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BOUNDARY FUNCTIONALS FOR THE SUPERPOSITION OF A RANDOM WALK AND A SEQUENCE OF INDEPENDENT RANDOM VARIABLES

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Abstract. For the superposition of a random walk and a sequence of independent random variables, we obtain the moment generating functions of the joint distribution of the first passage time and overshoot over a level, and those of the joint distribution of the first exit time from an interval and the value of the superposition at the exit time.

Introduction

Let $\{\xi_n\}$ and $\{\eta_n\}$, $n \in \mathbb{N}$, be two sequences of independent identically distributed integer-valued random variables. Assume that ξ_i and η_j are independent for all $i, j \in \mathbb{N}$. Put $\mathbb{Z} = \{0, \pm 1, \dots\}$ and denote by

$$\xi(0) = 0, \qquad \xi(n) = \xi_1 + \dots + \xi_n,$$
$$\eta_l(0) = l, \qquad \eta_l(n) = \eta_n, \qquad n \in \mathbb{N}, \quad l \in \mathbb{Z},$$

an integer-valued random walk $\xi(n)$, $n \in \mathbb{N} \cup 0$, and an integer-valued sequence of independent random variables $\eta_l(n)$, $n \in \mathbb{N} \cup 0$.

Definition 1. A random sequence $X_l(n) \in \mathbb{Z}$, $n \in \mathbb{N} \cup 0$, $l \in \mathbb{Z}$, defined by the following stochastic recurrent equation,

$$X_l(0) = l,$$

$$X_l(n) = \begin{cases} X_l(n-1) + \xi_n, & \text{with probability } \lambda, \\ \eta_n, & \text{with probability } 1 - \lambda, \end{cases} \qquad n \in \mathbb{N}, \quad \lambda \in [0, 1], \tag{1}$$

is called the superposition of the random walk $\xi(n)$, $n \in \mathbb{N} \cup 0$, and the sequence of independent random variables $\eta_l(n)$, $n \in \mathbb{N} \cup 0$.

Any superposition of a random walk and a sequence of independent random variables is a homogeneous Markov chain. If $\lambda = 1$ in equality (1), then Definition 1 implies that $X_l(n)$ has the same distribution as $l + \xi(n)$ and, in particular, the sequence $X_0(n)$, $n \in \mathbb{N} \cup 0$, is a random walk $\xi(n)$, $n \in \mathbb{N} \cup 0$. If $\lambda = 0$ in Definition 1, then $X_l(n)$, $n \in \mathbb{N} \cup 0$, is a sequence of independent random variables $\eta_l(n)$, $n \in \mathbb{N} \cup 0$.

For the superposition $X_l(n)$, $n \in \mathbb{N} \cup 0$, $l \in \mathbb{Z}$, we obtain the moment generating function of the joint distribution of the first passage time and overshoot over a level at the first passage time in Section 1. In Section 2, we obtain the moment generating

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function of the joint distribution of the first exit time from an interval and the value of the sequence at the first exit time for the superposition $X_l(n)$.

1. The first passage time

Let $l \in \mathbb{Z}$, $k \in \mathbb{N} \cup 0$, and let

$$\tau^{kl} = \inf\{n > 0 \colon X_{l-k}(n) > l\}, \qquad T^{kl} = X_{l-k}\big(\tau^{kl}\big) - l \in \mathbb{N}$$

be the first moment when the level $l \in \mathbb{Z}$ is crossed by the sequence $X_{l-k}(n)$ and overshoots over the level l at the first passage time, respectively. The random variable T^{kl} can be defined arbitrarily on the event $\{\tau^{kl} = \infty\}$; say, one can put $T^{kl} = \infty$. Our current aim is to evaluate

$$\mathsf{E}\big[t^{\tau^{kl}} z^{T^{kl}}; \tau^{kl} < \infty\big], \qquad l \in \mathbb{Z},\ k \in \mathbb{N} \cup 0,\ t \in (0,1],\ |z| \leq 1,$$

that is, the moment generating function of the joint distribution of the pair of random variables $\{\tau^{kl}, T^{kl}\}$.

To evaluate the moment generating function of the joint distribution of $\{\tau^{kl}, T^{kl}\}$, one needs the moment generating functions of the corresponding distributions for the random walk. For all $k \in \mathbb{N} \cup 0$, consider the following random variables:

$$\tau^k = \inf\{n \colon \xi(n) > k\}, \qquad T^k = \xi\big(\tau^k\big) - k,$$
$$\tau_k = \inf\{n \colon \xi(n) < -k\}, \qquad T_k = -\xi(\tau_k) - k.$$

In other words, these random variables are the moment of the first overshoot over the upper level k for the random walk $\xi(\cdot)$, the overshoot over the level k, the moment of the first overshoot over the lower level $-k$, and the overshoot over the level $-k$, respectively. The values of the random variables T^k and T_k on the events $\{\tau^k = \infty\}$ and $\{\tau_k = \infty\}$ do not matter. For example, one can put $T^k = T_k = \infty$. We also assume that $t^{\tau^k} z^{T^k} = 0 = t^{\tau_k} z^{T_k}$ on these events if $t \in (0,1)$ and $|z| \leq 1$. Note that the first passage times for random walks are Markovian [2].

The following Spitzer factorization identity is important for studies of boundary functionals of random walks:

$$\mathsf{E}\, z^{\xi(\nu_t)} = \frac{1-t}{1 - t\,\mathsf{E}\, z^{\xi_1}} = \mathsf{E}\, z^{\xi^+(\nu_t)}\, \mathsf{E}\, z^{\xi^-(\nu_t)}, \qquad |z| = 1,$$

where ν_t is a random variable, independent of the random walk $\xi(n)$, $n \in \mathbb{N} \cup 0$, and having a geometric distribution with parameter $t \in (0,1)$: $\mathsf{P}[\nu_t = n] = (1-t)t^n$, $n \in \mathbb{N} \cup 0$. Here

$$\xi^+(n) = \max_{m \leq n} \xi(m), \qquad \xi^-(n) = \min_{m \leq n} \xi(m), \qquad m, n \in \mathbb{N} \cup 0,$$

$$\mathsf{E}\, z^{\xi^{\pm}(\nu_t)} = \exp\left\{\sum_{n>0} \frac{t^n}{n}\, \mathsf{E}\big[z^{\xi(n)} - 1; \pm\xi(n) > 0\big]\right\}, \qquad t \in (0,1),\ \pm|z| \leq \pm 1.$$

Lemma 1. *Let $\xi(n) \in \mathbb{Z}$, $n \in \mathbb{N} \cup 0$, be an integer-valued random walk. Then, for $t \in (0,1)$ and $k \in \mathbb{N} \cup 0$,*

1) *the moment generating functions of the joint distributions of*

$$\{\tau^k, T^k\} \quad \textit{and} \quad \{\tau_k, T_k\}$$

are such that

$$\mathsf{E}\big[t^{\tau^k} z^{T^k}\big] = \Big(\mathsf{E}\, z^{\xi^+(\nu_t)}\Big)^{-1} \mathsf{E}\big[z^{\xi^+(\nu_t) - k}; \xi^+(\nu_t) > k\big], \qquad |z| \leq 1, \tag{2}$$

$$\mathsf{E}\big[t^{\tau_k} z^{T_k}\big] = \Big(\mathsf{E}\, z^{-\xi^-(\nu_t)}\Big)^{-1} \mathsf{E}\big[z^{-\xi^-(\nu_t) - k}; -\xi^-(\nu_t) > k\big], \qquad |z| \leq 1;$$

2) *the moment generating function of the joint distribution of the pair* $\{\xi(\nu_t), \xi^{\pm}(\nu_t)\}$ *is such that*

$$\mathsf{E}\big[z^{\xi(\nu_t)}; \xi^+(\nu_t) \le k\big] = \mathsf{E}\, z^{\xi^-(\nu_t)}\, \mathsf{E}\big[z^{\xi^+(\nu_t)}; \xi^+(\nu_t) \le k\big], \qquad |z| \ge 1, \tag{3}$$
$$\mathsf{E}\big[z^{\xi(\nu_t)}; \xi^-(\nu_t) \ge -k\big] = \mathsf{E}\, z^{\xi^+(\nu_t)}\, \mathsf{E}\big[z^{\xi^-(\nu_t)}; \xi^-(\nu_t) \ge -k\big], \qquad |z| \le 1.$$

Proof. Equalities (2) and (3) are obtained in [1] for stochastic processes with independent increments. We prove these results for random walks by using the Spitzer factorization identity and simple probabilistic reasoning.

First we obtain equalities (2). Using the full probability formula, homogeneity of the random walk, and strong Markov property we obtain

$$\mathsf{E}\big[z^{\xi^+(\nu_t)}; \xi^+(\nu_t) > k\big] = \mathsf{E}\big[t^{\tau^k} z^{\xi(\tau^k)}\big]\, \mathsf{E}\, z^{\xi^+(\nu_t)}, \qquad |z| \le 1, \quad z \ne 0, \tag{4}$$

for all $k \in \mathbb{N} \cup 0$. Below is a short explanation of the reasoning behind the latter result. It is obvious that the event $\{\xi^+(n) > k\}$ is equivalent to the event $\{\tau^k \le n\}$. Thus

$$\mathsf{E}\big[z^{\xi^+(n)}; \xi^+(n) > k\big] = \mathsf{E}\big[z^{\xi^+(n)}; \tau^k \le n\big] = \mathsf{E}\big[z^{\xi(\tau^k)} z^{\xi^+(n-\tau^k)}; \tau^k \le n\big], \qquad |z| \le 1.$$

Since τ^k is a Markov moment, the random variable $\xi^+(n-\tau^k)$ is independent of the σ-algebra $\mathfrak{B}_{\tau^k}$ generated by the events $\{\xi(m) < l\} \cap \{\tau^k > m\}$, $m \in \mathbb{N} \cup 0$, $l \in \mathbb{Z}$. Now we apply the full probability formula:

$$\mathsf{E}\big[z^{\xi(\tau^k)} z^{\xi^+(n-\tau^k)}; \tau^k \le n\big] = \sum_{m=1}^{n} \mathsf{E}\big[z^{\xi(m)}; \tau^k = m\big]\, \mathsf{E}\, z^{\xi^+(n-m)}, \qquad |z| \le 1.$$

Substituting the right hand side of this equality into the preceding formula we get

$$\mathsf{E}\big[z^{\xi^+(n)}; \xi^+(n) > k\big] = \sum_{m=1}^{n} \mathsf{E}\big[z^{\xi(m)}; \tau^k = m\big]\, \mathsf{E}\, z^{\xi^+(n-m)}, \qquad |z| \le 1.$$

Multiplying this equality by $(1-t)t^n = \mathsf{P}[\nu_t = n]$ and taking the sum over all $n \in \mathbb{N} \cup 0$ on both sides, we obtain relation (4). Now we divide both sides of (4) by $z^k\, \mathsf{E}\, z^{\xi^+(\nu_t)}$ and get

$$\mathsf{E}\big[t^{\tau^k} z^{T^k}\big] = \big(\mathsf{E}\, z^{\xi^+(\nu_t)}\big)^{-1} \mathsf{E}\big[z^{\xi^+(\nu_t)-k}; \xi^+(\nu_t) > k\big], \qquad |z| \le 1,$$

which is the moment generating function of the joint distribution of $\{\tau^k, T^k\}$. The first equality of (2) is proved. Applying the latter equality to the random walk $-\xi(n)$, $n \in \mathbb{N} \cup 0$, we prove the second equality of (2).

Now we prove equalities (3). Applying the full probability formula and space homogeneity of the walk we get

$$\mathsf{E}\, z^{\xi(\nu_t)} = \mathsf{E}\big[z^{\xi(\nu_t)}; \xi^+(\nu_t) \le k\big] + \mathsf{E}\big[t^{\tau^k} z^{\xi(\tau^k)}\big]\, \mathsf{E}\, z^{\xi(\nu_t)}, \qquad |z| = 1, \tag{5}$$

since τ^k is a Markov moment. This equality means that the walk is increasing on the interval $[0, \nu_t]$ if its trajectory either does not cross the upper level k (this corresponds to the first term on the right hand side of the equation) or if it crosses the level k and then increases on the interval $[0, \nu_t]$ (this corresponds to the second term on the right hand side).

To extend the above explanation we add the following reasoning. It is clear that the event $\{\tau^k > n\}$ is equivalent to the event $\{\xi^+(n) \le k\}$. Then

$$\begin{aligned}\mathsf{E}\, z^{\xi(n)} &= \mathsf{E}\big[z^{\xi(n)}; \tau^k > n\big] + \mathsf{E}\big[z^{\xi(n)}; \tau^k \le n\big] \\ &= \mathsf{E}\big[z^{\xi(n)}; \xi^+(n) \le k\big] + \mathsf{E}\big[z^{\xi(\tau^k)} z^{\xi(n)-\xi(\tau^k)}; \tau^k \le n\big], \qquad |z| = 1,\end{aligned}$$

by the full probability formula. Since τ^k is a Markov moment, the increment of the walk $\xi(n) - \xi(\tau^k) \doteq \xi(n-\tau^k)$ (the symbol $\doteq$ means that the corresponding random variables are identically distributed) does not depend on the σ-algebra $\mathfrak{B}_{\tau^k}$ generated by the events $\{\xi(m) < l\} \cap \{\tau^k > m\}$, $m \in \mathbb{N} \cup 0$, $l \in \mathbb{Z}$. Thus

$$\mathsf{E}\big[z^{\xi(\tau^k)} z^{\xi(n)-\xi(\tau^k)}; \tau^k \le n\big] = \sum_{m=1}^{n} \mathsf{E}\big[z^{\xi(m)}; \tau^k = m\big]\, \mathsf{E}\, z^{\xi(n-m)}.$$

Substituting this expression into the preceding equality we get

$$\mathsf{E}\, z^{\xi(n)} = \mathsf{E}\big[z^{\xi(n)}; \xi^+(n) \le k\big] + \sum_{m=1}^{n} \mathsf{E}\big[z^{\xi(m)}; \tau^k = m\big]\, \mathsf{E}\, z^{\xi(n-m)}.$$

Multiplying by $(1-t)t^n = \mathsf{P}[\nu_t = n]$ and taking the sum over all $n \in \mathbb{N} \cup 0$ we obtain equality (5). Substituting the expression for $\mathsf{E}[t^{\tau^k} z^{\xi(\tau^k)}]$ found from (4), equality (5) implies that

$$\mathsf{E}\big[z^{\xi(\nu_t)}; \xi^+(\nu_t) \le k\big] = \mathsf{E}\, z^{\xi^-(\nu_t)}\, \mathsf{E}\big[z^{\xi^+(\nu_t)}; \xi^+(\nu_t) \le k\big], \qquad |z| = 1.$$

Since the functions on both sides of this equality are analytic for $|z| \ge 1$, the equality holds for all $|z| \ge 1$. This proves the first equality of (3). Applying it to the random walk $-\xi(n)$, $n \in \mathbb{N} \cup 0$, we prove the second equality of (3). □

Therefore the moment generating functions for the joint distributions of $\{\tau^k, T^k\}$, $\{\tau_k, T_k\}$, and $\{\xi(\nu_t), \xi^\pm(\nu_t)\}$ are given in Lemma 1 in the case of an integer-valued random walk. The proof is based on the Spitzer factorization identity and two obvious stochastic equalities (4) and (5). We believe that this method is the simplest one for evaluating the above moment generating functions. The method used to prove equalities (2) and (3) is generalized to the case of homogeneous processes with independent increments without any difficulty. Below we present a simple proof of the Spitzer factorization identity.

Lemma 2. *Let $\xi(n) \in \mathbb{Z}$, $n \in \mathbb{N} \cup 0$, be an integer-valued random walk,*

$$\xi^+(n) = \max_{m \le n} \xi(m), \qquad \xi^-(n) = \min_{m \le n} \xi(m), \qquad m, n \in \mathbb{N} \cup 0,$$

and let $\nu_t \in \mathbb{N} \cup 0$ be a random variable, independent of the random walk and whose distribution is geometric with parameter $t \in (0,1)$. Then the Spitzer factorization identity [2] *holds for $t \in (0,1)$; that is,*

$$\mathsf{E}\, z^{\xi(\nu_t)} = \frac{1-t}{1 - t\, \mathsf{E}\, z^{\xi_1}} = \mathsf{E}\, z^{\xi^+(\nu_t)}\, \mathsf{E}\, z^{\xi^-(\nu_t)}, \qquad |z| = 1,$$

where the random variables $\xi^\pm(\nu_t)$ are infinitely divisible and their moment generating functions are of the following form:

$$\mathsf{E}\, z^{\xi^\pm(\nu_t)} = \exp\left\{ \sum_{n>0} \frac{t^n}{n}\, \mathsf{E}\big[z^{\xi(n)} - 1; \pm\xi(n) > 0\big] \right\}, \qquad t \in (0,1), \quad \pm|z| \le \pm 1.$$

Proof. Using the equality

$$\mathsf{E}\, z^{\xi(n)} = \mathbf{I}_{\{n=0\}} + \mathbf{I}_{\{n\in\mathbb{N}\}}\, \mathsf{E}\, z^{\xi_1}\, \mathsf{E}\, z^{\xi(n-1)}, \qquad |z| = 1,$$

and the expansion $\ln(1-x)^{-1} = x/1 + x^2/2 + \cdots$, $|x| < 1$, we derive the following equalities for $t \in (0,1)$:

$$\begin{aligned}\mathsf{E}\, z^{\xi(\nu_t)} &= \frac{1-t}{1-t\,\mathsf{E}\, z^{\xi_1}} = \exp\Big\{\sum_{n>0}\frac{t^n}{n}\big(\mathsf{E}\, z^{\xi(n)}-1\big)\Big\} \\ &= \exp\Big\{\sum_{n>0}\frac{t^n}{n}\,\mathsf{E}\big[z^{\xi(n)}-1;\xi(n)>0\big]\Big\} \\ &\quad\times\exp\Big\{\sum_{n>0}\frac{t^n}{n}\,\mathsf{E}\big[z^{\xi(n)}-1;\xi(n)<0\big]\Big\}, \qquad |z|=1.\end{aligned} \tag{6}$$

The first exponent on the right-hand side of the latter equality is an analytic function with respect to the argument z in the domain $|z| \le 1$. It can be expanded in the power series $\sum_{k=0}^{\infty} a_k z^k$ with nonnegative coefficients a_k, $a_0 > 0$. This exponent is the moment generating function of an integer-valued nonnegative random variable $\eta_t^+ \in \mathbb{N} \cup 0$:

$$\exp\Big\{\sum_{n>0}\frac{t^n}{n}\,\mathsf{E}\big[z^{\xi(n)}-1;\xi(n)>0\big]\Big\} = \mathsf{E}\, z^{\eta_t^+}, \qquad |z|\le 1, \quad \mathsf{P}[\eta_t^+ = 0] > 0.$$

It is clear that the random variable η_t^+, $t \in (0,1)$, is infinitely divisible; namely,

$$\mathsf{E}\, z^{\eta_t^+} = \Big(\mathsf{E}\, z^{\eta_t^+(m)}\Big)^m, \qquad |z| \le 1,$$

for an arbitrary $m \in \mathbb{N}$ where $\eta_t^+(m) \in \mathbb{N} \cup 0$, $\mathsf{P}[\eta_t^+(m) = 0] > 0$, is a random variable whose moment generating function is given by

$$\mathsf{E}\, z^{\eta_t^+(m)} = \exp\Big\{\frac{1}{m}\sum_{n>0}\frac{t^n}{n}\,\mathsf{E}\big[z^{\xi(n)}-1;\xi(n)>0\big]\Big\}, \qquad |z|\le 1,\ m\in\mathbb{N}.$$

Similarly, the second exponent on the right hand side of (6) is an analytic function in the domain $|z| \ge 1$. This function can be expanded in the power series $\sum_{k=0}^{\infty} b_k(1/z)^k$ with nonnegative coefficients b_k, $b_0 > 0$. This exponent is the moment generating function of a nonpositive integer-valued random variable η_t^-:

$$\exp\Big\{\sum_{n>0}\frac{t^n}{n}\,\mathsf{E}\big[z^{\xi(n)}-1;\xi(n)<0\big]\Big\} = \mathsf{E}\, z^{\eta_t^-}, \qquad |z|\ge 1, \quad \mathsf{P}[\eta_t^- = 0] > 0.$$

The random variable η_t^- also is infinitely divisible.

Now we explain the probabilistic meaning of the random variables η_t^+ and η_t^- in terms of boundary functionals of the random walk. First we note that

$$\mathsf{E}\, z^{\xi(\nu_t)} = \mathsf{E}\, z^{\eta_t^+}\,\mathsf{E}\, z^{\eta_t^-} = \mathsf{E}\big[z^{\xi(\nu_t)};\xi^+(\nu_t)\le k\big] + \mathsf{E}\big[t^{\tau^k} z^{\xi(\tau^k)}\big]\,\mathsf{E}\, z^{\xi(\nu_t)}, \qquad |z|=1,$$

for all $k \in \mathbb{N} \cup 0$. This equality looks similar to (5); however, the probabilistic meaning of the random variables η_t^+ and η_t^- is not clear yet. Using the first part of this equality we rewrite it in the following form:

$$\begin{aligned}&\mathsf{E}\big[z^{\eta_t^+-k};\eta_t^+>k\big] - \mathsf{E}\big[t^{\tau^k} z^{T^k}\big]\,\mathsf{E}\, z^{\eta_t^+} \\ &\quad= \mathsf{E}\big[z^{\xi(\nu_t)-k};\xi^+(\nu_t)\le k\big]\big(\mathsf{E}\, z^{\eta_t^-}\big)^{-1} - \mathsf{E}\big[z^{\eta_t^+-k};\eta_t^+\le k\big], \qquad |z|=1.\end{aligned}$$

The function on the left hand side of this equality is bounded, analytic in the unit circle $|z| < 1$, and continuous in the closed unit circle. Using this representation one can extend this function to the exterior of the circle in such a way that the function still is bounded and continuous. According to the Liouville theorem this function is equal to a constant $C(t)$ in the whole complex plane. Considering the case $z = 0$ we obtain from the left hand side of the latter equality that $C(t) = 0$.

The reasoning above is standard for proofs of factorization identities [3] and allows one to obtain two moment generating functions simultaneously from a factorization identity. Namely we obtain the moment generating function of the joint distribution of $\{\tau^k, T^k\}$ and that of $\{\xi(\nu_t), \xi^+(\nu_t)\}$ and express them in terms of the moment generating functions of random variables η_t^+ and η_t^-; that is,

$$\begin{aligned} &\mathsf{E}\big[t^{\tau^k} z^{T^k}\big] = \big(\mathsf{E}\, z^{\eta_t^+}\big)^{-1}\,\mathsf{E}\big[z^{\eta_t^+ - k}; \eta_t^+ > k\big], \qquad |z| \le 1, \\ &\mathsf{E}\big[z^{\xi(\nu_t)}; \xi^+(\nu_t) \le k\big] = \mathsf{E}\, z^{\eta_t^-}\ \mathsf{E}\big[z^{\eta_t^+}; \eta_t^+ \le k\big], \qquad |z| \ge 1. \end{aligned} \tag{7}$$

Now we are able to explain the meaning of the random variables. Putting $z = 1$ in the second equality we get

$$\mathsf{P}[\xi^+(\nu_t) \le k] = \mathsf{P}[\eta_t^+ \le k], \qquad k \in \mathbb{N} \cup 0.$$

Thus the random variable η_t^+ has the same distribution as $\xi^+(\nu_t)$. Applying this equality to the random walk $-\xi(n)$, $n \in \mathbb{N} \cup 0$, we prove that the random variable η_t^- has the same distribution as $\xi^-(\nu_t)$. $\square$

Remark 1. Relations (7) determine the moment generating functions of the joint distributions of

$$\{\tau^k, T^k\}, \qquad \{\xi(\nu_t), \xi^+(\nu_t)\}.$$

Therefore, as a byproduct of the proof of the Spitzer factorization identity in Lemma 2, we also obtain another method to derive the equalities of Lemma 1. This second proof of Lemma 1 uses some factorization tools (in contrast to the straightforward proof of Lemma 1).

We introduce the notation

$$\varphi_t^k(z) = \mathsf{E}\big[t^{\tau^k} z^{T^k}; \tau^k < \infty\big], \qquad \varphi_t^k = \varphi_t^k(1) = \mathsf{E}\big[t^{\tau^k}; \tau^k < \infty\big], \qquad k \in \mathbb{N} \cup 0.$$

Theorem 1. *Let $X_l(n) \in \mathbb{Z}$, $n \in \mathbb{N} \cup 0$, $l \in \mathbb{Z}$, be the superposition of a random walk and a sequence of independent random variables and let*

$$\tau^{kl} = \inf\{n \colon X_{l-k}(n) > l\}, \qquad T^{kl} = X_{l-k}\big(\tau^{kl}\big) - l \in \mathbb{N}, \qquad l \in \mathbb{Z},\ k \in \mathbb{N} \cup 0,$$

be the moment of the first overshoot over the level $l \in \mathbb{Z}$ and the overshoot over the level, respectively. Then

1) *the moment generating function of the joint distribution of the pair $\{\tau^{kl}, T^{kl}\}$ is such that*

$$\mathsf{E}\big[t^{\tau^{kl}} z^{T^{kl}}; \tau^{kl} < \infty\big] = \varphi_{t\lambda}^k(z) + t(1-\lambda)\big(1 - \varphi_{t\lambda}^k\big)\frac{\mathsf{E}\big[z^{\eta - l}; \eta > l\big] + \mathsf{E}\big[\varphi_{t\lambda}^{l-\eta}(z); \eta \le l\big]}{1 - t\lambda - t(1-\lambda)\,\mathsf{E}\big[(1 - \varphi_{t\lambda}^{l-\eta}); \eta \le l\big]}$$

for $|z| \le 1$, where $\eta \in \mathbb{Z}$ is a random variable having the same distribution as η_n, $n \in \mathbb{N}$;

2) *the first passage time τ^{kl} has a nondegenerate distribution for all $k \in \mathbb{N} \cup 0$ and $l \in \mathbb{Z}$ if $\lambda \in (0, 1)$.*

Proof. The trajectories of the sequence $X_l(n)$, $n \in \mathbb{N} \cup 0$, can be viewed as the trajectories of a random walk subject to the interventions of a sequence of random variables that occur at random times. Let τ be the first time of intervention of the sequence of random variables. It is clear that τ is a Markov moment and $\mathsf{P}[\tau = n] = (1-\lambda)\lambda^{n-1}$, $n \in \mathbb{N}$.

For the moment generating functions $\mathsf{E}[z^{T^{kl}};\tau^{kl}=n]$, $n\in\mathbb{N}$, $|z|\le 1$, we obtain for all $k\in\mathbb{N}\cup 0$ that

$$\begin{aligned}\mathsf{E}\big[z^{T^{kl}};\tau^{kl}=n\big] &\\ &=\lambda^n\,\mathsf{E}\big[z^{T^k};\tau^k=n\big]\\ &+(1-\lambda)\sum_{m=1}^{n-1}\lambda^{m-1}\,\mathsf{P}\big[\tau^k>m-1\big]\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\,\mathsf{E}\big[z^{T^{il}};\tau^{il}=n-m\big]\\ &+(1-\lambda)\lambda^{n-1}\,\mathsf{P}\big[\tau^k>n-1\big]\,\mathsf{E}\big[z^{\eta-l};\eta>l\big],\qquad l\in\mathbb{Z},\end{aligned}\tag{8}$$

by the full probability formula. This equality can be explained as follows. The event $\{\tau^{kl}=n\}$ occurs either on those trajectories of the random sequence $X_{l-k}(i)$, $i\le n$, where there is no intervention of the sequence of independent random variables $\eta_{l-k}(i)$, $i\le n$ (this case corresponds to the first term on the right hand side of the equation), or on the trajectories of the random sequence $X_{l-k}(i)$, $i\le n$, where the intervention of the sequence $\eta_{l-k}(i)$, $i\le n$, occurs at the moment $i=m$, $m\in\{1,\dots,n-1\}$ (this case corresponds to the second term on the right hand side of the equation), or on the trajectories of $X_{l-k}(i)$, $i\le n$, where the first intervention of the sequence $\eta_{l-k}(i)$ occurs at the moment $i=n$ (this case corresponds to the third term on the right hand side of the equation). Multiplying equality (8) by t^n and summing both sides of the equalities over $n\in\mathbb{N}$, we get the equation for the moment generating function $\mathsf{E}\big[t^{\tau^{kl}}z^{T^{kl}};\tau^{kl}<\infty\big]$ for $l\in\mathbb{Z}$, $k\in\mathbb{N}\cup 0$, and $|z|\le 1$:

$$\begin{aligned}\mathsf{E}\big[t^{\tau^{kl}}z^{T^{kl}};\tau^{kl}<\infty\big] &\\ &=\varphi_{t\lambda}^k(z)+t(1-\lambda)\frac{1-\varphi_{t\lambda}^k}{1-t\lambda}\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\,\mathsf{E}\big[t^{\tau^{il}}z^{T^{il}};\tau^{il}<\infty\big]\\ &+t(1-\lambda)\frac{1-\varphi_{t\lambda}^k}{1-t\lambda}\,\mathsf{E}\big[z^{\eta-l};\eta>l\big],\qquad l\in\mathbb{Z},\ k\in\mathbb{N}\cup 0.\end{aligned}\tag{9}$$

Put

$$A_l^t(z)=\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\,\mathsf{E}\big[t^{\tau^{il}}z^{T^{il}};\tau^{kl}<\infty\big],\qquad l\in\mathbb{Z}.$$

Multiplying equality (9) by $\mathsf{P}[\eta=l-k]$ and summing over $k\in\mathbb{N}\cup 0$, we get the linear equation for the function $A_l^t(z)$:

$$\begin{aligned}A_l^t(z)&=\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\varphi_{t\lambda}^i(z)+t(1-\lambda)\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\frac{1-\varphi_{t\lambda}^i}{1-t\lambda}A_l^t(z)\\ &+t(1-\lambda)\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\frac{1-\varphi_{t\lambda}^i}{1-t\lambda}\,\mathsf{E}\big[z^{\eta-l};\eta>l\big],\qquad l\in\mathbb{Z}.\end{aligned}$$

Solving this equation we find

$$A_l^t(z)=\frac{\mathsf{E}\big[z^{\eta-l};\eta>l\big]+\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\varphi_{t\lambda}^i(z)}{1-t(1-\lambda)\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\frac{1-\varphi_{t\lambda}^i}{1-t\lambda}}-\mathsf{E}\big[z^{\eta-l};\eta>l\big],\qquad l\in\mathbb{Z}.$$

Substituting the expression for $A_l^t(z)$ into equation (9), we determine the moment generating function of the joint distribution of $\{\tau^{kl},T^{kl}\}$:

$$\begin{aligned}\mathsf{E}\big[t^{\tau^{kl}}z^{T^{kl}};\tau^{kl}<\infty\big] &\\ &=\varphi_{t\lambda}^k(z)+t(1-\lambda)\big(1-\varphi_{t\lambda}^k\big)\frac{\mathsf{E}\big[z^{\eta-l};\eta>l\big]+\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\varphi_{t\lambda}^i(z)}{1-t\lambda-t(1-\lambda)\sum_{i=0}^{\infty}\mathsf{P}[\eta=l-i]\big(1-\varphi_{t\lambda}^i\big)}.\end{aligned}\tag{10}$$

Thus part 1) of Theorem 1 is proved. Now we show that the random variable τ^{kl}, $l \in \mathbb{Z}$, $k \in \mathbb{N} \cup 0$, has a nondegenerate distribution if $\lambda \in (0,1)$. Putting $z = 1$ in equality (10) yields

$$\mathsf{E}\big[t^{\tau^{kl}}; \tau^{kl} < \infty\big] = \varphi_{t\lambda}^k + t(1-\lambda)\big(1-\varphi_{t\lambda}^k\big)\frac{\mathsf{P}[\eta > l] + \sum_{i=0}^{\infty}\mathsf{P}[\eta = l-i]\varphi_{t\lambda}^i}{1 - t\lambda - t(1-\lambda)\sum_{i=0}^{\infty}\mathsf{P}[\eta = l-i]\big(1-\varphi_{t\lambda}^i\big)}.$$

Passing to the limit as $t \nearrow 1$ we get

$$\mathsf{P}\big[\tau^{kl} < \infty\big] = \varphi_{\lambda}^k + \big(1-\varphi_{\lambda}^k\big)\frac{\mathsf{P}[\eta > l] + \sum_{i=0}^{\infty}\mathsf{P}[\eta = l-i]\varphi_{\lambda}^i}{1 - \sum_{i=0}^{\infty}\mathsf{P}[\eta = l-i]\big(1-\varphi_{\lambda}^i\big)} = \varphi_{\lambda}^k + \big(1-\varphi_{\lambda}^k\big) = 1.$$

This means that the random variable τ^{kl} has a nondegenerate distribution. $\square$

Remark 2. If $\lambda = 1$, then definition (1) implies that $X_{l-k}(n) = l - k + \xi(n)$, $n \in \mathbb{N} \cup 0$. Then we derive from formula (10) for $\lambda = 1$ that

$$\mathsf{E}\big[t^{\tau^{kl}}; \tau^{kl} < \infty\big] = \varphi_t^k = \mathsf{E}\big[t^{\tau^k}; \tau^k < \infty\big].$$

Therefore the random variable τ^{kl}, $l \in \mathbb{Z}$, has the same distribution as τ^k. Thus the distribution of τ^{kl} is nondegenerate or degenerate depending on the distribution of the random variable τ^k.

Remark 3. If $\lambda = 0$, then Definition (1) implies that $X_{l-k}(n) = \eta_{l-k}(n)$ for all $n \in \mathbb{N} \cup 0$ where $\eta_l(n)$, $n \in \mathbb{N} \cup 0$, $l \in \mathbb{Z}$, is a sequence of independent random variables. Using equality (10) for $\lambda = 0$ we obtain that

$$\mathsf{E}\big[t^{\tau^{kl}} z^{T^{kl}}\big] = \frac{t\,\mathsf{E}[z^{\eta}; \eta > l]}{1 - t\,\mathsf{P}[\eta \le l]},$$
$$\mathsf{P}\big[\tau^{kl} = n, T^{kl} = m\big] = (\mathsf{P}[\eta \le l])^{n-1}\,\mathsf{P}[\eta = l + m]$$

for all $n, m \in \mathbb{N}$. Hence

$$\mathsf{P}\big[\tau^{kl} < \infty\big] = \sum_{n,m \in \mathbb{N}} \mathsf{P}\big[\tau^{kl} = n, T^{kl} = m\big] = 1$$

for $l \in \mathbb{Z}$ such that $\mathsf{P}[\eta > l] > 0$, whence we deduce that τ^{kl} is a random variable with a nondegenerate distribution. If $l \in \mathbb{Z}$ is such that

$$\mathsf{P}[\eta > l] = 0,$$

then $\mathsf{P}[\tau^{kl} = n] = 0$ for all $n \in \mathbb{N}$.

2. Exit from an interval

Let $B \in \mathbb{N} \cup 0$ and $l \in \mathbb{Z}$ be fixed and let

$$\chi^l(k) = \min\{n > 0\colon X_{l-k}(n) \notin [l-B, l]\}, \qquad Y^l(k) = X_{l-k}(\chi^l(k)), \quad k \in 0, \dots, B,$$

be the first exit time of the sequence $X_{l-k}(n)$, $n \in \mathbb{N} \cup 0$, from the interval $[l-B, l]$ and its value at the first exit time. We evaluate the moment generating function of the joint distribution of the first exit time of the sequence $X_{l-k}(n)$, $n \in \mathbb{N} \cup 0$, from the interval $[l-B, l]$ and the value of $X_{l-k}(n)$ at the first exit time; that is,

$$\mathsf{E}\big[t^{\chi^l(k)} z^{Y^l(k)}; \chi^l(k) < \infty\big], \qquad l \in \mathbb{Z},\ k = 0, \dots, B,\ t \in (0,1],\ |z| = 1.$$

We express the solution of this problem in terms of the moment generating function of the joint distribution of the first exit time for the random walk $\xi(n)$, $n \in \mathbb{N} \cup 0$, and the value of the walk at the first exit time.

Let $B \in \mathbb{N} \cup 0$ be fixed, $k = 0, \dots, B$, $r = B - k$, and let

$$\chi(k) = \min\{n > 0\colon \xi(n) \notin [-r, k]\}$$

be the first exit time of the random walk $\xi(n)$ from the interval $[-r,k]$. The random variable χ is a Markov moment such that $\mathsf{P}[\chi<\infty]=1$. The random walk $\xi(n)$, $n\in\mathbb{N}\cup 0$, exits the interval either through the upper boundary k or through the lower boundary $-r$. Consider the random events: $A^k=\{\xi(\chi(k))>k\}$ and $A_r=\{\xi(\chi(k))<-r\}$, meaning that the random walk exits the interval $[-r,k]$ through the upper boundary k and through the lower boundary $-r$, respectively. Let $\mathbf{I}_A=\mathbf{I}_A(\omega)$ be the indicator of an event A and let

$$X=(\xi(\chi(k))-k)\mathbf{I}_{A^k}+(-\xi(\chi(k))-r)\mathbf{I}_{A_r}\in\mathbb{N},\qquad \mathsf{P}[A^k+A_r]=1,$$

be the overshoot over the boundary at the exit time.

Theorem 2 ([4]). *Let $\xi(n)\in\mathbb{Z}$, $n\in\mathbb{N}\cup 0$, be an integer-valued random walk. Then the moment generating functions of the joint distributions of $\{\chi(k),X\}$ are such that*

$$\begin{aligned}\mathsf{E}\big[t^{\chi(k)};X=m,A^k\big]&=\sum_{i=1}^{\infty}\mathsf{E}\big[t^{\tau^k};T^k=i\big]\mathbf{K}_+^t(i,m)\\&\quad-\sum_{i=1}^{\infty}\mathsf{E}\big[t^{\tau_r};T_r=i\big]\sum_{\nu=1}^{\infty}\mathsf{E}\big[t^{\tau^{i+B}};T^{i+B}=\nu\big]\mathbf{K}_+^t(\nu,m),\\\mathsf{E}\big[t^{\chi(k)};X=m,A_r\big]&=\sum_{i=1}^{\infty}\mathsf{E}\big[t^{\tau_r};T_r=i\big]\mathbf{K}_-^t(i,m)\\&\quad-\sum_{i=1}^{\infty}\mathsf{E}\big[t^{\tau^k};T^k=i\big]\sum_{\nu=1}^{\infty}\mathsf{E}\big[t^{\tau_{i+B}};T_{i+B}=\nu\big]\mathbf{K}_-^t(\nu,m)\end{aligned}\tag{11}$$

for all $m\in\mathbb{N}$ where $K_\pm^{(0)}(i,m,t)=\delta_{im}$,

$$\mathbf{K}_\pm^t(i,m)=\sum_{n=0}^{\infty}K_\pm^{(n)}(i,m,t)$$

is the series of successive iterations, δ_{im} is the Kronecker symbol,

$$K_\pm^{(n)}(i,m,t)=\sum_{\nu=1}^{\infty}K_\pm^{(n-1)}(i,\nu,t)K_\pm(\nu,m,t),\qquad n\in\mathbb{N},$$

are successive iterations of the kernels $K_\pm(i,m,t)$ defined by

$$\begin{aligned}K_+(i,m,t)&=\sum_{\nu=1}^{\infty}\mathsf{E}\big[t^{\tau_{i+B}};T_{i+B}=\nu\big]\,\mathsf{E}\big[t^{\tau^{\nu+B}};T^{\nu+B}=m\big],\qquad i,m\in\mathbb{N},\\K_-(i,m,t)&=\sum_{\nu=1}^{\infty}\mathsf{E}\big[t^{\tau^{i+B}};T^{i+B}=\nu\big]\,\mathsf{E}\big[t^{\tau_{\nu+B}};T_{\nu+B}=m\big],\qquad i,m\in\mathbb{N},\end{aligned}$$

and where the moment generating functions of the distributions of $\{\tau^k,T^k\}$ and $\{\tau_r,T_r\}$ are defined by equalities (2) of Lemma 1.

Proof. Applying the full probability formula and the space homogeneity of the random walk, we prove that the moment generating functions

$$\mathsf{E}\big[t^{\chi(k)};X=m,A^k\big]\quad\text{and}\quad\mathsf{E}\big[t^{\chi(k)};X=m,A_r\big],\qquad m\in\mathbb{N},$$

satisfy the following system of equations:

$$
\begin{aligned}
\mathsf{E}\big[t^{\tau^k}; T^k = m\big] &= \mathsf{E}\big[t^{\chi(k)}; X = m, A^k\big] \\
&\quad + \sum_{i=1}^{\infty} \mathsf{E}\big[t^{\chi(k)}; X = i, A_r\big]\, \mathsf{E}\big[t^{\tau^{i+B}}; T^{i+B} = m\big], \\
\mathsf{E}\big[t^{\tau_r}; T_r = m\big] &= \mathsf{E}\big[t^{\chi(k)}; X = m, A_r\big] \\
&\quad + \sum_{i=1}^{\infty} \mathsf{E}\big[t^{\chi(k)}; X = i, A^k\big]\, \mathsf{E}\big[t^{\tau_{i+B}}; T_{i+B} = m\big],
\end{aligned} \tag{12}
$$

since $\chi(k)$ is a Markov moment.

The first equation of this system means that the random walk $\xi(n)$, $n \in \mathbb{N} \cup 0$, crosses the upper boundary k for the first time if either the random walk does not cross the lower boundary $-r$ (this case corresponds to the first term on the right hand side of the equation) or it crosses the boundary $-r$ and then crosses the upper boundary k (this case corresponds to the second term on the right hand side of the equation). One can add the following short explanation to the above reasoning. It is clear that

$$
\begin{aligned}
\mathsf{E}\big[t^{\tau^k}; T^k = m, \tau^k < \tau_r\big] &= \mathsf{E}\big[t^{\chi(k)}; X = m, A^k\big], \\
\mathsf{E}\big[t^{\tau_r}; T_r = m, \tau_r < \tau^k\big] &= \mathsf{E}\big[t^{\chi(k)}; X = m, A_r\big].
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathsf{E}\big[t^{\tau^k}; T^k = m\big] &= \mathsf{E}\big[t^{\tau^k}; T^k = m, \tau^k < \tau_r\big] + \mathsf{E}\big[t^{\tau^k}; T^k = m, \tau_r < \tau^k\big] \\
&= \mathsf{E}\big[t^{\chi(k)}; X = m, A^k\big] + \mathsf{E}\big[t^{\chi(k)} t^{\tau^{X+B}}; T^{X+B} = m, A_r\big]
\end{aligned}
$$

by the full probability formula. Since $\chi(k)$ is a Markov moment, the random variables τ^{X+B} and T^{X+B} do not depend on the σ-algebra $\mathfrak{B}_{\chi(k)}$. Thus

$$
\mathsf{E}\big[t^{\chi(k)} t^{\tau^{X+B}}; T^{X+B} = m, A_r\big] = \sum_{i=1}^{\infty} \mathsf{E}\big[t^{\chi(k)}; X = i, A_r\big]\, \mathsf{E}\big[t^{\tau^{i+B}}; T^{i+B} = m\big].
$$

Substituting this expression into the latter equality, we prove the first equation of system (12). The second equation of the system can be derived in a similar manner.

Now we solve this linear system of two equations with two unknowns. We find the expression for the moment generating function $\mathsf{E}[t^{\chi(k)}; X = m, A_r]$ from the second equation and substitute it into the first equation. Then

$$
\begin{aligned}
&\mathsf{E}\big[t^{\chi(k)}; X = m, A^k\big] \\
&\quad = \mathsf{E}\big[t^{\tau^k}; T^k = m\big] - \sum_{i=1}^{\infty} \mathsf{E}\big[t^{\tau_r}; T_r = i\big]\, \mathsf{E}\big[t^{\tau^{i+B}}; T^{i+B} = m\big] \\
&\quad\quad + \sum_{\nu=1}^{\infty} \sum_{i=1}^{\infty} \mathsf{E}\big[t^{\chi(k)}; X = i, A^k\big]\, \mathsf{E}\big[t^{\tau_{i+B}}; T_{i+B} = \nu\big]\, \mathsf{E}\big[t^{\tau^{\nu+B}}; T^{\nu+B} = m\big], \qquad m \in \mathbb{N}.
\end{aligned}
$$

Changing the order of summation in the third term on the right-hand side gives us

$$
\begin{aligned}
\mathsf{E}\big[t^{\chi(k)}; X = m, A^k\big] &= \mathsf{E}\big[t^{\tau^k}; T^k = m\big] - \sum_{i=1}^{\infty} \mathsf{E}\big[t^{\tau_r}; T_r = i\big]\, \mathsf{E}\big[t^{\tau^{i+B}}; T^{i+B} = m\big] \\
&\quad + \sum_{i=1}^{\infty} \mathsf{E}\big[t^{\chi(k)}; X = i, A^k\big] K_+(i, m, t), \qquad m \in \mathbb{N}.
\end{aligned} \tag{13}
$$

This equation, considered with respect to the function $\mathsf{E}[t^{\chi(k)}; X = m, A^k]$, $m \in \mathbb{N}$, is a discrete analog of the linear integral equation. The kernel $K_+(i, m, t)$ of equation (3) is such that

$$\begin{aligned} K_+(i,m,t) &= \sum_{\nu=1}^{\infty} \mathsf{E}\big[t^{\tau_{i+B}}; T_{i+B} = \nu\big]\, \mathsf{E}\big[t^{\tau^{\nu+B}}; T^{\nu+B} = m\big] \\ &\le \sum_{\nu=1}^{\infty} \mathsf{E}\big[t^{\tau_{i+B}}; T_{i+B} = \nu\big]\, \mathsf{E}\, t^{\tau^{\nu+B}} \le \mathsf{E}\, t^{\tau^B} \sum_{\nu=1}^{\infty} \mathsf{E}\big[t^{\tau_{i+B}}; T_{i+B} = \nu\big] \\ &\le \mathsf{E}\, t^{\tau^B}\, \mathsf{E}\, t^{\tau_B} \le \lambda < 1 \end{aligned}$$

for all $i, m \in \mathbb{N}$ and $t \in (0, t_0)$ where $\lambda = \mathsf{E}\, t_0^{\tau^B}\, \mathsf{E}\, t_0^{\tau_B} < 1$, $t_0 \in (0, 1)$. Here we used the equalities

$$\begin{aligned} \mathsf{E}\, t^{\tau^B} - \mathsf{E}\, t^{\tau^{i+B}} &= \mathsf{P}[\xi^+(\nu_t) \in [B+1, i+B]] \ge 0, \\ \mathsf{E}\, t^{\tau_B} - \mathsf{E}\, t^{\tau_{i+B}} &= \mathsf{P}[-\xi^-(\nu_t) \in [B+1, i+B]] \ge 0 \end{aligned}$$

for $i \in \mathbb{N}$ that follow from equalities (2) of Lemma 1. If

$$K_+^{(0)}(i,m,t) = \delta_{im}, \qquad K_+^{(n)}(i,m,t) = \sum_{\nu=1}^{\infty} K_+^{(n-1)}(i,\nu,t) K_+(\nu,m,t), \quad n \in \mathbb{N},$$

denotes the sequence of iterations of the kernel $K_+(i, m, t)$, then an induction proves that $K_+^{(n)}(i, m, t) < \lambda^n$, $n \in \mathbb{N}$, for all $i, m \in \mathbb{N}$ and $t \in (0, t_0)$. Thus the series of successive iterations

$$\mathbf{K}_+^t(i,m) = \sum_{n=0}^{\infty} K_+^{(n)}(i,m,t) < (1-\lambda)^{-1}$$

converges uniformly in $i, m \in \mathbb{N}$ and $t \in (0, t_0)$. Now we apply the successive iterations method [5] to solve equation (13) and to obtain the first equality in (11). The second equality can be proved in a similar way. □

The moment generating function of the joint distribution of the pair $\{\chi(k), \xi(\chi(k))\}$ is such that

$$\mathsf{E}\big[t^{\chi(k)} z^{\xi(\chi(k))}\big] = z^k\, \mathsf{E}\big[t^{\chi(k)} z^X; A^k\big] + z^{-r}\, \mathsf{E}\big[t^{\chi(k)} z^{-X}; A_r\big], \qquad r = B - k, \quad |z| = 1.$$

For every $l \in \mathbb{Z}$, fix an interval $[l - B, l]$ and put

$$\chi(l,k) = \min\{n > 0 : l - k + \xi(n) \notin [l-B, l]\}, \qquad k \in \{0, \dots, B\}.$$

Since the random walk is space homogeneous,

$$\mathsf{E}\big[t^{\chi(l,k)} z^{\xi(\chi(l,k))}\big] = z^l\, \mathsf{E}\big[t^{\chi(k)} z^X; A^k\big] + z^{l-B}\, \mathsf{E}\big[t^{\chi(k)} z^{-X}; A_r\big] \overset{\text{def}}{=} \psi_t^k(l,z), \qquad |z| = 1.$$

Let

$$\psi_t^k = \psi_t^k(l,1) = \mathsf{E}\big[t^{\chi(k)}; A^k\big] + \mathsf{E}\big[t^{\chi(k)}; A_r\big] = \mathsf{E}\, t^{\chi(k)}.$$

Theorem 3. *Let $X_l(n)$, $n \in \mathbb{N} \cup 0$, $l \in \mathbb{Z}$, be the superposition of a random walk and a sequence of independent random variables. Let*

$$\chi^l(k) = \min\{n > 0 \colon X_{l-k}(n) \notin [l-B, l]\}, \qquad Y^l(k) = X_{l-k}(\chi^l(k)), \quad k = 0, \dots, B,$$

be the first exit time of the sequence $X_{l-k}(n)$, $n \in \mathbb{N} \cup 0$, from the interval $[l - B, l]$ and *its value at the first exit time, respectively. Then*

1) *the moment generating function of the joint distribution of the pair* $\{\chi^l(k), Y^l(k)\}$ *is such that*

$$\begin{aligned}&\mathsf{E}\big[t^{\chi^l(k)}z^{Y^l(k)};\chi^l(k)<\infty\big]\\&\quad=\psi^k_{t\lambda}(l,z)+t(1-\lambda)\big(1-\psi^k_{t\lambda}\big)\frac{\mathsf{E}[z^\eta;\eta\notin[l-B,l]]+\mathsf{E}\big[\psi^{l-\eta}_{t\lambda}(l,z);\eta\in[l-B,l]\big]}{1-t\lambda-t(1-\lambda)\,\mathsf{E}\big[1-\psi^{l-\eta}_{t\lambda};\eta\in[l-B,l]\big]}\end{aligned}$$

for $|z|=1$. *In particular,*

$$\begin{aligned}&\mathsf{E}\big[t^{\chi^l(k)};Y^l(k)=l+m\big]\\&\quad=\mathsf{E}\big[(t\lambda)^{\chi(k)};X=m,A^k\big]+t(1-\lambda)\big(1-\psi^k_{t\lambda}\big)\\&\qquad\times\frac{\mathsf{P}[\eta=l+m]+\sum_{i=0}^B\mathsf{P}[\eta=l-i]\,\mathsf{E}\big[(t\lambda)^{\chi(i)};X=m,A^k\big]}{1-t\lambda-t(1-\lambda)\,\mathsf{E}\big[1-\psi^{l-\eta}_{t\lambda};\eta\in[l-B,l]\big]},\qquad m\in\mathbb{N},\end{aligned}$$

where the function $\mathsf{E}[t^{\chi(k)};X=m,A^k]$, $m\in\mathbb{N}$, *is defined by the first equality in* (11)*;*

2) *if* $\lambda\in(0,1]$, *then the random variable* $\chi^l(k)$ *has a nondegenerate distribution for* $l\in\mathbb{Z}$ *and* $k=0,\dots,B$.

Proof. Since the time of the first intervention of a sequence of independent random variables is Markovian, the functions $\mathsf{E}[z^{Y^l(k)};\chi^l(k)=n]$, $l\in\mathbb{Z}$, $k=0,\dots,B$, are such that

$$\begin{aligned}&\mathsf{E}\big[z^{Y^l(k)};\chi^l(k)=n\big]\\&\quad=\lambda^n\,\mathsf{E}\big[z^{\xi(\chi(l,k))};\chi(k)=n\big]\\&\qquad+(1-\lambda)\sum_{m=1}^{n-1}\lambda^{m-1}\,\mathsf{P}[\chi(k)>m-1]\sum_{i=0}^{B}\mathsf{P}[\eta=l-i]\,\mathsf{E}\big[z^{Y^l(i)};\chi^l(i)=n-m\big]\\&\qquad+(1-\lambda)\lambda^{n-1}\,\mathsf{P}[\chi(k)>n-1]\,\mathsf{E}\big[z^\eta;\eta\notin[l-B,l]\big],\qquad n\in\mathbb{N},\ |z|=1,\end{aligned}\tag{14}$$

by the full probability formula. This equation means that the event $\{\chi^l(k)=n\}$ occurs if either there is no intervention of the sequence of independent random variables $\eta_{l-k}(i)$, $i\le n$ (this case corresponds to the first term on the right hand side of the equation), or the first intervention occurs at moments $i=m$, $m\in\{1,\dots,n-1\}$ (this case corresponds to the second term on the right hand side of the equation), or the intervention happened at the time $i=n$ (this case corresponds to the third term on the right hand side of the equation).

Multiplying equation (14) by t^n and summing the results over $n\in\mathbb{N}$ we obtain the equation for the moment generating function $\mathsf{E}[t^{\chi^l(k)}z^{Y^l(k)};\chi^l(k)<\infty]$ for $l\in\mathbb{Z}$ and $k=0,\dots,B$:

$$\begin{aligned}&\mathsf{E}\big[t^{\chi^l(k)}z^{Y^l(k)};\chi^l(k)<\infty\big]\\&\quad=\psi^k_{t\lambda}(z)+t(1-\lambda)\frac{1-\psi^k_{t\lambda}}{1-t\lambda}\sum_{i=0}^{B}\mathsf{P}[\eta=l-i]\,\mathsf{E}\big[t^{\chi^l(i)}z^{Y^l(i)};\chi^l(i)<\infty\big]\\&\qquad+t(1-\lambda)\frac{1-\psi^k_{t\lambda}}{1-t\lambda}\,\mathsf{E}\big[z^\eta;\eta\notin[l-B,l]\big],\qquad|z|=1.\end{aligned}\tag{15}$$

Let

$$A^t_l(z,B)=\sum_{i=0}^{B}\mathsf{P}[\eta=l-i]\,\mathsf{E}\big[t^{\chi^l(i)}z^{Y^l(i)};\chi^l(k)<\infty\big],\qquad l\in\mathbb{Z},\ |z|=1.$$

Multiplying equation (15) by $\mathsf{P}[\eta = l-k]$ and summing the results over $k = 0, \dots, B$, we get the linear equation for the function $A_l^t(z,B)$:

$$A_l^t(z,B) = \sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\psi_{t\lambda}^i(l,z) + t(1-\lambda)\sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\frac{1-\psi_{t\lambda}^i}{1-t\lambda}A_l^t(z,B)$$
$$+\, t(1-\lambda)\sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\frac{1-\psi_{t\lambda}^i}{1-t\lambda}\,\mathsf{E}[z^\eta; \eta \notin [l-B,l]], \qquad l \in \mathbb{Z},\ |z| = 1.$$

Solving this equation yields

$$A_l^t(z,B) = \frac{\mathsf{E}[z^\eta; \eta \notin [l-B,l]] + \sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\psi_{t\lambda}^i(l,z)}{1 - t(1-\lambda)\sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\frac{1-\psi_{t\lambda}^i}{1-t\lambda}} - \mathsf{E}[z^\eta; \eta \notin [l-B,l]].$$

Substituting the expression for the function $A_l^t(z,B)$ into equation (15), we obtain the moment generating function of the joint distribution of the pair $\{\chi^l(k), Y^l(k)\}$:

(16)
$$\mathsf{E}\big[t^{\chi^l(k)} z^{Y^l(k)}; \chi^l(k) < \infty\big]$$
$$= \psi_{t\lambda}^k(l,z) + t(1-\lambda)\big(1-\psi_{t\lambda}^k\big)\frac{\mathsf{E}\big[z^\eta; \eta \notin [l-B,l]\big] + \sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\psi_{t\lambda}^i(l,z)}{1 - t\lambda - t(1-\lambda)\sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\big(1-\psi_{t\lambda}^i\big)},$$

$|z| = 1$. The first equation of the theorem is proved. Comparing the coefficients of z^{l-B-m}, $m \in \mathbb{N}$, on both sides of equality (16), we find

$$\mathsf{E}\big[t^{\chi^l(k)}; Y^l(k) = l-B-m\big]$$
$$= \mathsf{E}\big[(t\lambda)^{\chi(k)}; X = m, A_r\big]$$
$$+\, t(1-\lambda)\big(1-\psi_{t\lambda}^k\big)\frac{\mathsf{P}[\eta = l-B-m] + \sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\,\mathsf{E}[(t\lambda)^{\chi(i)}; X = m, A_r]}{1 - t\lambda - t(1-\lambda)\sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\big(1-\psi_{t\lambda}^i\big)},$$

where the function $\mathsf{E}[(t\lambda)^{\chi(k)}; X = m, A_r]$ is defined by equality (11). The second equality of the theorem can be obtained in a similar fashion.

Now we find a condition for the random variable $\chi^l(k)$, $l \in \mathbb{Z}$, $k = 0, \dots, B$, to have a nondegenerate distribution. Substituting $z = 1$ in equality (16), we get

$$\mathsf{E}\big[t^{\chi^l(k)}; \chi^l(k) < \infty\big]$$
$$= \mathsf{E}\big[(t\lambda)^{\chi(k)}; \chi(k) < \infty\big] \tag{17}$$
$$+\, t(1-\lambda)\big(1-\psi_{t\lambda}^k\big)\frac{\mathsf{P}[\eta \notin [l-B,l]] + \sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\psi_{t\lambda}^i}{1 - t\lambda - t(1-\lambda)\sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\big(1-\psi_{t\lambda}^i\big)}.$$

Note that $\mathsf{P}[\chi(k) < \infty] = 1$, where $\chi(k)$ is the first exit time from the interval $[-r,k]$ for the random walk $\xi(n)$, $n \in \mathbb{N} \cup 0$. Then, equality (17) implies for $\lambda = 1$ that

$$\mathsf{E}\big[t^{\chi^l(k)}; \chi^l(k) < \infty\big] = \mathsf{E}\big[t^{\chi(k)}; \chi(k) < \infty\big].$$

Thus the random variable $\chi^l(k)$ has for all $l \in \mathbb{Z}$ the same distribution as $\chi(k)$ and this distribution is nondegenerate. If $\lambda \in (0,1)$, then we pass to the limit (17) as $t \nearrow 1$ and prove that

$$\mathsf{P}\big[\chi^l(k) < \infty\big] = \psi_\lambda^k + (1-\psi_\lambda^k)\frac{\mathsf{P}[\eta \notin [l-B,l]] + \sum_{i=0}^{B} \mathsf{P}[\eta = l-i]\psi_\lambda^i}{1 - \sum_{i=0}^{B} \mathsf{P}[\eta = l-i](1-\psi_\lambda^i)} = 1.$$

Thus $\chi^l(k)$, $l \in \mathbb{Z}$, $k = 0, \dots, B$, has a nondegenerate distribution. □

Remark 4. Equality (17) implies for $\lambda = 0$ that

$$\mathsf{E}\big[t^{\chi^l(k)}; \chi^l(k) < \infty\big] = \frac{t\,\mathsf{P}[\eta \notin [l-B,l]]}{1 - t\,\mathsf{P}[\eta \in [l-B,l]]}, \qquad l \in \mathbb{Z},\ k = 0,\dots,B.$$

If $l \in \mathbb{Z}$ and $B \in \mathbb{N}\cup 0$ are such that $\mathsf{P}[\eta \notin [l-B,l]] > 0$, then the random variable $\chi^l(k)$, $k = 0,\dots,B$, has a geometric distribution:

$$\mathsf{P}[\chi^l(k) = n] = (\mathsf{P}[\eta \in [l-B,l]])^{n-1}\,\mathsf{P}[\eta \notin [l-B,l]], \qquad n \in \mathbb{N}.$$

On the other hand, if $l \in \mathbb{Z}$ and $B \in \mathbb{N} \cup 0$ are such that $\mathsf{P}[\eta \notin [l-B,l]] = 0$, then

$$\mathsf{P}\big[\chi^l(k) = \infty\big] = 1.$$

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