THE COUNTING PROCESS AND SUMMATION
OF A RANDOM NUMBER OF RANDOM VARIABLES

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Abstract. The behavior of the tail of the sum of a random number of random variables $F(x) = P\left(\sum_{i=1}^{\nu} \xi_i > x\right)$ is considered as $x \to \infty$. Estimates of the convergence of $F(x)$ to the limit function are constructed in terms of renewal theory. The estimates are based on the variance $\text{Var}_\nu(t)$ of the counting process $\nu(t) = \min\{n: \sum_{i=1}^{n} \xi_i > t\}$. A survey of bounds for $\text{Var}_\nu(t)$ is given for different sequences $\{\xi_i\}$, in particular, for the case where the terms of the sequence $\{\xi_i\}$ are not identically distributed.

1. Introduction and a survey of literature

Assume that $\{\xi_i, i \geq 1\}$ is a sequence of random variables such that $E \xi_i = m_i > 0$ and $E \xi_i^2 < \infty$ for all $i \geq 1$. Let $\nu$ be an integer-valued random variable independent of $\{\xi_i\}$ and whose second moment is finite, $E \nu^2 < \infty$. Put $P\{\nu = n\} = p_n, n \geq 0$. Denote by

$$\nu(t) = \min\left\{n: \sum_{i=1}^{n} \xi_i > t\right\}$$

the counting process generated by the sequence $\{\xi_i\}$. Let us also define the residual lifetime process

$$\zeta_t = \nu(t) \sum_{i=1}^{\nu(t)} \xi_i - t$$

and the renewal function and the variance of the counting process

$$H(t) = E \nu(t), \quad D(t) = \text{Var} \nu(t).$$

Set

$$\tau = \sum_{i=1}^{\nu} \xi_i.$$

We are interested in the behavior of $P\{\tau > x\}$ as $x \to \infty$. As is well known, the sequence $\{\xi_i\}$ uniquely determines the counting process containing, in a compact form, all information on hitting a set by the random walk $S_n = \sum_{i=1}^{n} \xi_i$. Therefore all bounds concerning the behavior of $P\{\tau > x\}$ are conveniently expressed using terminology in renewal theory. These results can be obtained by studying random walks generating the required type of renewal processes and by applying upper bounds for characteristics of these processes. In particular, this approach gives estimates for central moments of

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the counting process $\nu(t)$ and for moments of the residual lifetime $\zeta_i$. The paper by J. T. Chang [1] established moment inequalities and almost sure upper bounds for the residual lifetime of a usual renewal process. This paper also contains bounds for the tail $P\{\zeta_i > x\}$ and, in general, improves the results obtained earlier by G. Lorden [2]. Fundamental works by C. D. Fuh and T. L. Lai [3, 4] deal with Markov random walks. They prove the Wald equality and renewal theorem for Markov random walks and obtain representations for $E\zeta_i'$. As a by-product, they obtain bounds for $E\zeta_i$. The paper [5] contains moment inequalities for $\zeta_i$ in the case where the random variables $\{\xi_i\}$ are independent and identically distributed. The results are applied to the problem of summation of a geometric number of random variables. Our aim is to construct a general bound for the approximation of $P\{\tau > x\}$ by some limit function and to outline the range of problems where these bounds can be applied.

2. Main result

**Theorem 2.1.** Suppose that $p(x)$ is a convex differentiable nonincreasing function such that $p(n) = p_n$, $n \geq 0$. Denote $R(x) = -p'(x)$. Then for any number $\alpha$ belonging to the interval $(0, 1)$, we have the following inequality:

$$
|P\{\tau > x\} - P\{\nu > H(x)\}| \\
\leq \frac{1}{2} R(\alpha H(x))(D(x) + 1) + (2 + (H(x) + 1)p(H(x))) P\{\nu \leq \alpha H(x)\} \\
+ p(H(x)) + p(\alpha H(x)) \\
\leq 2p(\alpha H(x)) + \frac{3 + 2 E \nu}{(1 - \alpha)^2} D(x) + \frac{1}{2} R(\alpha H(x))(D(x) + 1).
$$

**Proof.** Let us write

$$
P\{\tau > x\} = P\left\{\sum_{i=1}^{\nu} \zeta_i > x\right\} = P\{\nu \geq \nu(x)\} = \sum_{i=0}^{\infty} P\{\nu = \nu(x) + i\} \\
= \sum_{i=0}^{\infty} E p_{\nu(x)+i} = E \sum_{i=\nu(x)}^{\infty} p_i.
$$

In a similar way,

$$
P\{\nu > H(x)\} = \sum_{i=0}^{\infty} P\{\nu = [H(x)] + 1 + i\} = E \sum_{i=[H(x)]+1}^{\infty} p_i.
$$

Then

$$
|P\{\tau > x\} - P\{\nu > H(x)\}| = \left| E \sum_{i=\nu(x)}^{\infty} p_i - E \sum_{i=[H(x)]+1}^{\infty} p_i \right| \\
\leq E \max\left(\left| \sum_{i=[H(x)]+1}^{\nu(x)-1} p_i \right|, \left| \sum_{i=\nu(x)}^{[H(x)]} p_i \right| \right) \\
\leq E \sum_{i=[H(x)]+1}^{\nu(x)} p_i.
$$

(4)

(If the upper bound $b$ in the sum on the right-hand side of (4) is less than the lower bound $a$, that is, $b < a$, we put $\sum_{a}^{b} p_i = -\sum_{a}^{b} p_i$.) Let us plot the function $y = p(x)$ and mark the points whose coordinates are $(a, p_a)$, $(a + 1, p_{a+1})$, ..., $(b + 1, p_{b+1})$, where $a, b \in N$, $a < b$. “Inscribe” and “circumscribe”
stepwise functions into the plot. These stepwise functions consist of rectangles (Figure 1).

By comparing areas, we obtain

$$\sum_{i=a+1}^{b+1} p(i) \leq \int_a^{b+1} p(s) \, ds \leq \sum_{i=a}^{b} p(i).$$

Or, in other words,

$$p(b) \leq \sum_{i=a}^{b} p(i) - \int_a^{b} p(s) \, ds \leq p(a).$$

Therefore formulas (4)–(5) imply that

$$|P\{\tau > x\} - P\{\nu > H(x)\}|$$

$$\leq \left| E \int_{i=[H(x)]+1}^{\nu(x)} p(s) \, ds \right| + p([H(x)]+1) + E p(\nu(x))$$

$$\leq \left| E \int_{i=[H(x)]+1}^{\nu(x)} p(s) \, ds \right| + p(H(x)) + E p(\nu(x)).$$

Let us transform the integral on the right-hand side of (6). Take an arbitrary number $\alpha$ belonging to the interval $(0, 1)$. Then

$$E \int_{[H(x)]+1}^{\nu(x)} p(s) \, ds$$

$$= E \left( \int_0^{\nu(x)} p(s) \, ds - \int_0^{[H(x)]+1} p(s) \, ds - p([H(x)]+1)(\nu(x) - [H(x)] - 1) \right)$$

$$\times I\{\nu(x) > \alpha H(x)\}$$

$$+ E \left( \int_0^{\nu(x)} p(s) \, ds - \int_0^{[H(x)]+1} p(s) \, ds - p([H(x)]+1)(\nu(x) - [H(x)] - 1) \right)$$

$$\times I\{\nu(x) \leq \alpha H(x)\}$$

$$= A + B.$$
Here, \( I\{S\} \) stands for the indicator of an event \( S \). We draw a bound for each term in (7) on its own. The absolute value \(|A|\) is evaluated by using the Taylor formula:

\[
|A| = \left| \mathbb{E} \frac{p'(u)}{2} (\nu(x) - [H(x)] - 1)^2 I\{\nu(x) > \alpha H(x)\} \right|
\]

where \( u \) is contained between \( \nu(x) \) and \( [H(x)] + 1 \). Since \( R(x) = -p'(x) \) is nonincreasing, we have

\[ R(u) \leq R(\min(\nu(x), [H(x)] + 1)). \]

Further, since \( \nu(x) > \alpha H(x) \), we obtain \( \min(\nu(x), [H(x)] + 1) > \alpha H(x) \). Therefore

\[
|A| \leq \frac{R(\alpha H(x))}{2} (D(x) + 1).
\]

Now we estimate \(|B|\):

\[
|B| \leq \mathbb{E} \left| \int_0^{\nu(x)} p(s) \, ds - \int_0^{[H(x)]+1} p(s) \, ds \right| I\{\nu(x) \leq \alpha H(x)\}
+ p([H(x)] + 1) \mathbb{E} |\nu(x) - [H(x)] - 1| I\{\nu(x) \leq \alpha H(x)\}.
\]

Inequality (5) implies that

\[
\int_0^\infty p(s) \, ds \leq \sum_{i=0}^\infty p_i = 1.
\]

Therefore the first term on the right-hand side of (9) does not exceed \( \mathbb{P}\{\nu(x) \leq \alpha H(x)\} \).

The second term yields the bound

\[
p([H(x)]) \mathbb{E} |\nu(x) - H(x) - 1| I\{\nu(x) \leq \alpha H(x)\}
\leq (H(x) + 1)p([H(x)]) \mathbb{P}\{\nu(x) \leq \alpha H(x)\}.
\]

The bound for \(|B|\) takes the form

\[
|B| \leq (1 + (H(x) + 1)p(H(x))) \mathbb{P}\{\nu(x) \leq \alpha H(x)\}.
\]

Observe that

\[
\mathbb{E} p(\nu(x)) \leq \mathbb{E} p(\nu(x)) I\{\nu(x) > \alpha H(x)\} + \mathbb{E} p(\nu(x)) I\{\nu(x) \leq \alpha H(x)\}
\leq p(\alpha H(x)) + \mathbb{P}\{\nu(x) \leq \alpha H(x)\}
\]

and apply bounds (6)–(8) and (10) to obtain the first inequality in the statement of Theorem 2.1. We obtain the second inequality by using the Chebyshev inequality:

\[
\mathbb{P}\{\nu(x) \leq \alpha H(x)\} \leq \mathbb{P}\{|\nu(x) - H(x)| \geq (1 - \alpha)H(x)\} \leq \frac{D(x)}{(1 - \alpha)^2 H^2(x)},
\]

since

\[
H(x)p(H(x)) < \frac{H(x)}{[H(x)]} p([H(x)]) < 2 \mathbb{E} \nu. \quad \square
\]

Remark 2.1. This argument was successfully applied to the problem of summation of a geometric number of independent random variables in the paper [6].
3. The variance of the counting process

Theorem 2.1 can be successfully used for obtaining upper bounds if a correct order in inequalities for $D(t)$ is known. Below we consider different cases.

1. If $\{\xi_i\}$ is a sequence of nonnegative independent identically distributed random variables with the mean $E \xi_i = m$, then it is well known that

$$D(t) \sim \frac{E \xi_i^2 - m^2}{m^3} t \text{ as } t \to \infty.$$ 

Thus one can readily see that

$$D(t) \leq \frac{4\kappa}{m} t + 2\kappa - 2\kappa, \quad \kappa = \frac{E \xi_1^2}{m^2}.$$ 

In this case, the renewal function satisfies the following inequality [2]:

$$\frac{t}{m} \leq H(x) \leq \frac{t}{m} + \kappa.$$ 

2. If $\{\xi_i\}$ is a sequence of nonnegative independent, but not identically distributed, random variables having the same mean $E \xi_i = m$ for all $i \geq 1$, then the bounds proved in [5] can be applied:

$$H(x) \leq \frac{t}{m} + \frac{1}{2} \sqrt{\frac{4}{m} \kappa t + \kappa^2 + \kappa}, \quad \kappa = \sup_i E \xi_i^2 / m^2,$$

$$D(t) \leq 4\kappa H(t) \leq 2\kappa \left( \frac{2}{m} t + \sqrt{\frac{4}{m} \kappa t + \kappa^2 + \kappa} \right).$$

3. Let $\{\xi_i\}$ be a sequence of independent identically distributed random variables assuming negative values and having the mean $E \xi_1 = m > 0$. It is well known [7] that in this setting $\nu(t)$ ranges not over all positive integers, but only over some values $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ called ladder epochs, while the random variables

$$X_1 = \sum_{i=1}^{\tau_1} \xi_i, \quad X_2 = \sum_{i=\tau_1+1}^{\tau_2} \xi_i, \quad \ldots, \quad X_n = \sum_{i=\tau_{n-1}+1}^{\tau_n} \xi_i, \quad \ldots,$$

called ladder variables, form a sequence of independent identically distributed random variables whose distribution equals $X = \sum_{i=1}^{\nu_0(0)} \xi_i$. It is clear that in this case

$$0 < X \leq \xi^{+}_{\nu_0(0)}, \quad \xi^{+} = \begin{cases} \xi & \text{if } \xi \geq 0, \\ 0 & \text{if } \xi < 0. \end{cases}$$

The mean residual lifetime $E \zeta_t$ for such processes was already described by G. Lorden [2]:

$$E \zeta_t \leq \frac{E(\xi^{+})^2}{m}.$$ 

An approach similar to that used by G. Lorden can be used for obtaining bounds for higher moments of $\zeta_t$. If $\zeta_t$ is a usual renewal process, then

$$E \zeta_t^p \leq \frac{p + 2}{p + 1} \frac{E \xi^{p+1}}{E \xi}.$$
for $p > 0$ (see [2, 11, p. 1224]). Therefore these estimates also hold for the process generated by the sequence $\{X_n\}$. Hence, by (11) and by the Wald identity, we obtain

$$
E\zeta^p \leq \frac{p + 2 EX^{p+1}}{p + 1} \leq \frac{p + 2 E(\xi^{p+1}_n)}{p + 1} \leq \frac{p + 2 E(\sum_{i=1}^{\nu(0)} (\xi_i^+)_{p+1}^+) - \sum_{i=1}^{\nu(0)} (\xi_i^+)_{p+1}^+}{p + 1} \leq \frac{p + 2 E(\xi_i^+)_{p+1}^+}{p + 1}.
$$

Now we estimate the variance and higher moments of the counting process. Following the lines of [5], we write

$$
E(\nu(t) - H(t))^p = \frac{1}{m^p} E \left( m\nu(t) - t - E \zeta_t - \sum_{i=1}^{\nu(t)} (\xi_i^+)_{p+1}^+ \right)^p.
$$

By the inequality

$$
|a + b|^p \leq 2^{p-1} (|a|^p + |b|^p), \quad p \geq 2,
$$

we obtain

$$
E(\nu(t) - H(t))^p \leq \frac{2^{p-1}}{m^p} \left( E|\zeta_t - E\zeta_t|^p + E \left( \sum_{i=1}^{\nu(t)} (\xi_i - m) \right)^p \right).
$$

The first term is estimated by using (11) and (13):

$$
E|\zeta_t - E\zeta_t|^p \leq 2^p E\zeta_t^p \leq \frac{2p + 2 E(\xi^+)_{p+1}^+}{p + 1}.
$$

A bound for the second term is obtained using the following theorem.

**Theorem 3.1** ([5]). Assume that $\zeta_1, \zeta_2, \ldots$ is a sequence of random variables, and $\mathcal{A}_0, \mathcal{A}_1, \ldots$ is a sequence of nondecreasing $\sigma$-algebras such that $\zeta_n$ is measurable with respect to $\mathcal{A}_n$. Let $\nu$ be a stopping time, that is, $\{\nu \leq k\} \in \mathcal{A}_k, k \geq 0$. Put

$$Z_0 = 0, \quad Z_k = \sum_{i=1}^{k} \zeta_i.
$$

(I) Let

$$E(\zeta_k / \mathcal{A}_{k-1}) = 0, \quad E(\delta^+ / \mathcal{A}_{k-1}) \leq A_\delta, \quad E(\delta / \mathcal{A}_0) \leq B_\delta
$$

for all $k$. Then

$$E(|Z_\nu|^{\delta}/\mathcal{A}_0) \leq c_1 A_\delta B_1 + c_2 A_\delta^{3/2} B_\delta^{1/2}
$$

for $\delta > 2$. If $\delta = 2$, then

$$E(Z_\nu^2 / \mathcal{A}_0) \leq A_2 B_1.
$$

(II) Let $\zeta_k \geq 0$ almost surely. Then

$$E(Z_\nu^2 / \mathcal{A}_0) \leq c_3 (A_\delta B_1 + A_\delta^2 B_\delta), \quad \delta \geq 1.
$$

In particular,

$$E Z_\nu \leq A_1 B_1.
$$

Here, the $c_i$ are constants that depend on $\delta$ only.
It follows from the proof of Theorem 3.1 in [5] that the constants \( c_1 \) and \( c_2 \) can be taken to be
\[
c_1 = 2 \cdot 4^3, \quad c_2 = 4 \cdot 2^{(4\delta^2-3\delta-2)/\delta} \delta^{-3}(\delta - 2)/(\delta - 2)/\delta.
\]
Put \( \zeta_i = \xi_i - m, \ i \geq 1 \). Let \( \mathcal{G}_0 = \{ \Omega, \emptyset \} \) be the trivial \( \sigma \)-algebra, and let
\[
\mathcal{G}_i = \sigma(\zeta_1, \ldots, \zeta_i)
\]
be the \( \sigma \)-algebra generated by the random variables \( \zeta_1, \ldots, \zeta_i, \ i \geq 1 \). Since the \( \xi_i \) are independent, we have
\[
E(\zeta_i / \mathcal{G}_{i-1}) = E(\zeta_i - m) = 0, \quad E(|\zeta_i|^p / \mathcal{G}_{i-1}) = E|\xi_i - m|^p \leq 2^{p-1}(E|\xi|^p + m^p),
\]
By substituting (15) and (16) into (14), the following statement is proved.

Lemma 3.1. In our setting, the following inequality holds for the central moments of the counting process if \( p > 2 \):
\[
E(\nu(t) - H(t))^p \leq \frac{2^{p-1}}{m^p} \left( \frac{2^p p^2 + 2 E(\xi^+)^{p+1}}{p+1} m + c_1 2^{p-1}(E|\xi|^p + m^p)H(t) + c_2 (E \xi^2 + m^2)^{p/2} E \nu(t)^{p/2} \right)
\]
where the constants \( c_1 \) and \( c_2 \) are the same as defined in inequality (16). In particular, if \( p = 2 \), then
\[
D(t) \leq \frac{2}{m^2} \left( \frac{16 E(\xi^+)^3}{m} + 2 (E \xi^2 + m^2) H(t) \right)
\leq \frac{4}{m^2} \left( (E \xi^2 + m^2) t + \frac{8}{3} E(\xi^+) + (E \xi^2 + m^2) E(\xi^+)^2 \right).
\]
An exact bound for the renewal function can be found in [3]:
\[
|H(t) - \frac{t}{m} - \frac{1}{2} E \xi^2| \leq \frac{1}{2} E(\xi^+)^2 m^2.
\]
4. Now suppose that \( \{\xi_i\} \) is a sequence of independent nonidentically distributed random variables that may attain negative values and such that
\[
E \xi_i = m > 0, \quad i \geq 1.
\]
It turns out that the argument used in case 3 can also be applied in the nonhomogenous case almost literally. We make similar estimates:
\[
(17) \quad E(\nu(t) - H(t))^p \leq \frac{2^{p-1}}{m^p} \left( E|\zeta_i - \xi| + E \sum_{i=1}^{\nu(t)} (\xi_i - m) \right)^p.
\]
The second term in (17) can be bounded using estimate (16); the proof follows the lines of that in case 3. A bound for $E\xi_p$ can be obtained using the argument in [5]. As we have already done in case 3, one can suppose that the process $\zeta$ is generated by the sequence of nonnegative random variables

$$X_1 = \sum_{i=1}^{\tau_1} \xi_i, \quad X_2 = \sum_{i=\tau_1+1}^{\tau_2} \xi_i, \quad \ldots, \quad X_n = \sum_{i=\tau_{n-1}+1}^{\tau_n} \xi_i, \quad \ldots,$$

where the $\{\tau_k\}$ are similar ladder epochs:

$$\tau_1 = \left\{ \min n: \sum_{i=1}^{n} \xi_i > 0 \right\}, \quad \ldots, \quad \tau_k = \left\{ \min n: \sum_{i=\tau_{k-1}+1}^{n} \xi_i > 0 \right\}, \quad \ldots.$$

Denote by $\tilde{\nu}(t) = \{\min n: \sum_{i=1}^{n} X_i > t\}$ the counting process generated by the sequence $\{X_i\}$. Suppose also that $M_p = \sup_{n \geq 1} E X_n p < \infty$. In contrast to case 3, the $\{X_i\}$ are not only nonidentically distributed, but also dependent. Nevertheless this does not affect the general form of the process $\zeta^{-1}_x, \ p \geq 2$, which remains piecewise continuous. By integrating this process with respect to $x$ on the segment $[0, t]$ and by taking the mathematical expectation on both sides of the equality, we have

$$0 \leq \int_0^t E \zeta_x^{-1} dx = \frac{1}{p} E \sum_{i=1}^{\tilde{\nu}(t)} X_i^{-1} - \frac{1}{p} E \zeta_t^{-1}.$$

Therefore

$$E \zeta_t^{-1} \leq E \sum_{i=1}^{\tilde{\nu}(t)} X_i^{-1}.$$

Now apply part (II) of Theorem 3.1 to obtain

$$E \zeta_t^{-1} \leq E \tilde{\nu}(t) \sup_{n \geq 1} E X_n^{-1} \leq H(t) \sup_{n \geq 1} E (\xi_n^+)^p.$$

This gives the following result.

**Lemma 3.2.** Under the assumptions of case 4, the central moments of the counting process satisfy the following inequality for $p > 2$:

$$E(\nu(t) - H(t))^p \leq \frac{2p-1}{mp} \left( 2p H(t) e_p^+ + c_1 2p-1 (e_p^+ + m^p) H(t) + c_2 \left( \sup_n E \xi_n^2 + m^2 \right)^{p/2} \right)^{p/2} E(\nu(t))^{p/2},$$

where $e_p^+ = \sup_{n \geq 1} E (\xi_n^+)^p$ and where $c_1$ and $c_2$ are the same constants as in inequality (16).

**BIBLIOGRAPHY**


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