4] and [6].

In this paper we extend several results of [5] and derive the limiting behaviour of arbitrary moments

$$\mathsf{E}\big(T_n^k\big), \qquad k \in \mathbb{N}.$$

This is achieved by using an integral representation of $\mathsf{E}(T_n^k)$ in terms of the Laplace transform of X_1, which is derived in Section 2.

Most of our results will be derived under the condition that X_1 satisfies

$$1 - F(x) \sim x^{-\alpha}\ell(x), \qquad x \uparrow \infty, \tag{2}$$

where $\alpha > 0$ and $\ell(x)$ is slowly varying, i.e.

$$\lim_{x\to\infty} \ell(tx)/\ell(x) = 1$$

for all $t > 0$; see e.g. [3]. It is well known that condition (2) appears as the essential condition in the domain of attraction problem of extreme value theory. For a recent treatment, see [2]. A distribution satisfying (2) is called of *Pareto-type* with *index* α.

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When $\alpha < 2$, then the condition coincides with the domain of attraction condition for weak convergence to a nonnormal stable law. It is then obvious that for $\beta > 0$,

$$\mathsf{E}\big(X_1^\beta\big) := \mu_\beta = \beta \int_0^\infty x^{\beta-1}(1-F(x))\,dx \le \infty \tag{3}$$

will be finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$. For convenience, we define $\mu_0 := 1$ and $\mu := \mu_1$.

The results of this paper are based on the theory of functions of regular variation (see e.g. [3]). Clearly, if $\mathsf{E}(X_1) = \infty$, both the numerator and the denominator in (1) will exhibit an erratic behaviour, whereas for $\mathsf{E}(X_1) < \infty$ and $\mathsf{E}(X_1^2) = \infty$, this is the case only for the numerator. The results in Section 3 quantify this effect.

As a by-product, the results of this paper suggest a new method for estimating the extreme value index of Pareto-type distributions from a data set of observations, which is discussed in Section 4.

The quantity T_n is a basic ingredient in the study of the sample coefficient of variation of a given set of independent observations $X_1, \dots, X_n$ from a random variable X, which is a frequently used risk measure in practical applications. In [1], this connection will be used to derive asymptotic properties of the sample coefficient of variation, including a distributional approach.

2. Preliminaries

Let $\varphi(s) := \mathsf{E}(e^{-sX_1}) = \int_0^\infty e^{-sx}\,dF(x)$, $s \ge 0$, denote the Laplace transform of X_1. Then, following an idea of [5], one can use the identity

$$\frac{1}{x^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-sx} s^{\beta-1}\,ds, \qquad \beta > 0,$$

and Fubini's theorem to deduce that

$$\mathsf{E}\,\frac{1}{X_1^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1}\varphi(s)\,ds.$$

More generally, for i.i.d. random variables $X_1, \dots, X_n$, one obtains the representation formula

$$\mathsf{E}\,\frac{\prod_{i=1}^n X_i^{k_i}}{(X_1 + X_2 + \dots + X_n)^\beta} = \frac{(-1)^{k_1+\dots+k_n}}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} \prod_{i=1}^n \frac{\partial^{k_i}\varphi(s)}{\partial s^{k_i}}\,ds, \tag{4}$$

for nonnegative integers k_i, $i = 1, \dots, n$.

In particular, by symmetry,

$$\mathsf{E}(T_n) = \mathsf{E}\,\frac{X_1^2 + X_2^2 + \dots + X_n^2}{(X_1 + X_2 + \dots + X_n)^2} = n \int_0^\infty s\varphi''(s)\varphi^{n-1}(s)\,ds, \tag{5}$$

which formed the basis for the analysis in [5]. The representation (5) can be generalized in the following way:

Lemma 2.1. *For an arbitrary positive integer k,*

$$\mathsf{E}\left(T_n^k\right) = \sum_{r=1}^k \sum_{\substack{k_1,\dots,k_r \ge 1 \\ k_1+\dots+k_r = k}} \frac{k!}{k_1! \cdots k_r!} B(n, k_1, \dots, k_r) \tag{6}$$

with

$$B(n, k_1, \dots, k_r) = \frac{\binom{n}{r}}{\Gamma(2k)} \int_0^\infty s^{2k-1} \varphi^{(2k_1)}(s) \cdots \varphi^{(2k_r)}(s) \varphi^{n-r}(s)\,ds.$$

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Proof. For an arbitrary positive integer k we have

$$\mathsf{E}\left(T_n^k\right) = \mathsf{E}\,\frac{\left(X_1^2+X_2^2+\cdots+X_n^2\right)^k}{(X_1+X_2+\cdots+X_n)^{2k}} = \sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n=k}} \frac{k!}{k_1!\cdots k_n!}\,\mathsf{E}\,\frac{X_1^{2k_1}X_2^{2k_2}\cdots X_n^{2k_n}}{(X_1+X_2+\cdots+X_n)^{2k}},$$

where $k_i \leq k$ are nonnegative integers. Choose an n-tuple $(k_1,\ldots,k_n)$ in the above sum and let r denote the number of its nonzero elements $(k_{i_1},\ldots,k_{i_r})$ (clearly $1\leq r\leq k$). There are exactly $\binom{n}{r}$ possibilities of extending $(k_{i_1},\ldots,k_{i_r})$ to an n-tuple by filling in $n-r$ zeroes; each of the resulting n-tuples leads to the same summand in (6). Thus we can write

$$\mathsf{E}\left(T_n^k\right) = \sum_{r=1}^{k}\sum_{\substack{k_1,\ldots,k_r\geq 1\\ k_1+\cdots+k_r=k}} \frac{k!}{k_1!\cdots k_r!}\underbrace{\binom{n}{r}\,\mathsf{E}\,\frac{X_1^{2k_1}X_2^{2k_2}\cdots X_n^{2k_r}}{(X_1+X_2+\cdots+X_n)^{2k}}}_{:=B(n,k_1,\ldots,k_r)}, \tag{7}$$

so that (6) holds in view of (4). □

3. Main results

As promised, we will assume in the sequel that X_1 satisfies condition (2). Recall that when $\alpha > 1$, then $\mu < \infty$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. The finiteness of μ and/or μ_2 has its influence on the asymptotic behaviour of the summands that make up the statistic T_n. It is therefore not surprising that our results will be heavily dependent on the range of α. We state a first and general result.

Lemma 3.1. *If X_1 has a regularly varying tail with index $\alpha > 0$, i.e.*

$$1 - F(x) \sim x^{-\alpha}\ell(x),$$

then the asymptotic behaviour of the m-th derivative of the Laplace transform $\varphi(s)$ as $s \downarrow 0$ is given by

$$\varphi^{(m)}(s) \sim (-1)^m \alpha\Gamma(m-\alpha)s^{\alpha-m}\ell(1/s), \qquad m > \alpha. \tag{8}$$

Proof. Let $\chi(s) := \int_0^\infty e^{-sx}(1-F(x))\,dx$. Since $1-F(x)\sim x^{-\alpha}\ell(x)$, it follows that for $k > \alpha - 1$,

$$(-1)^k\chi^{(k)}(s) = \int_0^\infty x^k e^{-sx}(1-F(x))\,dx \sim \Gamma(k+1-\alpha)s^{-k-1}\left(1-F\left(\frac{1}{s}\right)\right) \quad \text{as } s\to 0.$$

Since $\varphi(s) = 1 - s\chi(s)$, we have for $m \geq 1$,

$$\varphi^{(m)}(s) = -m\chi^{(m-1)}(s) - s\chi^{(m)}(s),$$

so that for $m > \alpha$,

$$\begin{aligned}\frac{s^m\varphi^{(m)}(s)}{1-F(1/s)} &= -m\frac{s^m\chi^{(m-1)}(s)}{1-F(1/s)} - \frac{s^{m+1}\chi^{(m)}(s)}{1-F(1/s)}\\ &\sim (-1)^m\big(m\Gamma(m-\alpha)-\Gamma(m+1-\alpha)\big) = (-1)^m\alpha\Gamma(m-\alpha),\end{aligned}$$

from which the assertion follows. □

Theorem 3.1. *If X_1 belongs to the domain of attraction of a stable law with index α, $0<\alpha<1$, then for all $k\geq 1$,*

$$\lim_{n\to\infty}\mathsf{E}\left(T_n^k\right) = \frac{k!}{\Gamma(2k)}\sum_{r=1}^{k}\frac{\alpha^{r-1}}{r\Gamma(1-\alpha)^r}G(r,k), \tag{9}$$

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where $G(r,k)$ is the coefficient of x^k in the polynomial

$$\left(\sum_{j=1}^{k-r+1} \frac{\Gamma(2j-\alpha)}{j!} x^j\right)^r.$$

Proof. From $1-F(x) \sim x^{-\alpha}\ell(x)$ it follows that $1-\varphi(s) \sim \Gamma(1-\alpha)s^\alpha \ell(1/s)$ (see e.g. Corollary 8.1.7 in [3]). Moreover, for any sequence $(a_n)_{n\geq 1}$ with $a_n \to \infty$ we have

$$\begin{aligned}\varphi^n\left(\frac{s}{a_n}\right) &= \exp\{n \log \varphi(s/a_n)\} \sim \exp\{-n(1-\varphi(s/a_n))\}\\ &\sim \exp\left\{-n\left(\frac{s}{a_n}\right)^\alpha \ell\left(\frac{a_n}{s}\right)\Gamma(1-\alpha)\right\}.\end{aligned}$$

Choose $(a_n)_{n\geq 1}$ such that

$$na_n^{-\alpha}\ell(a_n)\Gamma(1-\alpha) \to 1 \quad \text{as } n \to \infty. \tag{10}$$

Then for all $s \geq 0$,

$$\lim_{n\to\infty} \varphi^n\left(\frac{s}{a_n}\right) = e^{-s^\alpha}.$$

We will now make use of the representation (6) for $\mathsf{E}(T_n^k)$. We have to investigate the asymptotic behaviour of $B(n,k_1,\ldots,k_r)$. The change of variables $s = t/a_n$ together with an application of Potter's theorem [3, Th. 1.5.6], Lebesgue's dominated convergence theorem and Lemma 3.1 leads to

$$\begin{aligned}B(n,k_1,\ldots,k_r) &= \frac{\binom{n}{r}}{a_n\Gamma(2k)}\int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{a_n}\right)\cdots\varphi^{(2k_r)}\left(\frac{t}{a_n}\right)\underbrace{\varphi^{n-r}\left(\frac{t}{a_n}\right)}_{\to e^{-t^\alpha}} dt\\ &\sim \frac{\alpha^r\binom{n}{r}}{a_n\Gamma(2k)}\int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1}\left(\frac{t}{a_n}\right)^{r\alpha-2k} \ell^r\left(\frac{a_n}{t}\right)\left(\prod_{j=1}^r \Gamma(2k_j-\alpha)\right)e^{-t^\alpha}\,dt\\ &\sim \frac{\alpha^r\prod_{j=1}^r\Gamma(2k_j-\alpha)}{\Gamma(2k)} \underbrace{\frac{\binom{n}{r}\ell^r(a_n)}{a_n{}^{r\alpha}}}_{\to\Gamma(1-\alpha)^{-r}/r!} \underbrace{\int_0^\infty t^{r\alpha-1}e^{-t^\alpha}\,dt}_{=(r-1)!/\alpha}\\ &\sim \frac{\alpha^{r-1}\prod_{j=1}^r\Gamma(2k_j-\alpha)}{r\Gamma(1-\alpha)^r\Gamma(2k)}.\end{aligned}$$

Summing over all $r = 1,\ldots,k$ in (6), we arrive at

$$\lim_{n\to\infty}\mathsf{E}\left(T_n^k\right) = \frac{k!}{(2k-1)!}\sum_{r=1}^k \frac{\alpha^{r-1}}{r\Gamma(1-\alpha)^r}\sum_{\substack{k_1,\ldots,k_r\geq 1\\ k_1+\cdots+k_r=k}}\prod_{j=1}^r \frac{\Gamma(2k_j-\alpha)}{k_j!}. \tag{11}$$

Now observe that

$$G(r,k) := \sum_{\substack{k_1,\ldots,k_r\geq 1\\ k_1+\cdots+k_r=k}}\prod_{j=1}^r \frac{\Gamma(2k_j-\alpha)}{k_j!}$$

can be determined by generating functions. Concretely, if we look at the r-fold product

$$\left(\Gamma(2-\alpha)x + \frac{\Gamma(4-\alpha)}{2!}x^2 + \cdots + \frac{\Gamma(2m-\alpha)}{m!}x^m\right)^r$$

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for m sufficiently large, then $G(r,k)$ can be read off as its coefficient of x^k, since the kth power exactly comprises all contributions of combinations $k_1,\dots,k_r \geq 1$ with

$$k_1 + \cdots + k_r = k$$

in the above sum. It suffices to choose $m = k-r+1$, since larger powers do not contribute to the coefficient of x^k any more. Hence Theorem 3.1 follows from (11). ☐

Remark 3.1. For $k=1$, we obtain $\lim_{n\to\infty} \mathsf{E}(T_n) = 1-\alpha$, which is Theorem 5.3 of [5]. The limit of moments of higher order can now be calculated from (9):

$$\lim_{n\to\infty} \mathsf{E}\left(T_n^2\right) = \frac{1}{3}(1-\alpha)(3-2\alpha),$$

$$\lim_{n\to\infty} \mathsf{E}\left(T_n^3\right) = \frac{1}{15}(1-\alpha)\left(15-17\alpha+5\alpha^2\right),$$

$$\lim_{n\to\infty} \mathsf{E}\left(T_n^4\right) = \frac{1}{105}(1-\alpha)\left(105-155\alpha+79\alpha^2-14\alpha^3\right),$$

$$\lim_{n\to\infty} \mathsf{E}\left(T_n^5\right) = \frac{1}{945}(1-\alpha)\left(945-1644\alpha+1106\alpha^2-344\alpha^3+42\alpha^4\right).$$

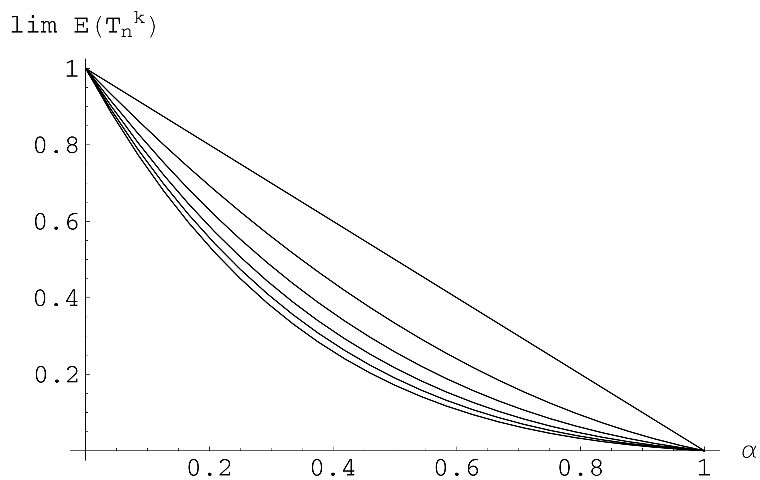

FIGURE 1. $\lim_{n\to\infty} \mathsf{E}(T_n^k)$ as a function of α ($k=1,\dots,5$ from top to bottom)

The following result generalizes Theorem 5.5 of [5], where the case $k=1$ was covered:

Theorem 3.2. *If X_1 belongs to the domain of attraction of a stable law with index $\alpha = 1$ and $\mathsf{E}(X_1) = \infty$, then for all $k \geq 1$,*

$$\mathsf{E}\left(T_n^k\right) \sim \frac{1}{2k-1}\frac{\ell(a_n)}{\tilde{\ell}(a_n)}, \tag{12}$$

where $\tilde{\ell}(x) = \int^x (\ell(t)/t)\,dt$ and $(a_n)_{n\geq 1}$ is a sequence satisfying $a_n \sim n\tilde{\ell}(a_n)$.

Proof. Since X_1 belongs to the domain of attraction of a stable law with index $\alpha = 1$, we have $1-F(x) \sim x^{-1}\ell(x)$ for some slowly varying function $\ell(x)$. Moreover

$$1-\varphi(s) \sim s\tilde{\ell}\left(\frac{1}{s}\right)$$

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with $\tilde{\ell}(x) = \int^x (\ell(t)/t)\,dt$ (see e.g. [3]). Note that $\tilde{\ell}(x)$ is again a slowly varying function. For any sequence $(a_n)_{n\geq 1}$ with $a_n \to \infty$ we have

$$\varphi^n\left(\frac{s}{a_n}\right) = \exp\{n\log\varphi(s/a_n)\} \sim \exp\{-n(1-\varphi(s/a_n))\} \sim \exp\left\{-n\left(\frac{s}{a_n}\right)\tilde{\ell}\left(\frac{a_n}{s}\right)\right\}.$$

If we choose a_n such that

$$na_n^{-1}\tilde{\ell}(a_n) \to 1 \quad \text{as } n\to\infty, \tag{13}$$

then

$$\lim_{n\to\infty}\varphi^n\left(\frac{s}{a_n}\right) = e^{-s}. \tag{14}$$

Take a_n as in (13) and replace s by t/a_n in the representation (6). An application of Potter's theorem, Lebesgue's dominated convergence theorem and Lemma 3.1 yields

$$\begin{aligned} B(n,k_1,\ldots,k_r) &= \frac{\binom{n}{r}}{a_n\Gamma(2k)}\int_0^\infty\left(\frac{t}{a_n}\right)^{2k-1}\varphi^{(2k_1)}\left(\frac{t}{a_n}\right)\cdots\varphi^{(2k_r)}\left(\frac{t}{a_n}\right)\underbrace{\varphi^{n-r}\left(\frac{t}{a_n}\right)}_{\to e^{-t}}dt \\ &\sim \frac{\binom{n}{r}}{a_n\Gamma(2k)}\int_0^\infty\left(\frac{t}{a_n}\right)^{2k-1}\left(\frac{t}{a_n}\right)^{r-2k}\ell^r\left(\frac{a_n}{t}\right)\left(\prod_{j=1}^r\Gamma(2k_j-1)\right)e^{-t}\,dt \\ &\sim \frac{\prod_{j=1}^r\Gamma(2k_j-1)}{r!\,\Gamma(2k)}\frac{n^r\ell^r(a_n)}{a_n{}^r}\underbrace{\int_0^\infty t^{r-1}e^{-t}\,dt}_{=(r-1)!} \\ &\sim \frac{\prod_{j=1}^r\Gamma(2k_j-1)}{r\Gamma(2k)}\left(\frac{\ell(a_n)}{\tilde{\ell}(a_n)}\right)^r. \end{aligned}$$

Note that $\ell(a_n)/\tilde{\ell}(a_n) \to 0$ for $n\to\infty$ and thus, opposed to the case $\alpha < 1$, only the summand with $r = 1$ contributes to the dominating asymptotic term of (6). Therefore we obtain

$$\mathsf{E}(T_n^k) \sim \frac{1}{2k-1}\frac{\ell(a_n)}{\tilde{\ell}(a_n)}. \qquad \square$$

Theorem 3.3. *Let X_1 belong to the domain of attraction of a stable law with index α, $1\leq\alpha<2$ and $\mu := \mathsf{E}(X_1) < \infty$. Then for all $k\geq 1$,*

$$\mathsf{E}\left(T_n^k\right) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\mu^\alpha}n^{1-\alpha}\ell(n). \tag{15}$$

Proof. Since μ is finite, it follows that

$$\lim_{n\to\infty}\varphi^n(t/n) = e^{-\mu t} \quad \text{for all } t\geq 0. \tag{16}$$

However, in view of (16), we will use the change of variables $s = t/n$ in the representation (6). By virtue of Potter's theorem, Lebesgue's dominated convergence theorem

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and Lemma 3.1 we then obtain

$$\begin{aligned}
B(n,k_1,\dots,k_r) &= \frac{\binom{n}{r}}{n\Gamma(2k)}\int_0^\infty \left(\frac{t}{n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{n}\right)\cdots\varphi^{(2k_r)}\left(\frac{t}{n}\right)\underbrace{\varphi^{n-r}\left(\frac{t}{n}\right)}_{\to e^{-\mu t}} dt \\
&\sim \frac{\alpha^r\binom{n}{r}}{n\Gamma(2k)}\int_0^\infty \left(\frac{t}{n}\right)^{2k-1}\left(\frac{t}{n}\right)^{r\alpha-2k}\ell^r\left(\frac{n}{t}\right)\left(\prod_{j=1}^r \Gamma(2k_j-\alpha)\right)e^{-\mu t}\,dt \\
&\sim \frac{\alpha^r\prod_{j=1}^r\Gamma(2k_j-\alpha)}{\Gamma(2k)}\quad\underbrace{\frac{\binom{n}{r}\ell^r(n)}{n^{r\alpha}}}_{\sim n^{r(1-\alpha)}\ell^r(n)/r!}\quad\underbrace{\int_0^\infty t^{r\alpha-1}e^{-\mu t}\,dt}_{=\Gamma(r\alpha)/\mu^{r\alpha}} \\
&\sim \frac{\alpha^r\Gamma(r\alpha)\prod_{j=1}^r\Gamma(2k_j-\alpha)}{r!\,\mu^{r\alpha}\Gamma(2k)}n^{r(1-\alpha)}\ell^r(n).
\end{aligned}$$

Hence the first-order asymptotic behaviour of (6) is solely determined by the term with $r=1$ and we obtain

$$\mathsf{E}\left(T_n^k\right) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\mu^\alpha}n^{1-\alpha}\ell(n). \qquad \square$$

Remark 3.2. For the special case $k=1$, (15) yields

$$\mathsf{E}\left(T_n\right) \sim \frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^\alpha}n^{1-\alpha}\ell(n),$$

which is Theorem 5.1 of [5].

We pass to the case $\alpha > 2$.

Theorem 3.4. *Let $1-F(x)\sim x^{-\alpha}\ell(x)$ for some slowly varying function $\ell(x)$ and $\alpha>2$. Then for all integers $k<\alpha-1$,*

$$\mathsf{E}\left(T_n^k\right) \sim \left(\frac{\mu_2}{\mu^2}\right)^k n^{-k} \tag{17}$$

and for $k>\alpha-1$,

$$\mathsf{E}\left(T_n^k\right) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\mu^\alpha}n^{1-\alpha}\ell(n). \tag{18}$$

If $k=\alpha-1$, then

(i) (17) *holds if $\ell(x)=o(1)$ (and in particular if $\mathsf{E}(X_1^{k+1})<\infty$),*

(ii)
$$\mathsf{E}(T_n^k) \sim \left(\left(\frac{\mu_2}{\mu^2}\right)^k + C\frac{\Gamma(k-1)\Gamma(k+2)}{\Gamma(2k)\mu^{1+k}}\right)n^{-k}$$
holds if $\ell(x)\sim C$ for a constant C,

(iii) *else* (18) *holds.*

Proof. Let us look at the quantity $B(n,k_1,\dots,k_r)$. By Lemma 3.1 and the Bingham–Doney lemma (see e.g. [3, Th. 8.1.6]) the asymptotic behaviour of $\varphi^{(m)}(s)$ at the origin is given by

$$(-1)^m\varphi^{(m)}(s) \sim \begin{cases} \alpha\Gamma(m-\alpha)s^{\alpha-m}\ell(1/s) & \text{if } m>\alpha, \\ \alpha\tilde{\ell}(1/s) & \text{if } m=\alpha \text{ and } \mathsf{E}(X_1^m)=\infty, \\ \mu_m & \text{if } m\le\alpha \text{ and } \mathsf{E}(X_1^m)<\infty, \end{cases}$$

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where $\tilde{\ell}(x) = \int_0^x (\ell(u)/u)\, du$ is itself a slowly varying function. For simplicity, let us first assume that $\alpha \notin \mathbb{N}$. Then one can conclude in an analogous way as in the proof of Theorem 3.3 that the asymptotic behaviour of $B(n, k_1, \ldots, k_r)$ is given by

$$B(n, k_1, \ldots, k_r) \sim C_1 n^{r-\alpha r_1 - 2(k-u_1)} \ell^{r_1}(n),$$

where r_1 is the number of integers among $k_1, \ldots, k_r$ that are greater than $\alpha/2$, u_1 is the sum of these and C_1 is some constant. It remains to determine the dominating asymptotic term among all possible $B(n, k_1, \ldots, k_r)$: If $r_1 > 0$, then $r_1 = 1$, $u_1 = k$ and thus $r = 1$ yields the largest exponent, so that the asymptotic order is $n^{1-\alpha}\ell(n)$. Note that $r_1 > 0$ is possible for $2k > \alpha$ only. For $r_1 = 0$, on the other hand, $r = k$ and thus $k_1 = \cdots = k_r = 1$ dominates, leading to asymptotic order n^{-k}. Hence the asymptotically dominating power among $B(n, k_1, \ldots, k_r)$ is given by $\max(1-\alpha, -k)$. From this we see that for $k < \alpha - 1$, $r = k$ dominates and we obtain from (6),

$$\mathsf{E}(T_n^k) \sim k!\, \frac{n^k \mu_2^k \Gamma(2k)}{k!\, \Gamma(2k) n^{2k} \mu^{2k}} \sim \left(\frac{\mu_2}{\mu^2}\right)^k n^{-k}.$$

Alternatively, if $k > \alpha - 1$, the term with $r = 1$ dominates and we obtain (18) in just the same way as in Theorem 3.3.

Finally, the above conclusions also hold for $\alpha \in \mathbb{N}$ except when $k = \alpha - 1$. In the latter case the slowly varying function $\ell(x)$ determines which of the two terms $n^{1-\alpha}\ell(n)$ (corresponding to $r = 1$) and n^{-k} (corresponding to $r = k$) dominates the asymptotic behaviour: if $\ell(x) = o(1)$ (which due to $\mathsf{E}(X_1^{k+1}) \sim (k+1)\int_0^n x^{-1}\ell(x)\, dx$ is in particular fulfilled for $\mathsf{E}(X_1^{k+1}) < \infty$), the second one dominates. If $\ell(x) \sim$ const, then both terms matter and the assertion of the theorem follows. $\square$

Corollary 3.1. *If $1 - F(x) \sim x^{-2}\ell(x)$, then for $k \geq 2$,*

$$\mathsf{E}(T_n^k) \sim \frac{1}{(k-1)(2k-1)\mu^2} \frac{\ell(n)}{n}$$

and

$$\mathsf{E}(T_n) \sim \begin{cases} \frac{\mu_2}{\mu^2 n} & \text{if } \mathsf{E}(X_1^2) < \infty, \\ \frac{2}{\mu^2} \frac{\tilde{\ell}(n)}{n} & \text{if } \mathsf{E}(X_1^2) = \infty. \end{cases}$$

Proof. One can easily verify that Theorem 3.4 remains true for $\alpha = 2$ except for $k = 1$ in the case $\mathsf{E}(X_1^2) = \infty$. In the latter case obviously $r = 1$ and one obtains (using $\varphi''(s) \sim 2\tilde{\ell}(1/s)$)

$$\mathsf{E}(T_n) \sim B(n,1) \sim \frac{2n\tilde{\ell}(n)}{n^2} \int_0^\infty t e^{-\mu t}\, dt \sim \frac{2}{\mu^2} \frac{\tilde{\ell}(n)}{n},$$

which is already contained in [5, Theorem 5.2]. $\square$

Remark 3.3. One might wonder whether a general limit result for $\mathsf{E}(T_n^k)$ for X_1 in the domain of attraction of a normal law (in the spirit of Theorem 5.2 of [5] for $k = 1$) can be obtained with the integral representation approach used in this paper. This is however not the case: From $\int_0^x y^2\, dF(y) \sim \ell_2(x)$ (where $\ell_2(x)$ is a slowly varying function) it follows by partial integration that $\varphi^{(2k)}(s)/\ell_2(1/s) = o(s^{2-2k})$ for $k > 1$ as $s \to 0$, but the latter is not strong enough to identify the dominating term among the $B(n, k_1, \ldots, k_r)$ without any further assumptions on the distribution of X_1.

As an illustration of the results of this paper, Table 1 gives the first order asymptotic terms of $\mathsf{E}(T_n)$, $\mathrm{Var}(T_n)$ and the dispersion $\mathrm{Var}(T_n)/\mathsf{E}(T_n)$ as a function of α. Note that the entries for $\alpha > 2$ have been obtained by calculating second-order asymptotic terms. The result for $\alpha > 4$ in the table actually holds whenever $\mu_4 < \infty$, since in this case

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TABLE 1. First order asymptotic terms of $\mathsf{E}(T_n)$, $\mathrm{Var}(T_n)$ and $\mathrm{Var}(T_n)/\mathsf{E}(T_n)$ for $1-F(x) \sim x^{-\alpha}\ell(x)$ as a function of α

	$\mathsf{E}(T_n)$	$\mathrm{Var}(T_n)$	$\mathrm{Var}(T_n)/\mathsf{E}(T_n)$
$0<\alpha<1$	$1-\alpha$	$\frac{\alpha(1-\alpha)}{3}$	$\frac{\alpha}{3}$
$\alpha=1$	$\frac{\ell(a_n)}{\tilde{\ell}(a_n)}(\to 0)$	$\frac{1}{3}\frac{\ell(a_n)}{\tilde{\ell}(a_n)}(\to 0)$	$\frac{1}{3}$
$1<\alpha<2$	$\frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^2}n^{1-\alpha}\ell(n)$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\mu^\alpha}n^{1-\alpha}\ell(n)$	$\frac{(3-\alpha)(2-\alpha)}{6}$
$\alpha=2$	$\frac{2}{\mu^2}\frac{\tilde{\ell}(n)}{n}$	$\frac{\ell(n)}{3n\mu^2}$	$\frac{1}{6}\frac{\ell(n)}{\tilde{\ell}(n)}(\to 0)$
$2<\alpha<4$	$\frac{\mu_2}{\mu^2 n}$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\mu^\alpha}n^{1-\alpha}\ell(n)$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\mu^{\alpha-2}\mu_2}n^{2-\alpha}\ell(n)$
$\alpha\ge 4$	$\frac{\mu_2}{\mu^2 n}$	$\frac{\mu_4\mu^2-\mu_2^2\mu^2+4\mu_2^3-4\mu\mu_2\mu_3}{\mu^6}\frac{1}{n^3}$	$\frac{\mu_4\mu^2/\mu_2-\mu_2\mu^2+4\mu_2^2-4\mu\mu_3}{\mu^4}\frac{1}{n^2}$

the derivation of second-order terms does not rely on the assumption of regular variation and one obtains

$$\mathsf{E}\left(T_n^2\right) = \frac{\mu_2^2}{\mu^4}\frac{1}{n^2} + \left(\frac{10\mu_2^3 - 3\mu_2^2\mu^2 - 8\mu\mu_2\mu_3 + \mu^2\mu_4}{\mu^6}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$\mathsf{E}^2(T_n) = \frac{\mu_2^2}{\mu^4}\frac{1}{n^2} + \left(\frac{6\mu_2^3 - 4\mu\mu_2\mu_3 - 2\mu^2\mu_2^2}{\mu^6}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

From Table 1 we see that the dispersion of T_n is a continuous function in α with its maximum in $\alpha = 1$ (see Figure 2).

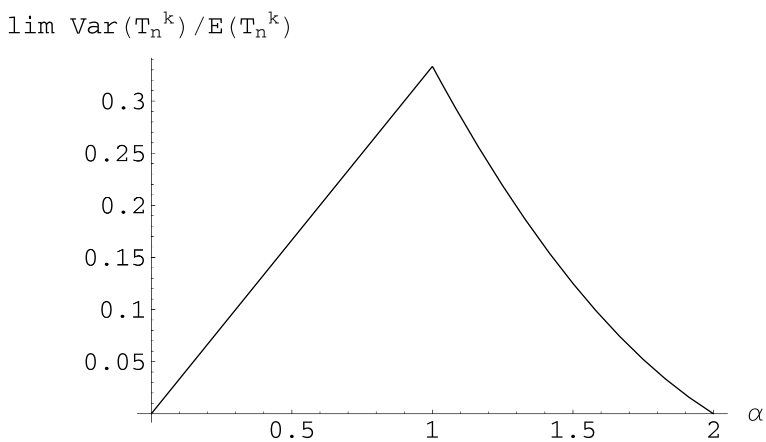

FIGURE 2. Limit of the dispersion of T_n as a function of α

4. Estimation of the extreme value index for Pareto-type tails

The results of Section 3 also give rise to an alternative and seemingly new method for estimating the extreme value index $1/\alpha$ for Pareto-type tails $1-F(x) \sim x^{-\alpha}\ell(x)$ with $0<\alpha<2$ from a given data set of independent observations (see e.g. [2] for other

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estimators of the extreme value index). In fact, plotting nT_n against n will tend to a line with slope $1-\alpha$, if $0 < \alpha < 1$ and plotting $\log(nT_n)$ against $\log n$ will tend to a line with slope $2 - \alpha$, if $1 < \alpha < 2$. The asymptotic behaviour of higher order moments of nT_n available from Section 3 can then be used to increase the efficiency of the estimation procedure.

At the same time, this provides a technique to test the finiteness of the mean of a distribution in the domain of attraction of a stable law.

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