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TAUBERIAN THEOREM FOR FIELDS WITH AN *OR* SPECTRUM. II

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Abstract. We consider homogeneous isotropic random fields whose spectra have some local singular properties. We prove Abelian and Tauberian theorems linking the local behavior of the spectral function and that of weighted integral functionals of random fields. Representations of weight functions in the form of the Hankel transform and series of functions are obtained. The asymptotic behavior is described in terms of functions of the class OR. Some examples are given.

1. Introduction

The problems of interest are to study the local behavior of the spectral functions and to obtain a relationship between this behavior and the asymptotics of some functionals of random fields. The behavior of the spectral function at zero is studied in [9]; Abelian and Tauberian theorems for the one-dimensional case and their applications are considered in [5]. We obtain a generalization of these results to the case of isotropic homogeneous fields. The conditions for these generalizations are expressed in terms of the variance of weighted integral functionals of random fields. In contrast to the paper [5], the conditions involve not only the integral functionals of the correlation functions but also the variances of certain weighted integrals of the random field itself. This simplifies applications of the results when studying the behavior of functionals of random fields.

Throughout the paper we use the notation introduced in [9]. As in [9], the symbols K and C with or without indices are used to denote constants (possibly dependent on the dimension of the space) whose exact values are not essential for our purposes. The same symbol may indicate different constants in the same proof. These constants are used without index n if the dimension of the space is obvious from the context.

2. Tauberian theorem and the variance of the integrals of random fields

We use some results from [9] to describe the local behavior of the spectral function $\Phi(\cdot)$ in a neighborhood of an arbitrary point $a \in [0, +\infty)$. We are interested in the asymptotic behavior of $\Phi(a+\lambda) - \Phi(a-\lambda)$ or $\Phi(a+\lambda) - \Phi(a)$ as $\lambda \to +0$.

2.1. **Case of $\Phi(a+\lambda) - \Phi(a-\lambda)$.** Consider the function $\Phi^a(\lambda)$, $\lambda \geq 0$, defined by

$$\Phi^a(\lambda) := \begin{cases} \Phi(a+\lambda) - \Phi(a-\lambda), & 0 \leq \lambda < a, \\ \Phi(a+\lambda), & \lambda \geq a. \end{cases} \tag{1}$$

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It is clear that $\Phi^a(\cdot)$ is a spectral function; thus the asymptotic behavior of

$$\Phi(a+\lambda) - \Phi(a-\lambda)$$

as $\lambda \to +0$ is determined by that of $\Phi^a(\lambda)$ at zero. In particular, the inclusion

$$\Phi\left(a+\frac{1}{\cdot}\right) - \Phi\left(a-\frac{1}{\cdot}\right) \in OR(\beta,\alpha)$$

is equivalent to $\Phi^a(1/\cdot) \in OR(\beta,\alpha)$. Therefore one can apply the results of [9] linking the asymptotic behavior of $\Phi^a(1/\cdot)$ at zero with the variance of integrals of the field over a sphere or ball.

Using the notation

$$\tilde{b}^a(r) := (2\pi)^n \int_0^\infty \frac{J^2_{n/2}(rx)}{(rx)^n}\, d\Phi^a(x),$$
$$\tilde{l}^a(r) := (2\pi)^n \int_0^\infty \frac{J^2_{(n-2)/2}(rx)}{(rx)^{n-2}}\, d\Phi^a(x),$$

we state the following result.

Theorem 1. *Let $0 > \alpha \geq \beta > -n$. Then the following conditions are equivalent:*

(i) $\Phi(a+\frac{1}{\cdot}) - \Phi(a-\frac{1}{\cdot}) \in OR(\beta,\alpha)$,

(ii) $\tilde{b}^a(\cdot) \in OR(\beta,\alpha)$,

(iii) $\tilde{b}^a(r) \asymp \Phi(a+1/r) - \Phi(a-1/r)$ *as $r \to \infty$ and there are constants C, C', and r_0 such that*

$$C'\left(\frac{r_1}{r}\right)^\beta \leq \frac{\Phi(a+1/r_1) - \Phi(a-1/r_1)}{\tilde{b}^a(r)} \leq C\left(\frac{r_1}{r}\right)^\alpha, \qquad r_1 \geq r \geq r_0.$$

If $0 > \alpha \geq \beta > 2-n$, then condition **(i)** *is equivalent to*

(iv) $\tilde{l}^a(\cdot) \in OR(\beta,\alpha)$,

(v) $\tilde{l}^a(r) \asymp \Phi(a+1/r) - \Phi(a-1/r)$ *as $r \to \infty$ and there are constants C, C', and r_0 such that*

$$C'\left(\frac{r_1}{r}\right)^\beta \leq \frac{\Phi(a+1/r_1) - \Phi(a-1/r_1)}{\tilde{l}^a(r)} \leq C\left(\frac{r_1}{r}\right)^\alpha, \qquad r_1 \geq r \geq r_0.$$

Rewrite $\tilde{b}^a(\cdot)$ with the help of definition (1) of the spectral function $\Phi^a(\cdot)$:

$$\begin{aligned}\tilde{b}^a(r) &= (2\pi)^n\left(\int_0^\infty \frac{J^2_{n/2}(r\lambda)}{(r\lambda)^n}\, d\Phi(a+\lambda) - \int_0^a \frac{J^2_{n/2}(r\lambda)}{(r\lambda)^n}\, d\Phi(a-\lambda)\right)\\ &= (2\pi)^n\left(\int_a^\infty \frac{J^2_{n/2}(r(\lambda-a))}{(r(\lambda-a))^n}\, d\Phi(\lambda) + \int_0^a \frac{J^2_{n/2}(r(a-\lambda))}{(r(a-\lambda))^n}\, d\Phi(\lambda)\right).\end{aligned}$$

Using formula (8) of [9], we prove that $J^2_{n/2}(\lambda)/\lambda^n$ is an even function, whence it follows that

$$\tilde{b}^a(r) = (2\pi)^n \int_0^\infty \frac{J^2_{n/2}(r(\lambda-a))}{(r(\lambda-a))^n}\, d\Phi(\lambda). \tag{2}$$

It is known (see [6, 10]) that

$$\tilde{b}^0(r) = \frac{1}{r^{2n}} \operatorname{Var}\left[\int_{v(r)} \xi(t)\, dt\right], \qquad \tilde{l}^0(r) = \frac{1}{r^{2(n-1)}} \operatorname{Var}\left[\int_{s(r)} \xi(t)\, dm(t)\right]$$

in the case of $a = 0$. The following question arises: is it possible to express the integral transforms $\tilde{b}^a(r)$ and $\tilde{l}^a(r)$ in terms of the variance of some integrals of random fields?

Theorem 2. *There exists a real-valued function $f_{r,a}(\cdot)$ such that*

$$\tilde{b}^a(r) = \mathrm{Var}\left[\int_{\mathbb{R}^n} f_{r,a}(|t|)\xi(t)\,dt\right]. \tag{3}$$

Moreover

$$f_{r,a}(|t|) = \frac{1}{|t|^{n/2-1}}\int_0^\infty \lambda^{n/2}\frac{J_{n/2}(r(\lambda-a))}{(r(\lambda-a))^{n/2}}J_{n/2-1}(|t|\lambda)\,d\lambda, \qquad |t|\neq r. \tag{4}$$

Proof. Assume that there exists a real-valued function $f_{r,a}(\cdot)$ such that

$$\tilde{b}^a(r) = \mathrm{Var}\left[\int_{\mathbb{R}^n} f_{r,a}(t)\xi(t)\,dt\right]. \tag{5}$$

The correlation function of the field $\xi(\cdot)$ is given by

$$\mathsf{B}_n(t-s) = \mathsf{B}_n(\|t-s\|) = \int_{\mathbb{R}^n} e^{i(x,t-s)}\,dF(x), \qquad t,s\in\mathbb{R}^n$$

(see [6, 10]); hence

$$\begin{aligned}\tilde{b}^a(r) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_{r,a}(t)f_{r,a}(s)\,\mathsf{B}_n(t-s)\,ds\,dt\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_{r,a}(t)e^{i(x,t)}\,dt\int_{\mathbb{R}^n} f_{r,a}(s)e^{i(x,-s)}\,ds\,dF(x).\end{aligned} \tag{6}$$

If $f_{r,a}(t)$ in (5) is a radial function, that is, $f_{r,a}(t_1) = f_{r,a}(t_2)$ for all $t_1,t_2\in\mathbb{R}^n$ such that $|t_1| = |t_2|$, then $\int_{\mathbb{R}^n} f_{r,a}(t)e^{i(x,t)}\,dt$ is also a radial function of the argument x. Using relation (4) of [10], we rewrite (6) as

$$\tilde{b}^a(r) = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f_{r,a}(t)e^{i(x,t)}\,dt\right)^2 dF(x) = \int_0^\infty\left(\int_{\mathbb{R}^n} f_{r,a}(t)e^{i(x,t)}\,dt\right)^2 d\Phi(|x|).$$

Defining $f_{r,a}(t)$ as a solution of the equation

$$\int_{\mathbb{R}^n} f_{r,a}(t)e^{i(x,t)}\,dt = (2\pi)^{n/2}\frac{J_{n/2}(r(|x|-a))}{(r(|x|-a))^{n/2}},$$

we derive representation (3) from equality (2). This suggests that $f_{r,a}(t)$ is the Fourier transform in the space $\mathbb{R}^n$ of the function $\frac{J_{n/2}(r(|x|-a))}{(r(|x|-a))^{n/2}}$, that is,

$$f_{r,a}(t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i(x,t)}\frac{J_{n/2}(r(|x|-a))}{(r(|x|-a))^{n/2}}\,dx.$$

Now we show that the above Fourier transform is well defined. It is sufficient to prove that

$$G_{r,a}(\cdot) := \frac{J_{n/2}(r(|\cdot|-a))}{(r(|\cdot|-a))^{n/2}} \in L_2(\mathbb{R}^n).$$

The limit of $G_{r,a}(|x|)$ as $|x|\to a$ exists by the definition of $G_{r,a}(\cdot)$ and Lemma 1 in [9]. Thus one can restrict the consideration to the asymptotic behavior of $G_{r,a}(\cdot)$ at infinity.

Let $\lambda := |x|$. Since

$$\int_{\mathbb{R}^n}\frac{J^2_{n/2}(r(|x|-a))}{(r(|x|-a))^n}\,dx = c\int_0^\infty \lambda^{n-1}\frac{J^2_{n/2}(r(\lambda-a))}{(r(\lambda-a))^n}\,d\lambda$$

and

$$\frac{J^2_{n/2}(r(\lambda-a))}{(r(\lambda-a))^n} = O\left(\frac{1}{\lambda^{n+1}}\right),$$

it follows that $G_{n,a}(\cdot)\in L_2(\mathbb{R}^n)$.

The function $G_{n,a}(\cdot)$ is radial; thus $f_{r,a}(\cdot)$ can be represented as the integral Hankel transform

$$f_{r,a}(|t|) = \frac{1}{|t|^{n/2-1}} \int_0^\infty \lambda^{n/2} \frac{J_{n/2}(r(\lambda-a))}{(r(\lambda-a))^{n/2}} J_{n/2-1}(|t|\lambda)\, d\lambda.$$

This result follows from the properties of the Fourier transform of radial functions that can be found in [3]. The theorem is proved. □

Remark 1. Integral transform (4) is well defined for all t, r, and a with the exception of $|t| = r$ and $ra \neq \pi k$, $k \in \mathbb{N} \cup \{0\}$. Indeed,

$$\begin{aligned} &\lambda^{n/2} \frac{J_{n/2}(r(\lambda-a))}{(r(\lambda-a))^{n/2}} J_{n/2-1}(|t|\lambda) \\ &\sim \frac{2}{\pi r^{n/2}\lambda} \cos\left(r(\lambda-a) - \frac{\pi n}{4} - \frac{\pi}{4}\right) \cos\left(|t|\lambda - \frac{\pi n}{4} + \frac{\pi}{4}\right) \\ &= \frac{1}{\pi r^{n/2}\lambda}\left[\cos\left((r-|t|)\lambda - ra - \frac{\pi}{2}\right) + \cos\left((r+|t|)\lambda - ra - \frac{\pi n}{2}\right)\right] \end{aligned}$$

as $\lambda \to \infty$ according to formula (10) in [9]. Therefore the integral in (3) diverges if $|t| = r$ and $\cos(ra + \frac{\pi}{2}) \neq 0$. Otherwise the integral is conditionally convergent. The function $f_{r,a}(t)$ can be defined for $|t| = r$ in an arbitrary way, since the values of $f_{r,a}(t)$ on a set of zero measure do not change the value of the integral in definition (3) of the function $\tilde{b}^a(r)$.

Remark 2. A similar approach applied to $\tilde{l}^a(r)$ does not result in the representation

$$\operatorname{Var}\left[\int_{\mathbb{R}^n} g_{r,a}(|t|)\xi(t)\, dt\right],$$

since $J_{(n-2)/2}(r(|\cdot| - a))(r(|\cdot| - a))^{(2-n)/2} \notin L_2(\mathbb{R}^n)$ and the integral

$$\int_0^\infty \lambda^{n/2} \frac{J_{(n-2)/2}(r(\lambda-a))}{(r(\lambda-a))^{(n-2)/2}} J_{n/2-1}(|t|\lambda)\, d\lambda$$

diverges.

2.2. **Case of $\Phi(a+\lambda) - \Phi(a)$.** Consider the function $\Phi_a(\lambda)$, $\lambda \geq 0$, defined by

$$\Phi_a(\lambda) := \Phi(a+\lambda) - \Phi(a), \qquad \lambda \geq 0. \tag{7}$$

It is clear that $\Phi_a(\cdot)$ is a spectral function. Thus the asymptotic behavior of

$$\Phi(a+\lambda) - \Phi(a)$$

as $\lambda \to +0$ is determined by that of $\Phi_a(\lambda)$ at zero. The results of [9] are helpful again.

Using the notation

$$\tilde{b}_a(r) := (2\pi)^n \int_0^\infty \frac{J^2_{n/2}(rx)}{(rx)^n}\, d\Phi_a(x), \qquad \tilde{l}_a(r) := (2\pi)^n \int_0^\infty \frac{J^2_{(n-2)/2}(rx)}{(rx)^{n-2}}\, d\Phi_a(x),$$

we state the following result.

Theorem 3. *Let $0 > \alpha \geq \beta > -n$. Then the following conditions are equivalent:*

- **(i)** $\Phi(a + \frac{1}{\cdot}) - \Phi(a) \in OR(\beta, \alpha)$,
- **(ii)** $\tilde{b}_a(\cdot) \in OR(\beta, \alpha)$,
- **(iii)** $\tilde{b}_a(r) \asymp \Phi(a + 1/r) - \Phi(a)$ *as $r \to \infty$ and there exist positive constants C, C', and r_0 such that*

$$C' \left(\frac{r_1}{r}\right)^\beta \leq \frac{\Phi(a+1/r_1) - \Phi(a)}{\tilde{b}_a(r)} \leq C \left(\frac{r_1}{r}\right)^\alpha, \qquad r_1 \geq r \geq r_0.$$

If $0 > \alpha \geq \beta > 2 - n$, *then condition* **(i)** *is equivalent to*

(iv) $\tilde{l}_a(\cdot) \in OR(\beta, \alpha)$,

(v) $\tilde{l}_a(r) \asymp \Phi(a + 1/r) - \Phi(a)$ *as* $r \to \infty$ *and there exist positive constants* C, C', *and* r_0 *such that*

$$C' \left(\frac{r_1}{r}\right)^\beta \leq \frac{\Phi(a + 1/r_1) - \Phi(a)}{\tilde{l}_a(r)} \leq C \left(\frac{r_1}{r}\right)^\alpha, \qquad r_1 \geq r \geq r_0.$$

We rewrite the expression for $\tilde{b}_a(\cdot)$ by using definition (7) of the spectral function $\Phi_a(\cdot)$:

$$\tilde{b}_a(r) = (2\pi)^n \int_0^\infty \frac{J^2_{n/2}(r\lambda)}{(r\lambda)^n} \, d\Phi(a + \lambda) = (2\pi)^n \int_a^\infty \frac{J^2_{n/2}(r(\lambda - a))}{(r(\lambda - a))^n} \, d\Phi(\lambda).$$

Similarly to Theorem 2 we obtain the integral transform of $\tilde{b}_a(r)$ expressed in terms of the variance of weighted integrals of random fields.

Theorem 4. *There exists a real-valued function* $g_{r,a}(\cdot)$ *such that*

$$\tilde{b}_a(r) = \mathrm{Var}\left[\int_{\mathbb{R}^n} g_{r,a}(|t|)\xi(t) \, dt\right]. \tag{8}$$

Moreover

$$g_{r,a}(|t|) = \frac{1}{|t|^{n/2-1}} \int_0^\infty (\lambda + a)^{n/2} J_{n/2-1}(|t|(\lambda + a)) \frac{J_{n/2}(r\lambda)}{(r\lambda)^{n/2}} \, d\lambda, \qquad |t| \neq r. \tag{9}$$

Proof. We follow the idea of the proof of Theorem 2. Assume that there exists a real-valued function $g_{r,a}(\cdot)$ satisfying (8). Then

$$\tilde{b}_a(r) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g_{r,a}(t) e^{i(x,t)} \, dt\right)^2 dF(x) = \int_0^\infty \left(\int_{\mathbb{R}^n} g_{r,a}(t) e^{i(x,t)} \, dt\right)^2 d\Phi(|x|).$$

Defining $g_{r,a}(t)$ as a solution of the equation

$$\int_{\mathbb{R}^n} g_{r,a}(t) e^{i(x,t)} \, dt = \begin{cases} (2\pi)^{n/2} \frac{J_{n/2}(r(|x|-a))}{(r(|x|-a))^{n/2}}, & |x| \geq a, \\ 0, & |x| < a, \end{cases}$$

we obtain representation (8).

Since $G_{r,a}(\cdot) \in L_2(\mathbb{R}^n)$, the function $g_{r,a}(t)$ is the Fourier transform in the space $\mathbb{R}^n$ of the function

$$\chi_{[a,\infty)}(|x|) \frac{J_{n/2}(r(|x| - a))}{(r(|x| - a))^{n/2}},$$

that is,

$$g_{r,a}(t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x,t)} \chi_{[a,\infty)}(|x|) \frac{J_{n/2}(r(|x| - a))}{(r(|x| - a))^{n/2}} \, dx.$$

The function $\chi_{[a,\infty)}(|x|) G_{n,a}(|x|)$ is radial; hence $g_{r,a}(\cdot)$ can be expressed as the integral Hankel transform

$$g_{r,a}(|t|) = \frac{1}{|t|^{n/2-1}} \int_a^\infty \lambda^{n/2} \frac{J_{n/2}(r(\lambda - a))}{(r(\lambda - a))^{n/2}} J_{n/2-1}(|t|\lambda) \, d\lambda.$$

The change of the variable results in representation (9) for $g_{r,a}(|t|)$. □

We describe the methods for the evaluation of functions $f_{r,a}(|t|)$ and $g_{r,a}(|t|)$ in the next sections. We also consider some examples.

3. Evaluation of $f_{r,a}(|t|)$

The author did not find in the literature the evaluation of integrals (4) and (9) for an arbitrary n (the case of $a = 0$ is an exemption). Handbooks and scientific papers devoted to the Hankel transform also do not contain any discussion of the methods for the evaluation of these integrals. Unfortunately, *Maple* and *Mathematica* do not evaluate these integrals either. Cumbersome expressions for the integrals obtained for some n in Section 5 lead to a conclusion that there is no simple formula for the integrals (4) and (9). In this section, we propose a method for evaluating $f_{r,a}(|t|)$ as the sum of some infinite function series. Note that the standard methods allowing one to express Bessel functions as power series (say, formula (8) in [9]) are not applicable in the case of integrals (4) and (9), since the term by term integration in our case leads to divergent integrals. In fact, the real reason for the problem of the evaluation of integrals (4) and (9) is that they are conditionally convergent and do not converge absolutely (see Remark 1 above).

In what follows, we use the notation ${}_2F_1(\alpha_1, \alpha_2; \beta; z)$ for the Gauss hypergeometric function [1]:

$$
{}_2F_1(\alpha_1, \alpha_2; \beta; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k}{(\beta)_k} \frac{z^k}{k!},
$$
$$
(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1), \quad k \in \mathbb{N}.
$$

Theorem 5. *The function $f_{r,a}(|t|)$ defined in Theorem 2 can be represented as the sum of the function series*

$$
\left(\frac{2}{ar^3}\right)^{n/2} \sum_{m=0}^{\infty} \frac{\left(\frac{n}{2}+m\right) C_{n+m-1}^{m} \Gamma(\frac{n+m}{2}) J_{n/2+m}(ra)\, {}_2F_1\left(\frac{n+m}{2}, -\frac{m}{2}; \frac{n}{2}; \left(\frac{|t|}{r}\right)^2\right)}{\Gamma(\frac{m}{2}+1)}
$$

for $|t| < r$ and

$$
\left(\frac{2}{ar|t|^2}\right)^{n/2} \Gamma\left(\frac{n}{2}\right) \sum_{m=0}^{\infty} \frac{r^{2m+1} C_{n+2m}^{2m+1} \Gamma(\frac{n}{2}+m+\frac{1}{2}) J_{n/2+2m+1}(ra)}{|t|^{2m+1} \Gamma(-m-\frac{1}{2}) \Gamma\left(\frac{n}{2}+2m+1\right)}
$$
$$
\times\, {}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}; \frac{n}{2}+2m+2; \left(\frac{r}{|t|}\right)^2\right)
$$

for $|t| > r$.

Proof. We make use of formula (4) from §11.41 of [4]:

$$
\frac{J_\nu(\omega)}{\omega^\nu} = 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m) \frac{J_{\nu+m}(Z)}{Z^\nu} \frac{J_{\nu+m}(z)}{z^\nu} C_m^\nu(\cos\varphi) \tag{10}
$$

where

$$
\omega = \sqrt{Z^2 + z^2 - 2zZ\cos\varphi}
$$

(the value of the square root is chosen such that $\omega \to +Z$ as $z \to 0$). Here $C_m^\nu(\cos\varphi)$ denotes the coefficient of α^m in the expansion of $(1 - 2\alpha\cos\varphi + \alpha^2)^{-\nu}$ in increasing powers of α, $\nu \neq 0, -1, -2, \ldots$.

Let $\nu = n/2$, $Z = r\lambda$, $z = ra$, and $\varphi = 0$. Then $C_m^\nu(\cos\varphi)$ is the coefficient of $(1-\alpha)^{-n}$ in the binomial series; thus $C_m^\nu(\cos\varphi) = C_{n+m-1}^m$.

It follows from (10) that

$$
\frac{J_{n/2}(r\lambda - ra)}{(r\lambda - ra)^{n/2}} = 2^{n/2} \Gamma\left(\frac{n}{2}\right) \sum_{m=0}^{\infty} \left(\frac{n}{2}+m\right) \frac{J_{n/2+m}(r\lambda)}{(r\lambda)^{n/2}} \frac{J_{n/2+m}(ra)}{(ra)^{n/2}} C_{n+m-1}^m. \tag{11}
$$

We use the representation (see §3.1 of [4])

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta$$

to obtain the bound

$$\left|\frac{J_\nu(z)}{z^\nu}\right| \le \frac{\sqrt{\pi}}{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}. \tag{12}$$

Therefore

$$\begin{aligned}\sum_{m=0}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{|J_{n/2+m}(r\lambda)|}{(r\lambda)^{n/2}}\frac{|J_{n/2+m}(ra)|}{(ra)^{n/2}}\\ \le \sum_{m=0}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{\pi}{2^{n+2m}\Gamma^2\left(\frac{n+1}{2}+m\right)}(r\lambda)^m(ra)^m.\end{aligned} \tag{13}$$

By the Stirling formula

$$\left(\frac{n}{2}+m\right)\frac{(n+m-1)!}{m!\,2^{2m}\Gamma^2\left(\frac{n+1}{2}+m\right)} \sim c\left(\frac{e^2}{4m}\right)^m \frac{1}{\left(\frac{n}{2}+m-1\right)^m} \sim c\left(\frac{e}{2m}\right)^{2m} \tag{14}$$

as $m\to\infty$; hence the series on the right hand side of (13) converges.

One can substitute the right hand side of (11) in integral (4) and integrate it term by term if the function series

$$2^{n/2}\Gamma\left(\frac{n}{2}\right)\sum_{m=0}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{J_{n/2+m}(r|x|)}{(r|x|)^{n/2}}\frac{J_{n/2+m}(ra)}{(ra)^{n/2}}$$

converges to

$$\frac{J_{n/2}(r|x|-ra)}{(r|x|-ra)^{n/2}}$$

pointwise and also in $L^2(\mathbb{R}^n)$. This implies the convergence of the corresponding Fourier transforms.

Therefore our next goal is to prove that

$$\int_{\mathbb{R}^n}\left(\sum_{m=M}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{J_{n/2+m}(r|x|)}{(r|x|)^{n/2}}\frac{J_{n/2+m}(ra)}{(ra)^{n/2}}\right)^2 dx \to 0, \qquad M\to\infty,$$

or

$$\int_0^\infty \lambda^{n-1}\left(\sum_{m=M}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{J_{n/2+m}(r\lambda)}{(r\lambda)^{n/2}}\frac{J_{n/2+m}(ra)}{(ra)^{n/2}}\right)^2 d\lambda \to 0, \qquad M\to\infty.$$

Let $K>0$. We split the integral into two terms:

$$\begin{aligned}&\int_0^\infty \lambda^{n-1}\left(\sum_{m=M}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{J_{n/2+m}(ra)}{(ra)^{n/2}}\frac{J_{n/2+m}(r\lambda)}{(r\lambda)^{n/2}}\right)^2 d\lambda\\ &= \underbrace{\int_0^K \lambda^{n-1}\left(\sum_{m=M}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{J_{n/2+m}(ra)}{(ra)^{n/2}}\frac{J_{n/2+m}(r\lambda)}{(r\lambda)^{n/2}}\right)^2 d\lambda}_{I_1}\\ &\quad + \frac{1}{r^n}\underbrace{\int_K^\infty \frac{1}{\lambda}\left(\sum_{m=M}^\infty \left(\frac{n}{2}+m\right) C^m_{n+m-1}\frac{J_{n/2+m}(ra)}{(ra)^{n/2}}J_{n/2+m}(r\lambda)\right)^2 d\lambda}_{I_2}.\end{aligned}$$

First we consider I_1. Inequality (12) yields

$$\frac{|J_{n/2+m}(ra)|}{(ra)^{n/2+m}}\frac{|J_{n/2+m}(r\lambda)|}{(r\lambda)^{n/2+m}} \le \frac{\pi}{2^{n+2m}\Gamma^2\left(\frac{n}{2}+m+\frac{1}{2}\right)}.$$

Applying relation (14), we get

$$\frac{\pi\left(\frac{n}{2}+m\right)C^m_{n+m-1}}{2^{n+2m}\Gamma^2\left(\frac{n}{2}+m+\frac{1}{2}\right)} \sim c\left(\frac{e}{2m}\right)^{2m}, \qquad m\to\infty.$$

Since the series

$$\sum_{n=M}^{\infty}\left(\frac{Ke^2ar^2}{4m^2}\right)^m$$

converges and approaches zero as $M\to\infty$, we obtain $I_1\to 0$ as $M\to\infty$.

Now we consider I_2. According to Lemma 3.3 of [8]

$$|J_{\frac{k}{2}}(z)| \le \frac{2^{k/2+1}}{\sqrt{\pi z}}, \qquad k\in\mathbb{N},\ z>0. \tag{15}$$

Applying inequality (15) for $k=n+2m$, we prove the bound

$$I_2 \le c\int_K^\infty \frac{1}{\lambda^2}\left(\sum_{m=M}^{\infty}\left(\frac{n}{2}+m\right)2^{n/2+m+1}C^m_{n+m-1}\frac{|J_{n/2+m}(ra)|}{(ra)^{n/2}}\right)^2 d\lambda$$

which together with (12) and Stirling formula implies that

$$\begin{aligned}&\left(\frac{n}{2}+m\right)2^{n/2+m+1}C^m_{n+m-1}\frac{|J_{n/2+m}(ra)|}{(ra)^{n/2}}\\ &\quad\le \left(\frac{n}{2}+m\right)\frac{\sqrt{\pi}(ra)^m 2^{n/2+m+1}}{2^{n/2+m}\Gamma\left(\frac{n}{2}+m+\frac{1}{2}\right)}\frac{(n+m-1)!}{m!\,(n-1)!}\\ &\quad\sim c\frac{(ra)^m\left(\frac{n}{2}+m\right)(n+m-1)^{n+m-1/2}e^{-n-m+1}}{\left(\frac{n}{2}+m-\frac{1}{2}\right)^{n/2+m}e^{-n/2-m+1/2}m^{m+1/2}e^{-m}} \sim c\frac{e^m(ra)^m}{m^{m-n/2}}, \qquad m\to\infty.\end{aligned}$$

Therefore the series

$$\sum_{m=M}^{\infty}\left(\frac{n}{2}+m\right)2^{n/2+m+1}C^m_{n+m-1}\frac{J_{n/2+m}(ra)}{(ra)^{n/2}}$$

converges absolutely and approaches zero as $M\to\infty$. This means that $I_2\to 0$ as $M\to\infty$.

Finally $I_1+I_2\to 0$ as $M\to\infty$ and one can integrate term by term. Using the Weber–Schafhetlin discontinuous infinite integral for $2\alpha>\gamma>-1$, we get (see (4) and (5) in §13.41 of [4])

$$\int_0^\infty \frac{J_{\alpha+p}(at)J_{\alpha-p-1}(bt)}{t^\gamma}\,dt = \begin{cases}\dfrac{b^{\alpha-p-1}\Gamma(\alpha-\frac{\gamma}{2})\,{}_2F_1\left(\alpha-\frac{\gamma}{2},-p-\frac{\gamma}{2};\alpha-p;\left(\frac{b}{a}\right)^2\right)}{2^\gamma a^{\alpha-p-\gamma}\Gamma(\alpha-p)\Gamma(p+\frac{\gamma}{2}+1)}, & b<a,\\ \dfrac{a^{\alpha+p}\Gamma(\alpha-\frac{\gamma}{2})\,{}_2F_1\left(\alpha-\frac{\gamma}{2},p-\frac{\gamma}{2}+1;\alpha+p+1;\left(\frac{a}{b}\right)^2\right)}{2^\gamma b^{\alpha+p-\gamma+1}\Gamma(\alpha+p+1)\Gamma(\frac{\gamma}{2}-p)}, & b>a.\end{cases} \tag{16}$$

Considering the case of $\alpha=(n+m)/2$, $p=m/2$, and $\gamma=0$, we obtain

$$\int_0^\infty J_{n/2+m}(r\lambda)J_{n/2-1}(|t|\lambda)\,d\lambda = \begin{cases}\dfrac{|t|^{n/2-1}\Gamma(\frac{n+m}{2})\,{}_2F_1\left(\frac{n+m}{2},-\frac{m}{2};\frac{n}{2};\left(\frac{|t|}{r}\right)^2\right)}{r^{\frac{n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2}+1)}, & |t|<r,\\ \dfrac{r^{n/2+m}\Gamma(\frac{n+m}{2})\,{}_2F_1\left(\frac{n+m}{2},\frac{m}{2}+1;\frac{n}{2}+m+1;\left(\frac{r}{|t|}\right)^2\right)}{|t|^{\frac{n}{2}+m+1}\Gamma(-\frac{m}{2})\Gamma(\frac{n}{2}+m+1)}, & |t|>r.\end{cases}$$

Remark 3. The integral vanishes if $m/2 \in \mathbb{N} \cup \{0\}$ and $|t| > r$. The asymptotic behavior of the Bessel functions (10) implies that the integral diverges for $|t| = r$ if m is odd (see [9]).

Therefore $f_{r,a}(|t|)$ can be defined up to a set of zero measure as follows:

$$c \sum_{m=0}^{\infty} \frac{\left(\frac{n}{2}+m\right) C_{n+m-1}^{m} \Gamma\left(\frac{n+m}{2}\right) J_{n/2+m}(ra)}{r^n \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}+1\right) (ra)^{n/2}} \, {}_2F_1\left(\frac{n+m}{2}, -\frac{m}{2}; \frac{n}{2}; \left(\frac{|t|}{r}\right)^2\right)$$

for $|t| < r$ and

$$c \sum_{m=0}^{\infty} \frac{C_{n+2m}^{2m+1} \Gamma\left(\frac{n}{2}+m+\frac{1}{2}\right) J_{n/2+2m+1}(ra) \, {}_2F_1\left(\frac{n}{2}+m+\frac{1}{2}, m+\frac{3}{2}; \frac{n}{2}+2m+2; \left(\frac{r}{|t|}\right)^2\right)}{a^{n/2} r^{n/2-2m-1} |t|^{n+2m+1} \Gamma\left(-m-\frac{1}{2}\right) \Gamma\left(\frac{n}{2}+2m+1\right)}$$

for $|t| > r$.

One can neglect the exact values of $f_{r,a}(|t|)$ on a set of zero measure, since they do not change the values of the integral in (3). For definiteness one can define $f_{r,a}(|t|)$ by the above equality everywhere. □

Remark 4. It is not surprising that the function series involved in the representation of the function $f_{r,a}(|t|)$ is undefined for $|t| = r$ (see Remark 3). Even in the known case of $a = 0$, the function

$$f_{r,0}(t) = \chi_{v(r)}(t)/r^n$$

is discontinuous for such points t.

4. Evaluation of $g_{r,a}(|t|)$

We propose a method to evaluate $g_{r,a}(|t|)$ as a sum of infinite function series.

Theorem 6. *If $n > 2$, then the function $g_{r,a}(|t|)$ defined in Theorem 4 can be represented in the form of the function series, namely*

$$\begin{aligned} g_{r,a}(|t|) = \frac{a^{n/2}\Gamma\left(\frac{n-2}{2}\right)}{2^{n/2}|t|^{n/2}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{n}{2}+m-1\right) C_{n+m-3}^{m} J_{(n-2)/2+m}(|t|a) \\ \times \sum_{k=0}^{n-1} C_{n-1}^{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \\ \times \begin{cases} \dfrac{{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right)}{|t|^k \Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m+n-k-1}{2}\right)}, & |t| > r, \\ \dfrac{|t|^{m+1} {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}; \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1}\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{n-m-k+1}{2}\right)}, & |t| < r. \end{cases} \end{aligned} \tag{17}$$

Proof. We use formula (10) for $\nu = \frac{n}{2} - 1$, $Z = |t|u$, $z = |t|a$, and $\varphi = \pi$:

$$\begin{aligned} &\frac{J_{n/2-1}(|t|u + |t|a)}{(|t|u + |t|a)^{n/2-1}} \\ &\quad = \frac{\Gamma\left(\frac{n-2}{2}\right)}{2^{1-n/2}} \sum_{m=0}^{\infty} \frac{\left(\frac{n-2}{2}+m\right) C_{n+m-3}^{m} J_{(n-2)/2+m}(|t|u) J_{(n-2)/2+m}(|t|a)}{(-1)^m (|t|u)^{n/2-1} (|t|a)^{n/2-1}}. \end{aligned} \tag{18}$$

Our method is to substitute the right hand side of equality (18) in integral (9) and integrate it term by term. In doing so we use the following result concerning the term-by-term integration of function series over an infinite interval.[1]

[1] *Translator's remark.* This result is Proposition C in §4, Supplement II to the Russian translation of the book [4]. The author refers to it in the Ukrainian original without actually stating it.

Proposition C. *Let $\int_1^x f_n(t)\,dt = g(x)$. Assume that the series $\sum f_n(x)$ uniformly converges in any finite interval (a,b). Assume also that the series $\sum g_n(x)$ converges uniformly in the infinite interval $a \ge x$. Then*

(i) $\sum\left[\int_a^\infty f_n(x)\,dx\right]$ *converges,*
(ii) $\int_a^\infty\left[\sum f_n(x)\right]dx$ *converges,*
(iii) *the values in* (i) *and* (ii) *coincide.*

We now apply this result.

Lemma 1. *Let $h_k(x) := \int_c^x w_k(u)\,du$, let the series $\sum_{k=0}^\infty w_k(u)$ converge uniformly in an arbitrary finite interval $[c,c')$, and let the series $\sum_{k=0}^\infty h_k(x)$ converge uniformly in the infinite interval $[c,\infty)$. Then*

(a) *the series $\sum_{k=0}^\infty \int_c^\infty w_k(u)\,du$ converges;*
(b) *the integral $\int_c^\infty \sum_{k=0}^\infty w_k(u)\,du$ converges;*
(c) *the series in* (a) *equals the integral in* (b).

Let $u \in [0,c')$. Then inequality (12) implies that

$$\sum_{m=0}^\infty \frac{\left(\frac{n-2}{2}+m\right)C_{n+m-3}^m\left|J_{(n-2)/2+m}(|t|u)J_{(n-2)/2+m}(|t|a)J_{n/2}(ru)\right|}{(|t|u)^{n/2-1}(|t|a)^{n/2-1}(ru)^{n/2}}(u+a)^{n-1}$$
$$\le \sum_{m=0}^\infty \frac{\pi^{3/2}\left(\frac{n-2}{2}+m\right)C_{n+m-3}^m(|t|a)^m(|t|u)^m(u+a)^{n-1}}{2^{n+2m-2}\Gamma^2\left(m+\frac{n}{2}-\frac12\right)2^{n/2}\Gamma\left(\frac{n}{2}+\frac12\right)}$$
$$\le \sum_{m=0}^\infty \frac{\pi^{3/2}\left(\frac{n-2}{2}+m\right)C_{n+m-3}^m(ac'|t|^2)^m(c'+a)^{n-1}}{2^{3n/2+2m-2}\Gamma^2\left(m+\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}.$$

We again use the Stirling formula:

$$\frac{\left(\frac{n-2}{2}+m\right)C_{n+m-3}^m}{2^{2m}\Gamma^2\left(m+\frac{n-1}{2}\right)}(ac'|t|^2)^m$$
$$\sim C\left(\frac{ac'|t|^2}{4}\right)^m\left(\frac{n-2}{2}+m\right)\frac{(n+m-3)^{n+m-3/2}e^{-n-m+3}}{m^{m+1/2}e^{-m}\left(m+\frac{n}{2}-\frac32\right)^{2m+n-2}e^{-2m-n+3}}$$
$$\sim C\left(\frac{ac'|t|^2e^2}{4m^2}\right)^m, \qquad m\to\infty.$$

Thus the series

$$\sum_{m=0}^\infty \frac{(-1)^m\left(\frac{n-2}{2}+m\right)C_{n+m-3}^m J_{(n-2)/2+m}(|t|u)J_{(n-2)/2+m}(|t|a)J_{n/2}(ru)}{(|t|u)^{n/2-1}(|t|a)^{n/2-1}(ru)^{n/2}}(u+a)^{n-1}$$

converges uniformly with respect to the argument u on $[0,c')$.

Consider the series

$$\sum_{m=0}^\infty \underbrace{\frac{\left(\frac{n-2}{2}+m\right)C_{n+m-3}^m J_{(n-2)/2+m}(|t|a)}{(-1)^m(|t|a)^{n/2-1}}}_{\Delta(m)} \times\int_0^x \underbrace{\frac{(u+a)^{n-1}J_{(n-2)/2+m}(|t|u)J_{n/2}(ru)}{(|t|u)^{n/2-1}(ru)^{n/2}}}_{\varphi_m(u)}\,du. \tag{19}$$

We show that the series $\sum_{m=0}^{\infty}|\Delta(m)|\left|\int_0^x \varphi_m(u)\,du\right|$ converges uniformly in the semiaxis $[0,\infty)$. The integral in (19) is estimated as

$$\left|\int_0^x \varphi_m(u)\,du\right| \le \left|\int_0^K \varphi_m(u)\,du\right| + \left|\int_K^x \varphi_m(u)\,du\right|$$

where $K \ge \max(1/r, 1/|t|)$.

Using bound (12), we get

$$\left|\int_0^K \varphi_m(u)\,du\right| \le \frac{\pi(K+a)^{n-1}|t|^m K^{m+1}}{2^{n+m-1}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}+m\right)}.$$

Bound (12) together with the Stirling formula yields

$$\begin{aligned}
&|\Delta(m)|\frac{\pi(K+a)^{n-1}|t|^m K^{m+1}}{2^{n+m-1}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}+m\right)}\\
&\quad\le \frac{\pi^{3/2}\left(\frac{n-2}{2}+m\right)(n+m-3)!}{\sqrt{2}(n-3)!\,m!\,\Gamma\left(\frac{n+1}{2}\right)\Gamma^2\left(\frac{n-1}{2}+m\right)}\frac{\left(a|t|^2\right)^m(K+a)^{n-1}K^{m+1}}{2^{n+2m-2}}\\
&\quad\sim C\left(\frac{aKe^2|t|^2}{4m^2}\right)^m, \qquad m\to\infty.
\end{aligned}$$

Therefore the series $\sum_{m=0}^{\infty}|\Delta(m)|\left|\int_0^K \varphi_m(u)\,du\right|$ converges.

Now we estimate the integral $\left|\int_K^x \varphi_m(u)\,du\right|$ from above:

$$\begin{aligned}
&\underbrace{\int_K^x \frac{(u+a)^{n-1}\left|\psi_m(|t|u)\psi_1(ru)\right|}{(|t|u)^{n/2-1}(ru)^{n/2}}\,du}_{I_3}\\
&\quad+\underbrace{\int_K^x \frac{\sqrt{2}(u+a)^{n-1}\left|\cos\left(ru-\frac{\pi(n+1)}{4}\right)\psi_m(|t|u)\right|}{\sqrt{\pi ru}(|t|u)^{n/2-1}(ru)^{n/2}}\,du}_{I_4}\\
&\quad+\underbrace{\int_K^x \frac{\sqrt{2}(u+a)^{n-1}\left|\cos\left(|t|u-\left(\frac{n-1}{2}+m\right)\frac{\pi}{2}\right)\psi_1(ru)\right|}{\sqrt{\pi|t|u}(|t|u)^{n/2-1}(ru)^{n/2}}\,du}_{I_5}\\
&\quad+\underbrace{\left|\int_K^x \frac{2(u+a)^{n-1}\cos\left(|t|u-\left(\frac{n-1}{2}+m\right)\frac{\pi}{2}\right)\cos\left(ru-\frac{\pi(n+1)}{4}\right)}{\pi u\sqrt{r|t|}(|t|u)^{n/2-1}(ru)^{n/2}}\,du\right|}_{I_6}
\end{aligned}$$

where

$$\psi_m(z) := J_{(n-2)/2+m}(z) - \sqrt{\frac{2}{\pi z}}\cos\left(z-\left(\frac{n-1}{2}+m\right)\frac{\pi}{2}\right).$$

Since

$$\begin{aligned}
I_6 &= \left|\int_K^x (u+a)^{n-1}\frac{\cos\left((|t|+r)u-(n+m)\frac{\pi}{2}\right)+\cos\left((|t|-r)u-(m-1)\frac{\pi}{2}\right)}{\pi r^{(n+1)/2}|t|^{(n-1)/2}u^n}\,du\right|\\
&\le \left|\int_K^x \frac{\cos\left((|t|+r)u-(n+m)\frac{\pi}{2}\right)}{\pi r^{(n+1)/2}|t|^{(n-1)/2}u^n(u+a)^{1-n}}\,du\right|\\
&\quad+\left|\int_K^x \frac{\cos\left((|t|-r)u-(m-1)\frac{\pi}{2}\right)}{\pi r^{(n+1)/2}|t|^{(n-1)/2}u^n(u+a)^{1-n}}\,du\right|
\end{aligned}$$

and the maximum of the integrals

$$\left|\int_K^x (u+a)^{n-1}\frac{\cos\left((|t|\pm r)u\right)}{\pi r^{(n+1)/2}|t|^{(n-1)/2}u^n}\,du\right| \quad \text{and} \quad \left|\int_K^x (u+a)^{n-1}\frac{\sin\left((|t|\pm r)u\right)}{\pi r^{(n+1)/2}|t|^{(n-1)/2}u^n}\,du\right|$$

with respect to $x \geq K$ is finite and does not depend on m if $|t| \neq r$, we obtain $\max_{m\geq 0,\, x\geq K} I_6 \leq C < \infty$.

To estimate the integrals I_3, I_4, and I_5, we need the following auxiliary result.

Lemma 2. *For every $\nu = k/2-1$, $k\in\mathbb{N}$, there exists a constant*

$$d_\nu \leq (\nu+1)\cdot 2^{\nu+12} \tag{20}$$

such that

$$\left|J_\nu(z)-\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right| \leq \frac{d_\nu}{z^{3/2}}$$

for all $z \geq 1$.

Proof. We use induction on ν.

First we prove a stronger result for $\nu = k-\frac{1}{2}$, $k\in\mathbb{N}\cup\{0\}$, namely,

$$d_\nu \leq \frac{\nu+1}{\sqrt{\pi}}2^{\nu+2}. \tag{21}$$

Since

$$J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\cos(z), \qquad J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin(z),$$

we have $d_{-\frac{1}{2}} = d_{1/2} = 0$. Thus inequality (21) holds for $\nu = \pm\frac{1}{2}$. Now we use the recurrence formula for the Bessel function (see (1) in §3.2 of [4]):

$$J_\nu(z)=\frac{2(\nu-1)}{z}J_{\nu-1}(z)-J_{\nu-2}(z) \tag{22}$$

and bound (15) for the maximum of the Bessel function whose index is half of an integer number. Assume that inequality (21) holds for $\nu-2$. Then (22) and (15) for $\nu = k+\frac{1}{2}$, $k\in\mathbb{N}$, imply that

$$\begin{aligned}
&\left|J_\nu(z)-\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right| \\
&\leq \left|J_{\nu-2}(z)-\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{(\nu-2)\pi}{2}-\frac{\pi}{4}\right)\right| + \left|\frac{2(\nu-1)\sqrt{z}J_{\nu-1}(z)}{z^{3/2}}\right| \\
&\leq \frac{d_{\nu-2}}{z^{3/2}}+\frac{2(\nu-1)2^\nu}{\sqrt{\pi}z^{3/2}} \leq \frac{3(\nu-1)2^\nu}{\sqrt{\pi}z^{3/2}} \leq \frac{(\nu+1)2^{\nu+2}}{\sqrt{\pi}z^{3/2}}.
\end{aligned} \tag{23}$$

Therefore inequality (21) holds for $\nu = k-\frac{1}{2}$, $k\in\mathbb{N}\cup\{0\}$.

Now let $\nu\in\mathbb{N}\cup\{0\}$. Applying the representation of Bessel functions of the first kind in terms of the Bessel functions of the third kind (see (1) in §3.61 of [4]) and the Weber expansion (see §7.33 in [4]) for $|z| > \frac{\nu}{2}-\frac{1}{4}$, we get

$$\begin{aligned}
J_\nu(z) &= \frac{H_\nu^{(1)}(z)+H_\nu^{(2)}(z)}{2} \\
&= \frac{1}{\sqrt{2\pi z}}\Big[\exp\left\{i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right\}\left(1+R_1^{(1)}\right) \\
&\qquad + \exp\left\{-i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\right\}\left(1+R_1^{(2)}\right)\Big].
\end{aligned}$$

As a corollary,

$$\left|J_\nu(z) - \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right| \le \frac{\left|R_1^{(1)}\right| + \left|R_1^{(2)}\right|}{\sqrt{2\pi z}}$$

where the remainder terms $R_1^{(1)}$ and $R_1^{(2)}$ can be estimated by formulas (3)–(5) in §7.33 of [4], namely

$$\max\left(\left|R_1^{(1)}\right|, \left|R_1^{(2)}\right|\right) \le \frac{\pi G^2\left|4\nu^2 - 1\right|}{2^3 z},$$
$$G = \begin{cases} \left(1 - \frac{\nu - 1/2}{2z}\right)^{-\nu-1/2}, & \nu > \frac{1}{2}, \\ \left(1 - \frac{\nu+3/2}{2z}\right)^{-\nu-5/2}\left(1 + \frac{2\nu+2}{z}\right), & \nu < \frac{1}{2}. \end{cases}$$

This implies for $z \ge 1$ that

$$\left|J_0(z) - \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right)\right| \le \frac{\sqrt{\pi}(1+2/z)^2}{2^{5/2}z^{3/2}\left(1 - 3/(4z)\right)^5} \le \frac{9\sqrt{\pi}2^{15/2}}{z^{3/2}},$$
$$\left|J_1(z) - \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}\right)\right| \le \frac{3\sqrt{\pi}}{2^{5/2}z^{3/2}\left(1 - 1/(4z)\right)^3} \le \frac{\sqrt{\pi}2^{7/2}}{9z^{3/2}},$$

whence bound (20) follows for $\nu = 0, 1$.

Assuming that bound (20) holds for $\nu - 2$, we apply (23) for $\nu = k+1$, $k \in \mathbb{N}$:

$$\left|J_\nu(z) - \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right| \le \frac{d_{\nu-2}}{z^{3/2}} + \frac{2(\nu-1)2^\nu}{\sqrt{\pi}z^{3/2}} \le \frac{(\nu-1)2^{\nu+10}}{z^{3/2}} + \frac{2(\nu-1)2^\nu}{\sqrt{\pi}z^{3/2}}$$
$$\le \frac{(\nu+1)2^{\nu+12}}{z^{3/2}}.$$

Thus bound (20) holds for $\nu = k - 1$, $k \in \mathbb{N}$. □

Recall that the constant K is chosen such that $\max(Kr, K|t|) \ge 1$. Applying Lemma 2, we obtain the following bounds for I_3 and I_4:

$$I_3 \le \int_K^x \frac{d_{n/2}d_{(n-2)/2+m}(u+a)^{n-1}\,du}{(|t|u)^{n/2-1}(ru)^{n/2}(|t|u)^{3/2}(ru)^{3/2}}$$
$$\le \frac{(n/2+m)(n/2+1)2^{n+m+23}}{|t|^{(n+1)/2}r^{(n+3)/2}}\int_K^\infty \frac{(u+a)^{n-1}}{u^{n+2}}\,du,$$
$$I_4 \le \int_K^x \frac{\sqrt{2}d_{(n-2)/2+m}(u+a)^{n-1}\,du}{\sqrt{\pi r u}(|t|u)^{n/2-1}(ru)^{n/2}(|t|u)^{3/2}} \le \frac{(n/2+m)2^{n/2+m+11}}{\sqrt{\pi}|t|^{(n+1)/2}r^{(n+1)/2}}\int_K^\infty \frac{(u+a)^{n-1}}{u^{n+1}}\,du,$$
$$I_5 \le \int_K^x \frac{\sqrt{2}d_{n/2}(u+a)^{n-1}\,du}{\sqrt{\pi|t|u}(|t|u)^{n/2-1}(ru)^{n/2}(ru)^{3/2}} \le \frac{(n/2+1)2^{n/2+12}}{\sqrt{\pi}|t|^{(n-1)/2}r^{(n+3)/2}}\int_K^\infty \frac{(u+a)^{n-1}}{u^{n+1}}\,du.$$

Hence

$$\max\left(\max_{x\ge K} I_3, \max_{x \ge K} I_4\right) \le C\left(\frac{n}{2} + m\right)2^m, \qquad \max_{m\ge 0,\, x\ge K} I_5 \le C < \infty,$$
$$|\Delta(m)|\left|\int_K^\infty \varphi_m(u)\,du\right| \le |\Delta(m)|\left(I_3 + I_4 + I_5 + I_6\right) \le C|\Delta(m)|\left(\left(\frac{n}{2} + m\right)2^m + 1\right).$$

We again use the Stirling formula and bound (12):

$$|\Delta(m)|\left(\left(\frac{n}{2}+m\right)2^m+1\right) \le \frac{\left(\frac{n-2}{2}+m\right)C^m_{n+m-3}\sqrt{\pi}(|t|a)^m}{2^{(n-2)/2+m}\Gamma(\frac{n-1}{2}+m)}\left(\left(\frac{n}{2}+m\right)2^m+1\right)$$
$$\sim C\frac{(ae|t|)^m}{m^{m-n/2+1}}\left(\frac{n}{2}+m+\frac{1}{2^m}\right) \sim C\frac{(ae|t|)^m}{m^{m-n/2}}, \qquad m\to\infty.$$

This implies that the series $\sum_{m=0}^{\infty}|\Delta(m)|\left|\int_K^x \varphi_m(u)\,du\right|$ converges uniformly on the infinite interval $[K,\infty)$.

Now Lemma 1 allows one to integrate the series step by step:

$$g_{r,a}(|t|) = \frac{2^{n/2-1}\Gamma\left(\frac{n-2}{2}\right)}{|t|^{n-2}a^{n/2-1}r^{n/2}} \times \sum_{m=0}^{\infty}(-1)^m\left(\frac{n}{2}+m-1\right)C^m_{n+m-3}J_{(n-2)/2+m}(|t|a) \times \int_0^\infty (\lambda+a)^{n-1}\frac{J_{(n-2)/2+m}(|t|\lambda)J_{n/2}(r\lambda)}{\lambda^{n-1}}\,d\lambda, \qquad |t|\ne r. \tag{24}$$

Formula (16) with $\alpha=\frac{n+m}{2}$, $p=\frac{m}{2}-1$, and $\gamma=n-k-1$ yields

$$\int_0^\infty \frac{J_{(n-2)/2+m}(|t|\lambda)J_{n/2}(r\lambda)}{\lambda^{n-k-1}}\,d\lambda = \frac{r^{n/2}|t|^{n/2}\Gamma\left(\frac{m+k+1}{2}\right)}{2^{n-k-1}}\begin{cases}\frac{{}_2F_1\left(\frac{m+k+1}{2},\frac{3+k-m-n}{2};\frac{n}{2}+1;\left(\frac{r}{|t|}\right)^2\right)}{|t|^{k+2}\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m+n-k-1}{2}\right)}, & |t|>r,\\ \frac{|t|^{m-1}{}_2F_1\left(\frac{m+k+1}{2},\frac{m+k+1-n}{2};\frac{n}{2}+m;\left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1}\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{n-m-k+1}{2}\right)}, & |t|<r.\end{cases} \tag{25}$$

Expanding $(\lambda+a)^{n-1}$ according to the binomial formula and substituting (25) in (24), we obtain representation (17). □

5. Examples

We consider some particular cases of the above general results in this section.

Example 1. Let $a=0$. The series for $f_{r,a}(|t|)$ obtained in Theorem 5 contains only one nonzero term corresponding to $m=0$ if $|t|<r$. This follows from the relation

$$\lim_{z\to 0}\frac{J_\nu(z)}{z^\nu} = \frac{1}{2^\nu\Gamma(\nu+1)} \tag{26}$$

(see formula (8) in [9]). The nonzero term equals

$$\frac{nC^0_{n-1}\Gamma(\frac{n}{2})\,{}_2F_1\left(\frac{n}{2},0;\frac{n}{2};(|t|/r)^2\right)}{2r^n\Gamma\left(\frac{n}{2}+1\right)} = r^{-n}$$

and this implies the known result $f_{r,0}(t)=\chi_{v(r)}(t)/r^n$.

Similarly, the series for $g_{r,a}(|t|)$ obtained in Theorem 6 contains only one nonzero term corresponding to the case of $k=n-1$ and $m=0$ if $|t|<r$. Using relation (26), we evaluate this term:

$$\Gamma\left(\frac{n-2}{2}\right)\left(\frac{n}{2}-1\right)C^0_{n-3}C^{n-1}_{n-1}\frac{{}_2F_1\left(\frac{n}{2},0;\frac{n}{2};(|t|/r)^2\right)}{r^n\Gamma\left(\frac{n}{2}\right)} = r^{-n}.$$

Again $g_{r,0}(t)=\chi_{v(r)}(t)/r^n$.

Example 2. Let $n = 3$. We have

$$\begin{aligned}\sqrt{|t|} f_{r,a}(|t|) &= \int_0^\infty \lambda^{3/2} \frac{J^{3/2}(r(\lambda - a))}{(r(\lambda - a))^{3/2}} J_{1/2}(|t|\lambda)\, d\lambda \\ &= \int_0^a (a-\lambda)^{3/2} \frac{J^{3/2}(r\lambda)}{(r\lambda)^{3/2}} J_{1/2}(|t|(a-\lambda))\, d\lambda \\ &\quad + \int_0^\infty (\lambda + a)^{3/2} \frac{J^{3/2}(r\lambda)}{(r\lambda)^{3/2}} J_{1/2}(|t|(\lambda + a))\, d\lambda.\end{aligned}$$

Since $J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z$ and $J^{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z\right)$, one can express $f_{r,a}(t)$ in terms of the special functions Si and Ci (this computation is obtained with the help of *Mathematica 5.0*):

$$\begin{aligned}\frac{1}{4r^{3/2}\sqrt{|t|}} \Bigg(& \frac{a\cos(a|t|)\left(4r|t| - (r^2 - |t|^2)\ln\left(\frac{r-|t|}{r+|t|}\right)^2\right)}{\pi\sqrt{r^3|t|}} \\ & + 2\cos(a|t|)\left(\frac{r + |t| - \big|r - |t|\big|}{\sqrt{r^3|t|}} + \frac{\operatorname{sign}(r - |t|) - 1}{\sqrt{r|t|}}\right) \\ & + \frac{a(r + |t|)\big(r - |t| + \big|r - |t|\big|\big)\sin(a|t|)}{\sqrt{r^3|t|}} + \frac{2\left(4r + |t|\ln\left(\frac{r-|t|}{r+|t|}\right)^2\right)\sin(a|t|)}{\pi\sqrt{r^3|t|}} \\ & - \frac{4\sin(a|t|)\left(2ar + a|t|\left(\operatorname{Ci}(a(r + |t|)) - \operatorname{Ci}(a(r - |t|)) + \ln\left|\frac{r-|t|}{r+|t|}\right|\right)\right)}{a\pi\sqrt{r^3|t|}} \\ & + \frac{8\sin(a|t|)\cos(a|t|)\sin(ar)}{a\pi\sqrt{r^3|t|}} \\ & - \frac{1}{a\pi\sqrt{r^3|t|}}\Bigg(2\cos(a|t|)\Bigg(a^2\Bigg(2r|t| + \left(r^2 - |t|^2\right)\Bigg(\operatorname{Ci}(a(r - |t|)) \\ & \qquad - \operatorname{Ci}(a(r + |t|)) - \ln\left|\frac{r - |t|}{r + |t|}\right|\Bigg)\Bigg) \\ & \qquad - 2a|t|\cos(a|t|)\sin(ar) + 2(ar\cos(ar) - \sin(ar))\sin(a|t|)\Bigg)\Bigg) \\ & + \frac{4\cos(a|t|)}{a\pi r^2|t|}\left(a\sqrt{r|t|^3}(\operatorname{Si}(a(r - |t|)) + \operatorname{Si}(a(r + |t|))) - 2\sqrt{r|t|}\sin(ar)\sin(a|t|)\right) \\ & + \frac{2\sin(a|t|)}{a\pi\sqrt{r^3|t|}}\Bigg(2\cos(a|t|)(ar\cos(ar) - \sin(ar)) + 2a|t|\sin(ar)\sin(a|t|) \\ & \qquad + a^2\Big(r^2\operatorname{Si}(a(r - |t|)) + |t|^2\operatorname{Si}(a(|t| - r)) \\ & \qquad + \left(r^2 - |t|^2\right)\operatorname{Si}(a(r + |t|))\Big)\Bigg)\Bigg)\end{aligned}$$

where

$$\operatorname{Si}(z) := \int_0^z \frac{\sin u}{u}\, du = -\operatorname{Si}(-z), \qquad \operatorname{Ci}(z) := -\int_z^\infty \frac{\cos u}{u}\, du = \operatorname{Ci}(-z), \quad z > 0$$

(see [1]).

The graph of the function $f_{r,a}(|t|)$ is depicted in Figures 1 and 2 for $r = 1$ and $a = 1.2$ and 15. In contrast to the case of $a = 0$, the function does not vanish for $|t| > r$ and may take negative values. The number of points where the graph of the function crosses the axis Ox on the interval $[0, r)$ increases if a increases.

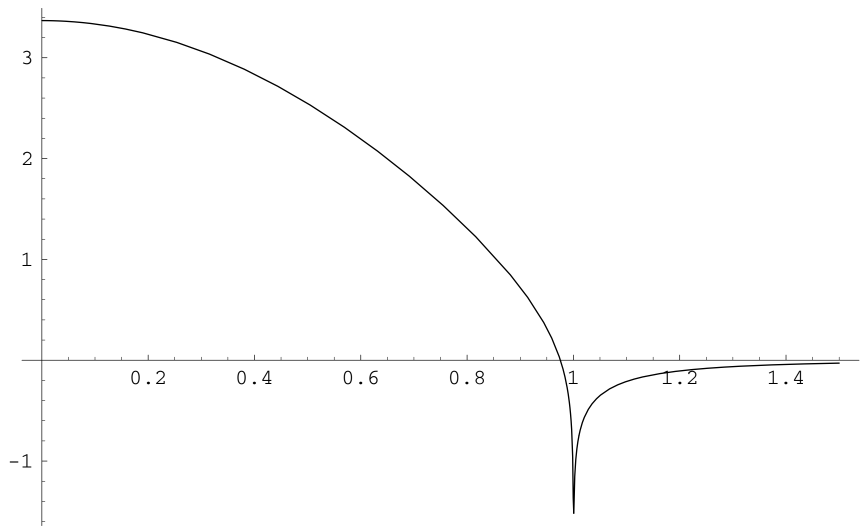

FIGURE 1. Graph of $f_{1,1.2}(|t|)$

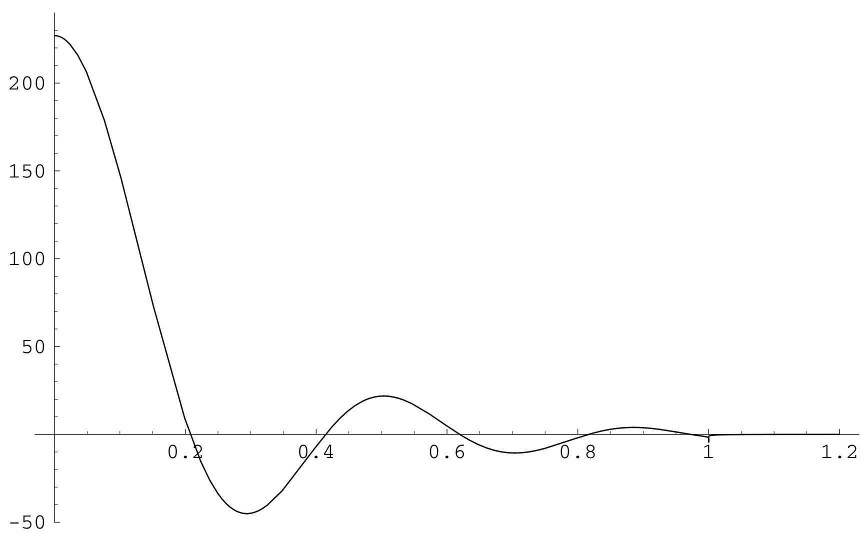

FIGURE 2. Graph of $f_{1,15}(|t|)$

Example 3. Let $n = 3$. The function $g_{r,a}(|t|)$ can be expressed in terms of the elementary functions (this result is also obtained with the help of *Mathematica 5.0*):

$$\frac{1}{2\pi r^3|t|}\begin{cases} a\cos(a|t|)\left(2r|t| + \left(|t|^2 - r^2\right)\ln\left(\frac{|t|-r}{r+|t|}\right)\right) & \\ \quad \times 2\sin(a|t|)\left(|t|\ln\left(\frac{|t|-r}{r+|t|}\right) + 2r\right), & |t| > r, \\ \cos(a|t|)\left(a\left(|t|^2 - r^2\right)\ln\left(\frac{r-|t|}{r+|t|}\right) + 2|t|(\pi + ar)\right) & \\ \quad \times \sin(a|t|)\left(4r + a\pi\left(r^2 - |t|^2\right) + 2|t|\ln\left(\frac{r-|t|}{r+|t|}\right)\right), & |t| < r. \end{cases}$$

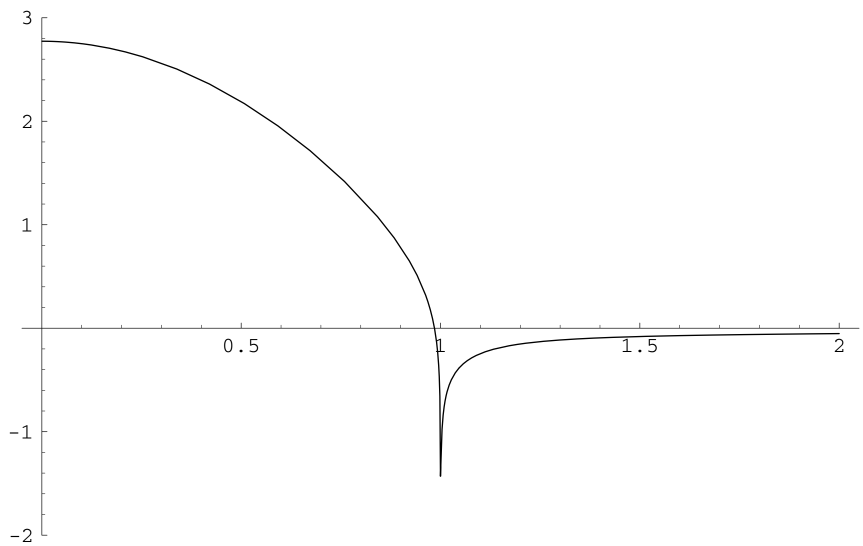

FIGURE 3. Graph of $g_{1,1}(|t|)$

The graph of the function $g_{r,a}(|t|)$ is depicted in Figures 3–5 for $r = 1$ and $a = 1$ and 10. Similarly to the preceding example and in contrast to the case of $a = 0$, the function does not vanish for $|t| > r$ and may take negative values. Note that the function is discontinuous at the points $|t| = 1$. As a increases, the frequency of oscillation increases and this results in the increase of the number of points where the function crosses the Ox axis. The graph approaches Ox as $|t|$ increases.

Remark 5. The functions $f_{r,a}(|t|)$ and $g_{r,a}(|t|)$ can be written explicitly in terms of the elementary and special functions in the case of $n = 2k + 1$, $k \in \mathbb{N}$. One can obtain this result similarly to Examples 2 and 3. However the expressions become very complicate if $n \geq 5$.

6. Concluding remarks

Abelian and Tauberian theorems are proved in the paper for functions possessing properties generalizing the regular variation. A relationship is studied between the local behavior of spectral functions and that of functionals of random fields. It is shown that the conditions can be given in terms of the variance of weighted integrals of random fields. A representation is found for the weight functions in terms of the Hankel transform and function series.

The following is the list of problems to be investigated elsewhere:

(1) Find a functional of the random field whose variance equals $\tilde{l}^a(r)$; in particular, does the functional

$$\int_{s(r)} \Psi_{r,a}(t)\xi(t)\,dm(t)$$

possess this property?

(2) Study the properties of the functions $f_{r,a}(|t|)$ and $g_{r,a}(|t|)$ that are similar to those mentioned in Examples 2 and 3.

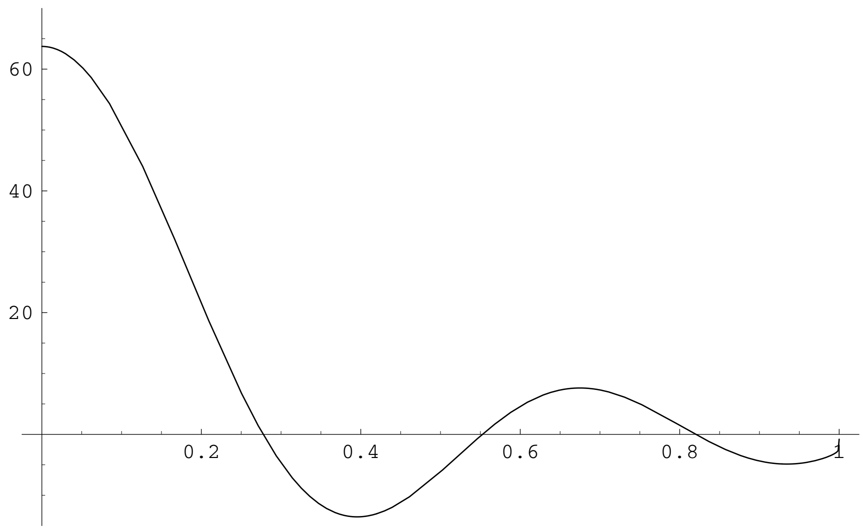

FIGURE 4. Graph of $f_{1,10}(|t|)$ for $|t| < 1$

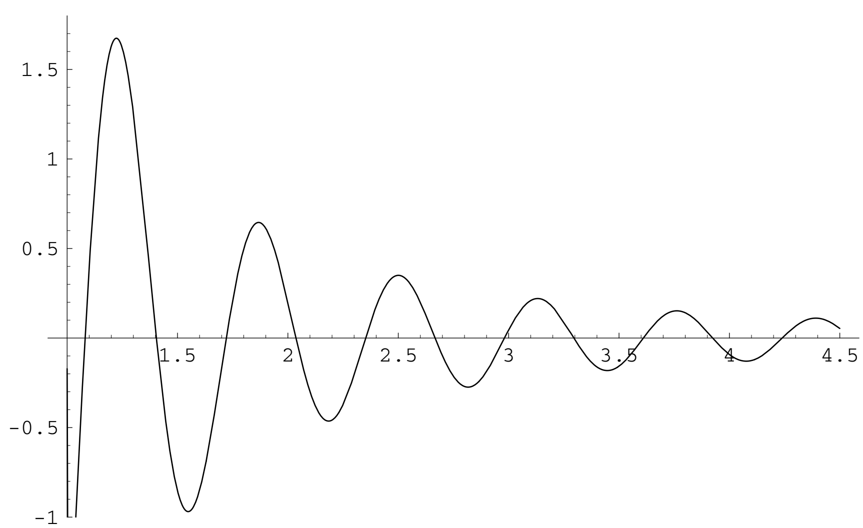

FIGURE 5. Graph of $f_{1,10}(|t|)$ for $|t| > 1$

(3) Prove an analog of Theorem 6 for the case of $n = 2$ (one should apply the Neumann addition theorem for $J_0(\cdot)$ (see §11.2 in [4]) instead of the Gegenbauer formula (10)).

(4) Apply the Tauberian theorems proved above to investigate the limit behavior of functionals of random fields.

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