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MIXED EMPIRICAL STOCHASTIC POINT PROCESSES IN COMPACT METRIC SPACES. I

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Abstract. We study models of finite mixed empirical ordered point processes in compact metric spaces constructed from samples without repetition. We introduce the notion of the generating sequence of the probability measure of an ordered point process. A multidimensional family of distributions is constructed that completely determines the probability distribution of an ordered point process. An example is considered where we evaluate multidimensional distributions.

Interest in the theory of stochastic point processes and marked point processes has been continually growing over the last two decades in view of their applications in stochastic geometry, economics, biology, ecology, cosmology, and stereology. However the models of marked point processes with statistical interactions between marked pairs, points of positions, and marks have not yet been studied in full detail. When simulating on computers some problems of spherical stochastic geometry, one faces the problem of constructing the trajectories of various types of point processes and marked point processes [6, 12].

In this paper, we consider models of point processes and marked point processes in compact metric spaces constructed from random samples without repetition. The models of this kind are called mixed empirical ordered point processes and marked point processes [10].

The main definitions of the theory of finite simple ordered point processes and marked point processes are given in Section 1. The structure of finite simple point processes and marked point processes in compact metric spaces is studied in Section 2. We introduce the generating sequence of the probability measure of a point process (marked point process). In Section 3, we construct a model of a finite simple mixed empirical ordered point process in a compact metric space whose trajectories are samples without repetition from a population X according to the distribution P_x on the σ-algebra $\mathfrak{A}_X$.

We construct a multidimensional family of distributions that completely determines the probability distribution P of a point process. We also consider an example and evaluate multidimensional distributions.

1. Basic notions of the theory of stochastic point processes and marked point processes

In what follows, we need some basic notions of the theory of stochastic point processes and marked point processes [5, 6, 9].

Let $(X, \mathfrak{A}_X)$ be a measurable space. Assume that the σ-algebra $\mathfrak{A}_X$ contains all the singleton subsets $\{x\}$ of the main space X. We say that the structure $\mathfrak{B}_X$ of bounded

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subsets is introduced in a measurable space $(X, \mathfrak{A}_X)$ if a family $\mathfrak{B}_X$ of subsets of X is given such that [13]

1) $\mathfrak{B}_X$ is a hereditary family, that is, if $E \in \mathfrak{B}_X$ and $F \in E$, then $F \in \mathfrak{B}_X$;
2) $\mathfrak{B}_X$ is closed under finite unions of its elements;
3) $\mathfrak{B}_X$ is a covering of the space X;
4) the family of bounded measurable sets $\mathfrak{C}_X = \mathfrak{A}_X \cap \mathfrak{B}_X$ is cofinal in $\mathfrak{B}_X$ under inclusion, that is, if $B_X \in \mathfrak{B}_X$, then there exists $F_X \in \mathfrak{C}_X$ for which $B_X \subset F_X$.

Definition 1 ([13])**.** A measurable space $(X, \mathfrak{A}_X)$ endowed with a structure of bounded sets $\mathfrak{B}_X$ is called a bounded space and is denoted by $(X, \mathfrak{A}_X, \mathfrak{B}_X)$.

Definition 2 ([13])**.** A multiset $E = (x_1, x_2, \dots, x_n, \dots)$ (where some points x_n may appear several times) of a bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$ is called a locally bounded set if its intersection with an arbitrary bounded set $B_X \in \mathfrak{C}_X$, $\mathfrak{C}_X = \mathfrak{A}_X \cap \mathfrak{B}_X$, contains only a finite number of elements: $N(E, B_X) = \operatorname{card}[E \cap B_X] < \infty$, where $\operatorname{card}[E \cap B_X]$ denotes the number of elements of the set $E \cap B_X$.

Definition 3. A locally bounded set E is called simple if all its elements are distinct, that is, $E = (x_1, \dots, x_i, \dots, x_j, \dots, x_n, \dots)$ and $x_i \neq x_j$ for $i \neq j$.

Denote by $\mathcal{E}$ the class of all locally bounded sets of a bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$. Let B_X be an arbitrary measurable bounded set, $B_X \in \mathfrak{C}_X$, and let $N(E, B_X)$ be a mapping defined on the class $\mathcal{E}$ such that

$$E \to N(E, B_X) \in Z_+ = \{0, 1, 2, \dots\}.$$

Viewing $\mathcal{E}$ as a new main space, denote by $\mathfrak{X}$ the minimal σ-algebra generated by subsets of the main space $\mathcal{E}$ such that all mappings $N_{B_X}(\cdot) = N(\cdot, B_X) \colon \mathcal{E} \to Z_+$ are measurable for all sets B_X of the class $\mathfrak{C}_X$. Consider the structure of the σ-algebra $\mathfrak{X}$ in more detail. Since Z_+ is a countable set, the σ-algebra $\mathfrak{A}_{Z_+}$ of measurable (Borel) sets in Z_+ is constituted by all subsets of Z_+: $\mathfrak{A}_{Z_+} = \{M \colon M \subset Z_+\}$. Denote by $N_{B_X}^{-1}(\mathfrak{A}_{Z_+})$ the preimage of the σ-algebra $\mathfrak{A}_{Z_+}$ that corresponds to the mapping $N_{B_X}(\cdot)$:

$$N_{B_X}^{-1}(\mathfrak{A}_{Z_+}) = \left\{ N_{B_X}^{-1} \colon M \subset Z_+ \right\}.$$

It is easy to see that $N_{B_X}^{-1}(\mathfrak{A}_{Z_+})$ is a σ-algebra of subsets of the space $\mathcal{E}$. By $\mathcal{T}$ we denote the union of all such σ-algebras $N_{B_X}^{-1}(\mathfrak{A}_{Z_+})$ where B_X runs over the whole class $\mathfrak{C}_X$:

$$\mathcal{T} = \bigcup_{B_X \in \mathfrak{C}_X} N_{B_X}^{-1}(\mathfrak{A}_{Z_+}). \tag{1}$$

It is easy to see that the σ-algebra $\mathfrak{X}$ coincides with the σ-algebra $\mathcal{S}(\mathcal{T})$ generated by the union $\mathcal{T}$ of σ-algebras $N_{B_X}^{-1}(\mathfrak{A}_{Z_+})$ for all $B_X \in \mathfrak{C}_X$: $\mathfrak{X} = \mathcal{S}(\mathcal{T})$.

Definition 4 ([11])**.** The probability space $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ where P is a probability measure defined on the σ-algebra $\mathfrak{X}$ is called a point process in the bounded space $(\mathfrak{X}, \mathfrak{A}_X, \mathfrak{B}_X)$.

Definition 5. A stochastic point process $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ is called simple in a bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$ if almost all trajectories $E \in \mathcal{E}$ of the process are simple locally bounded sets in the space X.

We also say [8] that a simple point process does not have multiple points with probability one: $\mathsf{P}\{E \colon N(E, \{x\}) \le 1\} = 1$ for all $x \in X$.

Definition 6. A stochastic point process $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ is called finite in a bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$ if the size of any trajectory E of the process is finite with probability one:

$$\mathsf{P}\{E \colon \operatorname{card}[E] = N(E, X) < \infty\} = 1.$$

Definition 7. A process $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ is called a finite simple ordered point process (ordered point process) in a bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$ if all its trajectories

$$E = (x_1, \dots, x_i, \dots, x_n)$$

can be ordered.

If an order does not exist in a set (trajectory) E, a simple finite point process is called an unordered point process. For an unordered point process, a trajectory E can be written as $E = \{x_{(1)}, \dots, x_{(n)}\}$ where the subscripts are not necessarily related to the order in the set E (we use the subscripts to distinguish between the elements of E).

Now we turn to the notion of a stochastic marked point process. We consider a partial case where the space of marks K is a subset of the real line $\mathbb{R}^1$: $K \subseteq \mathbb{R}^1$, since this case is the most important for applications. Let $Y = X \times K$ be the Cartesian product of the spaces X and K where X is the space of positions and K is the space of marks [6, 9]. For the σ-algebra $\mathfrak{A}_K$ in the space K we take the σ-algebra generated by the class $\mathcal{S}(\mathbb{R}^1)$ of all Borel sets of the real line $\mathbb{R}^1$:

$$\mathfrak{A}_K = K \cap \mathcal{S}(\mathbb{R}^1) = \big\{B_K \colon B_K = K \cap A, A \in \mathcal{S}(\mathbb{R}^1)\big\}.$$

As usual, for the σ-algebra in the space Y we take the Cartesian product $\mathfrak{A}_Y = \mathfrak{A}_X \otimes \mathfrak{A}_K$ of the σ-algebras $\mathfrak{A}_X$ and $\mathfrak{A}_K$. In the measurable space $(Y, \mathfrak{A}_Y)$ we introduce the structure of bounded sets $\mathfrak{B}_Y = \mathfrak{B}_X \odot \mathfrak{B}_K$ and define a set $B_Y \in Y$ to be bounded if there are bounded sets $B_X \in \mathfrak{B}_X$ and $B_K \in \mathfrak{B}_K$ such that $B_Y \subseteq B_X \times B_K$. It is easy to check that all axioms of the structure of bounded sets hold for $\mathfrak{B}_Y$ and thus $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ is a bounded space.

Definition 8. A stochastic point process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ in a bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ is called a marked point process. Here $\mathcal{E}^* = \{E^*\}$ is the class of all locally bounded sets E^* of the bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$.

Definition 9. A stochastic marked point process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ is called simple in the bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ if $x_i \neq x_j$, $i \neq j$, for almost all its trajectories

$$E^* = ([x_1; k_1], \dots, [x_n; k_n], \dots),$$

that is, for all $x \in X$

$$\mathsf{P}^*\{E^* \colon N^*(E^*, \{x\} \times K) \leq 1\} = 1,$$

where $N^*(E^*, B_Y)$ is a stochastic counting measure of the marked point process and where $B_Y \in \mathfrak{C}_Y = \mathfrak{A}_Y \cap \mathfrak{B}_Y$.

Definition 10. A stochastic marked point process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ is called strictly simple in a bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ if $x_i \neq x_j$ and $k_i \neq k_j$, $i \neq j$, for almost all its trajectories $E^* = ([x_1; k_1], \dots, [x_n; k_n], \dots)$, that is, for all $x \in X$ and $k \in K$

$$\mathsf{P}^*\{E^* \colon N^*(E^*, \{x\} \times K) \leq 1\} = 1,$$
$$\mathsf{P}^*\{E^* \colon N^*(E^*, X \times \{k\}) \leq 1\} = 1.$$

Definition 11. A stochastic marked point process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ is called finite in a bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ if the size of almost all trajectories

$$E^* = ([x_1; k_1], \dots, [x_n; k_n])$$

of the process is finite:

$$\mathsf{P}^*\big\{E^* \colon \operatorname{card}[E^*] = N^*(E^*, Y) < \infty\big\} = 1.$$

Definition 12 ([6, 11])**.** A process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ is called a finite simple ordered marked point process in a bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ if the projection

$$(\mathcal{E}, \mathfrak{X}, \mathsf{P}) = \mathrm{Pr}_X(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$$

to the space of positions X is a finite simple ordered point process in the bounded space of positions $(X, \mathfrak{A}_X, \mathfrak{B}_X)$. An arbitrary trajectory $E^* \in \mathcal{E}^*$ of an unordered marked point process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ is a finite simple ordered locally bounded set (a vector) of the phase space $Y = X \times K$:

$$E^* = (y_i \colon y_i = [x_i; k_i], i = 1, \dots, n),$$

where $X \ni x_i$ is a point of position and $K \ni k_i$ is the mark of the random event $y_i = [x_i; k_i]$.

Definition 13 ([6])**.** A process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ is called a finite simple unordered marked point process in a bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ if its projection $(\mathcal{E}, \mathfrak{X}, P) = \mathrm{Pr}_X(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ to the space of positions X is a finite simple unordered point process in bounded space of positions $(X, \mathfrak{A}_X, \mathfrak{B}_X)$. The trajectory of a unordered marked point process is a finite simple unordered locally bounded set $E^* = \{y_{(1)}, \dots, y_{(n)}\} = \{[x_{(1)}; k_{(1)}], \dots, [x_{(n)}; k_{(n)}]\}$ of the phase space $Y = X \times K$ (the subscripts are not necessarily related to the order in the locally bounded set E^*; we use them just to distinguish between the points of E^*).

2. Finite simple stochastic point processes and marked point processes in compact metric spaces and generating sequences

Finite simple stochastic point processes $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ in a bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$ have nice properties if the space of positions X is a compact metric space. In this case, the structure $\mathfrak{B}_X$ of bounded sets coincides with the family of all subsets of the space X; in particular, $X \in \mathfrak{B}_X$ and every simple locally bounded set $E \in \mathcal{E}$ is a finite subset of points of X: $E \subset X$ and $\mathrm{card}[E] < \infty$. Thus $\mathcal{E} = \{E\}$ is the family of all finite simple locally bounded sets E of X and the class $\mathcal{E}$ of locally bounded sets can be split into subclasses $\{\mathcal{E}_n \colon n = 0, 1, 2, \dots\}$ where every subclass $\mathcal{E}_n$ contains finite simple locally bounded sets E of size n: $\mathcal{E}_n = \{E \colon N(E, X) = n\}$. Moreover, for all $m, k \in Z_+$, $m \neq k$, the following conditions hold: $\mathcal{E}_m \cap \mathcal{E}_k = \varnothing$ and $\bigcup_{n=0}^{\infty} \mathcal{E}_n = \mathcal{E}$. Note that all subclasses $\mathcal{E}_n$ forming the covering of $\mathcal{E}$ are measurable: $\mathcal{E}_n \in \mathfrak{X}$. Indeed

$$\mathcal{E}_n = N_X^{-1}(n) = \{E \colon N(E, X) = n\},$$

whence $\mathcal{E}_n \in \mathcal{T} \subset \mathfrak{X}$ where the family of σ-algebras $\mathcal{T}$ is defined by (1). This implies that, on every subclass $\mathcal{E}_n$, a probability measure P is given. Put $p_n = \mathsf{P}\{\mathcal{E}_n\}$ for $n \in Z_+$. It is clear that the subclasses $\{\mathcal{E}_n\}_{n=0}^{\infty}$ are disjoint random events and

$$\sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \mathsf{P}\{\mathcal{E}_n\} = \mathsf{P}\left\{\bigcup_{n=0}^{\infty} \mathcal{E}_n\right\} = \mathsf{P}\{\mathcal{E}\} = 1.$$

We obtain a discrete probability mass $p_0, p_1, \dots, p_n, \dots$ and call it the generating sequence of the distribution P. The name is explained by the property that the probability measure P is expressed explicitly in terms of the generating sequence.

Similar reasoning applies for a marked point process $(\mathcal{E}^*, \mathfrak{X}^*, \mathsf{P}^*)$ in a bounded space $(Y, \mathfrak{A}_Y, \mathfrak{B}_Y)$ where Y is a compact metric space. If $\mathcal{E}_n^* = \{E^* \colon E^* \in \mathcal{E}^*, N^*(E^*, Y) = n\}$, then the subclasses $\{\mathcal{E}_n^*\}_{n=0}^{\infty}$ are disjoint random events in the space $\mathcal{E}^*$:

1) $\mathcal{E}_n^* \neq \varnothing$ for all $n = 0, \dots, \infty$,
2) $\mathcal{E}_m^* \cap \mathcal{E}_k^* = \varnothing$ for all $m, k \in Z_+$, $m \neq k$,
3) $\bigcup_{n=0}^{\infty} \mathcal{E}_n^* = \mathcal{E}^*$,
4) $\sum_{n=0}^{\infty} p_n^* = \sum_{n=0}^{\infty} \mathsf{P}^*\{\mathcal{E}_n^*\} = \mathsf{P}^*\{\bigcup_{n=0}^{\infty} \mathcal{E}_n^*\} = \mathsf{P}^*\{\mathcal{E}^*\} = 1,$

where $\{p_n^* = \mathsf{P}^*\{\mathcal{E}_n^*\}\}_{n=0}^{\infty}$ is the generating sequence of the probability measure P^* of the marked point process.

3. Mixed empirical ordered point processes in compact metric spaces

Assume that the space of positions X of a stochastic ordered point process $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ in a bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$ is a compact metric space equipped with the metric $\rho_X(x_1, x_2)$ and with natural structures of measurable sets $\mathfrak{A}_X$ and bounded sets $\mathfrak{B}_X$. We treat the space X as the sample space of a random variable x according to the probability distribution P_x on the σ-algebra $\mathfrak{A}_X$. Thus $(X, \mathfrak{A}_X, \mathsf{P}_x)$ is a sample probability space for the random variable x [2].

Assume that a number $n \in Z_+$ is chosen randomly (according to the probability distribution defined by the generating sequence $\{p_n\}$). Then every trajectory

$$E = (x_1, \dots, x_i, \dots, x_n)$$

of size n of the ordered point process in $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ is a result of n independent repetitions of the same random experiment G_1 which is a random choice without repetition of a point x_i in the space X. It is assumed that the experiment G_1 corresponds to the probability space $(X, \mathfrak{A}_X, \mathsf{P}_x)$. Then every trajectory E of the ordered point process $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ is a sample (a vector) without repetition of a finite random size n, $E = (x_1, \dots, x_i, \dots, x_j, \dots, x_n)$, $x_i \neq x_j$, $i \neq j$, from a population X of values of the random variable x whose distribution is P_x. It is easy to check that the random elements (random variables) $x_1, \dots, x_i, \dots, x_n$ are independent and identically distributed with the probability measure P_x where the nonnegative integer-valued random variable n does not depend on the random variables $x_1, x_2, \dots$. The process $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$ is called a mixed simple empirical ordered point process in the bounded space $(X, \mathfrak{A}_X, \mathfrak{B}_X)$ [10].

Now we give an expression for the probability distribution P_x of an arbitrary point of position (which also is denoted by x) of a trajectory E on the metric space X in terms of the probability measure P corresponding to the ordered point process $(\mathcal{E}, \mathfrak{X}, \mathsf{P})$. Namely

$$\begin{aligned}
\mathsf{P}_x(B_X) &= \mathsf{P}_x\{x \in B_X\} \\
&= \mathsf{P}\{E = (\{x\})\colon x \in B_X \mid \mathcal{E}_1\} \\
&= \mathsf{P}\{E = (\{x\})\colon N(E, B_X) = 1 \mid \mathcal{E}_1\}
\end{aligned}$$

for all $B_X \in \mathfrak{C}_X$.

Let

$$\mathsf{P}_x\{x \in B_X\} = \mathsf{P}_x(B_X) = p$$

for an arbitrary bounded measurable set B_X. If n is fixed, $N(E, X) = n$, $n \in Z_+$, and $B_X \in \mathfrak{C}_X$, then the conditional distribution of the nonnegative integer-valued counting measure $N(E, B_X)$ is the binomial $B(n, p)$ with parameter p:

$$\mathsf{P}\big\{E\colon N(E, B_X) = k \mid N(E, X) = n\big\} = \binom{n}{k} p^k q^{n-k}$$

where $N(E, B_X)$ is the frequency of the random event $\{x \in B_X\}$ in n independent trials, $q = 1 - p$, $k = 0, 1, 2, \dots, n$ (see [4, 9]). We propose the following method for constructing the probability distribution P on the σ-algebra $\mathfrak{X}$ of locally bounded sets. Let

$$\mathfrak{A}(B_X; k) = \{E\colon N(E, B_X) = k\}$$

for all $B_X \in \mathfrak{C}_X$, $k \in Z_+$. The full probability formula and the properties of the Bernoulli trials allow one to evaluate the distribution of the counting measure $N(E, B_X)$:

$$\begin{aligned} \mathsf{P}\{\mathfrak{A}(B_X;k)\} &= \sum_{n=k}^{\infty} p_n \, \mathsf{P}\bigl\{E\colon N(E,B_X)=k \mid N(E,X)=n\bigr\} \\ &= \sum_{n=k}^{\infty} p_n \binom{n}{k} p^k q^{n-k} \end{aligned} \tag{2}$$

where $\{p_n\}$ is the generating sequence of the distribution P. Distribution (2) is completely determined in terms of the measure $\mathsf{P}_x(B_X)=p$ and generating sequence $\{p_n\}$. Every term of series (2) is the product of two factors p_n and $\binom{n}{k}p^kq^{n-k}$. One can treat p_n as the probability of choosing a sample of size n from the population X (this probability is determined by the generating sequence $p_0, p_1, \dots, p_n, \dots$ of the distribution P). The term $\binom{n}{k}p^kq^{n-k}$ can be treated as the probability that the event $\{x_i \in B_X\}$ occurs exactly k times in n independent Bernoulli trials.

Remark 1. If an analog ϑ of the Lebesgue measure exists in the space X (this is the case, for example, if X is a compact differentiable manifold (say, a sphere or an ellipsoid)), then

$$p = \mathsf{P}_x\{B_X\} = \frac{\vartheta(B_X)}{\vartheta(X)},$$

$$\mathsf{P}\bigl\{E\colon N(E,B_X)=k \mid \mathcal{E}_n\bigr\} = \binom{n}{k}\left[\frac{\vartheta(B_X)}{\vartheta(X)}\right]^k \left[1-\frac{\vartheta(B_X)}{\vartheta(X)}\right]^{n-k},$$

$$\mathsf{P}\{\mathfrak{A}(B_X;k)\} = \sum_{n=k}^{\infty} p_n \binom{n}{k}\left[\frac{\vartheta(B_X)}{\vartheta(X)}\right]^k \left[1-\frac{\vartheta(B_X)}{\vartheta(X)}\right]^{n-k}.$$

Using probabilities (2), one can introduce the family of multidimensional distributions

$$p^{B_X^{(1)},\dots,B_X^{(m)}}\{(k_1,\dots,k_m)\} = \mathsf{P}\left\{\mathfrak{A}\left(B_X^{(1)},\dots,B_X^{(m)};k_1,\dots,k_m\right)\right\}$$

where $m \in \mathbb{N}$, $(k_1,\dots,k_m) \in Z_+^m$, $Z_+^m = \underbrace{Z_+ \times \cdots \times Z_+}_{m \text{ times}}$, and

$$\mathfrak{A}\left(B_X^{(1)},\dots,B_X^{(m)};k_1,\dots,k_m\right) = \left\{E\colon N\left(E,B_X^{(1)}\right)=k_1,\dots,N\left(E,B_X^{(m)}\right)=k_m\right\}.$$

The multidimensional distributions are defined for all sets $\left(B_X^{(1)},\dots,B_X^{(m)}\right) \subset \mathfrak{C}_X$. If the sets $B_X^{(1)},\dots,B_X^{(m)}$ are disjoint and $\bigcup_{i=1}^m B_X^{(i)} = X$, then

$$p^{B_X^{(1)},\dots,B_X^{(m)}}\{(k_1,\dots,k_m)\} = \prod_{i=1}^{m} \mathsf{P}\left\{E\colon N\left(E,B_X^{(i)}\right)=k_i\right\}. \tag{3}$$

Below we show that the family of distributions (3) determines the probability distribution P on the σ-algebra $\mathfrak{X}$ in the space of locally bounded sets $\mathcal{E}$.

Theorem 1. *The probability distribution* P *on the* σ*-algebra* $\mathfrak{X}$ *of the space of locally bounded sets* $\mathcal{E}$ *is determined by the family of distributions* (3)*:*

$$p^{B_X^{(1)},\dots,B_X^{(m)}}\{(k_1,\dots,k_m)\} = \prod_{i=1}^{m} \mathsf{P}\left\{E\colon N\left(E,B_X^{(i)}\right)=k_i\right\}$$

for all finite collections $\left(B_X^{(1)},\dots,B_X^{(m)}\right)$ *of disjoint sets belonging to the semiring* $\mathfrak{H}$ *that generates the* σ*-algebra* $\mathfrak{A}_X$ *and such that any set* $B_X \in \mathfrak{A}_X$ *can be covered by a finite family of sets of* $\mathfrak{H}$*.*

Proof. It is sufficient to check that the family of distributions (3) satisfies the assumptions of Theorem 1.4.1 of [3]. We recall Theorem 1.4.1 using our notation. Let the following hold.

1) $\mathfrak{B}$ is a ring of all bounded sets B_X of $\mathfrak{A}_X$.
2) $\mathfrak{H}$ is a semiring that generates $\mathfrak{B}$ and such that any set $B_X \in \mathfrak{B}$ can be covered by a finite family of sets of $\mathfrak{H}$.
3) To every finite sequence $B_X^{(1)}, \dots, B_X^{(m)}$ of disjoint sets of $\mathfrak{H}$ there corresponds a distribution $p^{B_X^{(1)},\dots,B_X^{(m)}}$ belonging to the space of all distributions $(\mathbb{P}, \mathfrak{O})$ defined on the σ-algebra $\mathfrak{M}$ of all counting measures [3]. This distribution is given by

$$p^{B_X^{(1)},\dots,B_X^{(m)}}\{(\ell_1,\dots,\ell_m)\} = \prod_{i=1}^m \mathsf{P}\left\{E\colon N\left(E, B_X^{(i)}\right) = \ell_i\right\} \tag{4}$$

for all $(\ell_1,\dots,\ell_m) \subset Z_+^m$.

Then the distribution P exists if and only if

a) for an arbitrary permutation $i_1,\dots,i_m$ of indices $1,\dots,m$ and for all numbers $\ell_1,\dots,\ell_m \in Z_+$, one has
$$p^{B_X^{(1)},\dots,B_X^{(m)}}\{(\ell_1,\dots,\ell_m)\} = p^{B_X^{(i_1)},\dots,B_X^{(i_m)}}\{(\ell_{i_1},\dots,\ell_{i_m})\};$$
b) for all numbers $\ell_1,\dots,\ell_{m-1} \in Z_+$,
$$\begin{aligned} &p^{B_X^{(1)},\dots,B_X^{(m)}}\{(\nu_1,\dots,\nu_m)\colon \nu_1 = \ell_1,\dots,\nu_{m-1} = \ell_{m-1}, \nu_m = Z_+\} \\ &\qquad = p^{B_X^{(1)},\dots,B_X^{(m-1)}}\{(\ell_1,\dots,\ell_{m-1})\};\end{aligned}$$
c) if the union B_X of sets $B_X^{(k)},\dots,B_X^{(m)}$ belongs to the space $\mathfrak{H}$, then
$$\begin{aligned} &p^{B_X^{(1)},\dots,B_X^{(k-1)},B_X}\{(\ell_1,\dots,\ell_k)\} \\ &\qquad = p^{B_X^{(1)},\dots,B_X^{(m)}}\left\{(\nu_1,\dots,\nu_m)\colon \nu_i = \ell_i, i = 1,\dots,k-1, \sum_{i=k}^m \nu_i = \ell_k\right\}\end{aligned}$$
for all numbers $\ell_1,\dots,\ell_k \in Z_+$;
d) for an arbitrary monotone sequence $\left\{B_X^{(n,1)} \cup \dots \cup B_X^{(n,m_n)}\right\}_{n=1,2,\dots}$ decreasing to the empty set $\varnothing$ and such that any element of the sequence is a finite union of disjoint sets of $\mathfrak{H}$, one has
$$\lim_{n\to\infty} p^{B_X^{(n,1)},\dots,B_X^{(n,m_n)}}\{(0,\dots,0)\} = 1.$$

Condition a) holds in view of the definition of the distribution $p^{B_X^{(1)},\dots,B_X^{(m)}}$ (see (3)). Further, definition (3) implies

$$\begin{aligned} &p^{B_X^{(1)},\dots,B_X^{(m)}}\{(\nu_1,\dots,\nu_m)\colon \nu_1 = \ell_1,\dots,\nu_{m-1} = \ell_{m-1}, \nu_m \in Z_+\} \\ &\quad = \mathsf{P}\left\{E\colon N\left(E, B_X^{(1)}\right) = \ell_1\right\} \dots \mathsf{P}\left\{E\colon N\left(E, B_X^{(m-1)}\right) = \ell_{m-1}\right\} \\ &\qquad \times \mathsf{P}\left\{E\colon N\left(E, B_X^{(m)}\right) \in Z_+\right\} \\ &\quad = \mathsf{P}\left\{E\colon N\left(E, B_X^{(1)}\right) = \ell_1\right\} \dots \mathsf{P}\left\{E\colon N\left(E, B_X^{(m-1)}\right) = \ell_{m-1}\right\} \\ &\quad = p^{B_X^{(1)},\dots,B_X^{(m-1)}}\{(\ell_1,\dots,\ell_{m-1})\}\end{aligned}$$

for all $(\nu_1,\dots,\nu_m) \subset Z_+^m$, since $\mathsf{P}\{E\colon N(E, B_X^{(m)}) \in Z_+\} = 1$. Therefore condition b) also holds.

Let the union

$$B_X = \bigcup_{i=k}^{m} B_X^{(i)}$$

of disjoint subsets $\left(B_X^{(k)}, \dots, B_X^{(m)}\right) \subset \mathfrak{H}$ belong to the semiring $\mathfrak{H}$ that generates the σ-algebra $\mathfrak{A}_X$. Since

$$\begin{aligned}
\{E\colon N(E,B_X)=\ell_k\} &= \left\{E\colon N\left(E, \bigcup_{i=k}^{m} B_X^{(i)}\right) = \ell_k\right\}\\
&= \left\{E\colon N\left(E, B_X^{(k)}\right)=\nu_k, \dots, N\left(E, B_X^{(m)}\right)=\nu_m, \nu_k+\dots+\nu_m=\ell_k\right\},\\
\mathsf{P}\{E\colon N(E,B_X)=\ell_k\}&\\
&= \mathsf{P}\left\{E\colon N\left(E, B_X^{(k)}\right)=\nu_k, \dots, N\left(E, B_X^{(m)}\right)=\nu_m, \nu_k+\dots+\nu_m=\ell_k\right\}
\end{aligned}$$

for all $\ell_1, \dots, \ell_k \in Z_+$, we have

$$\begin{aligned}
p^{B_X^{(1)},\dots,B_X^{(k-1)},B_X}\{(\ell_1,\dots,\ell_k)\} &= \prod_{i=1}^{k-1} \mathsf{P}\left\{E\colon N\left(E, B_X^{(i)}\right)=\ell_i\right\} \mathsf{P}\{E\colon N(E,B_X)=\ell_k\}\\
&= \prod_{i=1}^{k-1} \mathsf{P}\left\{E\colon N\left(E, B_X^{(i)}\right)=\ell_i\right\}\\
&\quad\times \mathsf{P}\left\{E\colon N\left(E, B_X^{(k)}\right)=\nu_k, \dots, N\left(E, B_X^{(m)}\right)=\nu_m, \nu_k+\dots+\nu_m=\ell_k\right\}\\
&= p^{B_X^{(1)},\dots,B_X^{(m)}}\left\{(\nu_1,\dots,\nu_m)\colon \nu_i=\ell_i (i=1,\dots,k-1), \sum_{i=k}^{m}\nu_i=\ell_k\right\},
\end{aligned}$$

whence condition c) follows.

To check condition d), we consider an arbitrary monotone sequence

$$\left\{B_X^{(n)}\right\}_{n=1,2,\dots} = \left\{B_X^{(n,1)} \cup \dots \cup B_X^{(n,m_n)}\right\}_{n=1,2,\dots}$$

that decreases to the empty set $\varnothing$:

$$\left\{B_X^{(n)}\right\}_{n=1,2,\dots} \searrow \varnothing, \qquad n\to\infty,$$

where every element $B_X^{(n)}$ of the sequence is a finite union of disjoint subsets

$$B_X^{(n,1)}, \dots, B_X^{(n,m_n)} \subset \mathfrak{H}.$$

Since

$$B_X^{(n)} = B_X^{(n,1)} \cup \dots \cup B_X^{(n,m_n)} \searrow \varnothing, \qquad n\to\infty,$$

there exists n_0 such that

$$N\left(E, B_X^{(n)}\right) = N\left(E, B_X^{(n,1)} \cup \dots \cup B_X^{(n,m_n)}\right) = 0$$

for all $n > n_0$. This implies that $N\left(E, B_X^{(n,i)}\right) = 0$, $i=1,\dots,m_n$, for all $n>n_0$, since $N(E,(\cdot))$ is a counting measure. Then

$$p^{B_X^{(n)}}\{0\} = p^{B_X^{(n,1)},\dots,B_X^{(n,m_n)}}\{(0,\dots,0)\}$$

and

$$\lim_{n\to\infty} p^{B_X^{(n)}}\{0\} = \lim_{n\to\infty} p^{B_X^{(n,1)},\dots,B_X^{(n,m_n)}}\{(0,\dots,0)\}.$$

Moreover equality (2) yields that

$$p^{B_X^{(n)}}\{0\} = \mathsf{P}\left\{E\colon N\left(E, B_X^{(n)}\right) = 0\right\} = \sum_{i=0}^{\infty} p_i \binom{i}{0} \widetilde{p}_n^{\,0} \widetilde{q}_n^{\,i} = \sum_{i=0}^{\infty} p_i \widetilde{q}_n^{\,i},$$

where $\widetilde{p}_n = \mathsf{P}_X\{B_X^{(n)}\}$ and $\widetilde{q}_n = 1 - \widetilde{p}_n$, whence

$$\lim_{n\to\infty} p^{B_X^{(n,1)},\dots,B_X^{(n,m_n)}}\{(0,\dots,0)\} = \sum_{i=0}^{\infty} p_i \lim_{n\to\infty} \widetilde{q}_n^{\,i} = \sum_{i=0}^{\infty} p_i = 1,$$

since $\widetilde{q}_n \to 1$ as $n \to \infty$.

Thus all the assumptions of Theorem 1.4.1 of [3] hold and therefore there exists a distribution P on the σ-algebra $\mathfrak{X}$ of the space of locally bounded sets $\mathcal{E}$ whose finite dimensional distributions coincide with $p^{B_X^{(1)},\dots,B_X^{(n)}}$ [3, 9].

The theorem is proved. $\square$

The most interesting case for applications is the one where the semiring $\mathfrak{H}$ generating the σ-algebra $\mathfrak{A}_X$ coincides with the σ-algebra of all Borel sets of the space X: $\mathfrak{H} = \mathfrak{A}_X$.

Consider an example where we construct and evaluate explicitly the finite dimensional distributions (3).

Example 1. Let X be the two dimensional sphere $\mathcal{S}^2$ endowed with the natural metric and topology; it is known that $X = \mathcal{S}^2$ is a compact metric space. The Lebesgue measure exists on the sphere $\mathcal{S}^2$ (the same holds for any other oriented differentiable manifolds). Thus every probability distribution P_x, absolutely continuous with respect to the Lebesgue measure, of a random variable x is defined on the σ-algebra $\mathfrak{A}_{\mathcal{S}^2}$ by the Radon–Nikodym density $f(\varphi, \theta)$ where φ and θ are the spherical coordinates on $\mathcal{S}^2$ (φ is the longitude and θ is the latitude). If $B_{\mathcal{S}^2}$ is an arbitrary measurable (Borel) subset of $\mathfrak{A}_{\mathcal{S}^2}$, then

$$p = \mathsf{P}_x(B_{\mathcal{S}^2}) = \iint_{B_{\mathcal{S}^2}} f(\varphi,\theta)\, d\varphi\, d\theta.$$

Assume that the generating sequence $\{p_n\}$ is such that $p_n = 1/2^{n+1}$, $n = 0, 1, 2, \dots$, and $p \in (0,1)$. Then

$$\begin{aligned}\mathsf{P}\{\mathfrak{A}(B_{\mathcal{S}^2}; k)\} &= \sum_{n=k}^{\infty} \frac{1}{2^{n+1}} \binom{n}{k} p^k q^{n-k} \\ &= \frac{1}{2k!}\left(\frac{p}{q}\right)^k \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\left(\frac{q}{2}\right)^n\end{aligned} \tag{5}$$

for all $k \in Z_+$ where $q = 1 - p$.

To evaluate the sum of series (5), consider the function

$$f(t) = \frac{t}{1-t} = t + t^2 + \cdots + t^m + \cdots, \qquad 0 < t < 1. \tag{6}$$

Differentiating the power series (6) k times on the interval $(0,1)$, we obtain the expression for the k-th derivative $f^{(k)}(t)$:

$$f^{(k)}(t) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) t^{n-k}.$$

On the other hand, differentiating the expression $f(t) = t/(1-t)$, we obtain

$$f^{(k)}(t) = \frac{k!}{(1-t)^{k+1}}.$$

Thus

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)t^{n-k} = \frac{k!}{(1-t)^{k+1}}$$

and

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)t^{n} = \frac{t^k k!}{(1-t)^{k+1}}. \tag{7}$$

Considering $t = q/2$, we obtain from (3), (5), and (7) that

$$\mathsf{P}\{\mathfrak{A}(B_{\mathcal{S}^2};k)\} = \frac{p^k}{(1+p)^{k+1}},$$

$$p^{B^{(1)}_{\mathcal{S}^2},\dots,B^{(m)}_{\mathcal{S}^2}}\{(k_1,\dots,k_m)\} = \prod_{i=1}^{m} \frac{p_i^{k_i}}{(1+p_i)^{k_i+1}}$$

for all $m \in \mathbb{N}$, $(k_1,\dots,k_m) \in Z_+^m$, $B_{\mathcal{S}^2} \in \mathfrak{A}_{\mathcal{S}^2}$, and all disjoint sets $\left(B^{(1)}_{\mathcal{S}^2},\dots,B^{(m)}_{\mathcal{S}^2}\right) \subset \mathfrak{A}_{\mathcal{S}^2}$ where

$$p_i = \mathsf{P}_x\left(B^{(i)}_{\mathcal{S}^2}\right) = \iint_{B^{(i)}_{\mathcal{S}^2}} f(\varphi,\theta)\,d\varphi\,d\theta, \qquad i = 1,\dots,m.$$

Similar results can be proved for an arbitrary compact oriented differentiable manifold, since there exists an analog of the Lebesgue measure on this metric space and one can apply the Lebesgue theory of integration [1, 7].

Concluding remarks

Some models of mixed finite simple empirical ordered point processes are considered in compact metric spaces by applying the Moyal ideas for constructing point processes on abstract spaces [11]. A trajectory of a random size n of such a process is a result of n independent repetitions of the same random experiment which is the random choice without repetition of a point of the compact metric space X according to the probability measure P_x. For ordered point processes, the family of multidimensional distributions is constructed and it is proved that it determines the probability measure of the ordered point process.

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