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THE ORDINAL CONVERGENCE AND GLIVENKO–CANTELLI TYPE THEOREMS IN $L_p(-\infty,\infty)$

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I. K. MATSAK

Abstract. Let $F(t)$ be a distribution function and $F_n(t)$ the corresponding empirical distribution function. We find necessary and sufficient conditions for the ordinal convergence o-lim $F_n = F$ in the spaces $L_p(-\infty,\infty)$.

1. Introduction and main results

Let ξ, ξ_1, $\xi_2,\dots$ be independent identically distributed random variables in $\mathbf{R}^1$ with the distribution function $F(t)$. Consider the corresponding empirical distribution function

$$F_n(t)=\frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,t)}(\xi_i),\qquad -\infty<t<\infty.$$

Glivenko [1] proves that

$$\sup_{-\infty<t<\infty}|F_n(t)-F(t)|\to 0\quad\text{a.s.}$$

(this is the well-known Glivenko–Cantelli theorem; it is sometimes called the main theorem of statistics). The shorthand "a.s." here and in what follows means "almost surely". Further studies of the asymptotic behavior of the random function $F_n(t)$ are related to the so-called empirical stochastic process

$$\beta_n(t)=\sqrt{n}(F_n(t)-F(t)),\qquad -\infty<t<\infty.$$

Surveys of results in this direction can be found in [2]–[4].

We treat the stochastic processes

$$X_n(t)=I_{(-\infty,t)}(\xi_n)-F(t),\qquad -\infty<t<\infty, \tag{1}$$

as random elements assuming values in the space $L_p(-\infty,\infty)$, $1\le p<\infty$. The well-known laws of large numbers in Banach spaces [5] imply that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|F_n(t)-F(t)|^p\,dt=0\quad\text{a.s.} \tag{2}$$

if

$$\mathsf{E}\,\|X_n\|_{L_p}<\infty. \tag{3}$$

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One can improve relation (2) as follows:

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\left|\sup_{n>m}F_n(t)-F(t)\right|^p dt=0,\tag{4}$$
$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\left|\inf_{n>m}F_n(t)-F(t)\right|^p dt=0 \quad \text{a.s.}$$

The ordinal law of large numbers is studied in the author papers [6, 7] for Banach lattices. The ordinal law of large numbers can be rewritten in the form of (4) for the random element X_n defined by equality (1) in the space $L_p(-\infty,\infty)$, $1\le p<\infty$. On the other hand, equalities (4) can be viewed as an ordinal variant of the Glivenko–Cantelli theorem in the space $L_p(-\infty,\infty)$.

Below are the main results of the current paper.

Theorem 1. *The empirical distribution function $F_n(t)$ satisfies the ordinal law of large numbers* (4) *for $1<p<\infty$ if and only if*

$$\mathsf{E}\,|\xi|^{1/p}<\infty.\tag{5}$$

Theorem 2. *If $p=1$ and $0<F(t)<1$ for all $t\in(-\infty,\infty)$, then the condition*

$$\int_0^{\infty}(1-F(t))\ln\left(\frac{1}{1-F(t)}\right)dt+\int_{-\infty}^{0}F(t)\ln\left(\frac{1}{F(t)}\right)dt<\infty\tag{6}$$

is sufficient for the ordinal law of large numbers (4).

Remark 1. The ordinal law of large numbers (4) holds if $|\xi|<C<\infty$ almost surely. This follows from Theorems 1 and 2. Another way to prove the same result is to use the Glivenko–Cantelli theorem.

Related to Theorem 2 is the question of whether or not condition (3) (or equivalent condition (5)) is sufficient for the ordinal law of large numbers (4) in the space $L_p(-\infty,\infty)$ if $p=1$?

The answer to this question is negative. A corresponding example is given at the end of this paper.

2. Ordinal law of large numbers in Banach lattices

Let B be a separable Banach lattice equipped with a norm $\|\cdot\|$ and modulus $|\cdot|$ and let X_i, $i\ge 1$, be independent copies of a random element X assuming values in B, $S_n=\sum_{i=1}^n X_i$.

Along with the convergence in the norm, one can study the ordinal convergence in Banach lattices. We recall the corresponding definition (see [8, 9]).

A sequence (x_n) of elements of a Banach lattice B is called o-convergent to an element x, $x=\text{o-}\lim_{n\to\infty}x_n$, if there exists a sequence (v_n) such that $|x-x_n|<v_n$ and $v_n\downarrow 0$, that is, $v_1\ge v_2\ge\dots$ and $\inf_{n\ge1}v_n=0$.

We say that a random element X with $\mathsf{E}\,X=0$ satisfies the ordinal law of large numbers if

$$\text{o-}\lim_{n\to\infty}\frac{S_n}{n}=0 \quad \text{a.s.}\tag{7}$$

It is known that the ordinal convergence of a sequence (x_n) to an element x is equivalent to

$$\limsup_{n\to\infty}x_n=\liminf_{n\to\infty}x_n=x,$$

where

$$\limsup_{n\to\infty} x_n = \operatorname{o-lim}_{m\to\infty}\left(\sup_{n\ge m} x_n\right),$$
$$\liminf_{n\to\infty} x_n = \operatorname{o-lim}_{m\to\infty}\left(\inf_{n\ge m} x_n\right).$$

Thus equality (7) can be rewritten in the case of a separable, σ-complete, and ordinally continuous Banach lattice B as follows:

$$\lim_{m\to\infty}\left\|\sup_{n\ge m}\frac{S_n}{n}\right\| = \lim_{m\to\infty}\left\|\inf_{n\ge m}\frac{S_n}{n}\right\| = 0 \quad \text{a.s.}$$

This implies relation (4) for the space $L_p(-\infty,\infty)$.

The general results on the ordinal law of large numbers in Banach lattices obtained in the papers [6, 7] will essentially be used in the proof of equalities (4) below.

Theorem A ([7])**.** *Let B be a separable Banach lattice that does not contain uniformly ℓ_1^n and let X be a random element assuming values in B and such that $\mathsf{E}\,X = 0$. Then the following conditions are equivalent:*

(i) *X satisfies the b-law of large numbers, that is,*

$$\lim_{n\to\infty}\frac{\|S_n\|}{n} = 0 \quad \textit{a.s.};$$

(ii) *X satisfies the ordinal law of large numbers* (7)*;*

(iii) *the first moment exists,*

$$\mathsf{E}\,\|X\| < \infty. \tag{8}$$

Let the element $\mathfrak{S}_\psi(X)$ of a Banach lattice be defined by

$$\mathfrak{S}_\psi(X) = \sup(x\in K_\psi),$$
$$K_\psi = (\mathsf{E}\,\eta X\colon\ \mathsf{E}\,\psi^*(\eta)\le 1),$$

where

$$\psi^*(t) = \sup_{s\in\mathbf{R}^1}(st - \psi(s))$$

is the convex conjugate function to the N-function $\psi(t)$ [9]. The element $\mathfrak{S}_\psi(X)$ is called the mean ψ-deviation of the random element X (see [6]).

Theorem B ([6])**.** *Let B be a separable* 1*-concave Banach lattice, X a random element in B such that $\mathsf{E}\,X = 0$, and $\psi(t) = |t|\ln(1+|t|)$. Then X satisfies the ordinal law of large numbers* (7) *if the mean ψ-deviation $\mathfrak{S}_\psi(X)$ exists in B.*

We also need a general criterion for the ordinal law of large numbers. Put

$$\bar{X}_n = X_n\,\mathbb{I}(\|X_n\|\le n), \qquad \tilde{X}_n = X_n\,\mathbb{I}(\|X_n\| > n),$$
$$\bar{S}_n = \sum_{i=1}^n \bar{X}_i, \qquad \tilde{S}_n = \sum_{i=1}^n \tilde{X}_i,$$

where the symbol $\mathbb{I}(A)$ stands for the indicator of a random event A.

Theorem 3. *Let B be a separable σ-complete Banach lattice and let X be a random element in B that satisfies* (8) *and such that $\mathsf{E}\,X = 0$. Then the following conditions are equivalent:*

(i) *X satisfies the ordinal law of large numbers* (7)*;*

(ii) *the following moment exists:*

$$\mathsf{E}\left\|\sup_{n\geq 1}\left|\frac{\bar{S}_n}{n}\right|\right\| < \infty. \tag{9}$$

Proof of Theorem 3. The reasoning given in the paper [7] proves that if inequality (8) holds, then the condition

$$\left\|\sup_{n\geq 1}\left|\frac{S_n}{n}\right|\right\| < \infty \tag{10}$$

is necessary and sufficient for the ordinal law of large numbers (7).

Moreover if inequality (8) holds, then

$$\sum_{n\geq 1}\mathsf{P}(\|X_n\|\geq n) < \infty.$$

The Borel–Cantelli lemma implies that only a finite number of random elements $\tilde{X}_n$ are nonzero almost surely, that is,

$$\left\|\sup_{n\geq 1}\frac{|\tilde{S}_n|}{n}\right\| < \infty \quad \text{a.s.}$$

Since $S_n = \bar{S}_n + \tilde{S}_n$, condition (10) is equivalent to

$$Z = \left\|\sup_{n\geq 1}\frac{|\bar{S}_n|}{n}\right\| < \infty \quad \text{a.s.} \tag{11}$$

It remains to check the implication (11) $\Rightarrow$ (9), since the converse is obvious.

We follow the Yurinskiĭ method [10] and apply it to sums of independent random elements in Banach spaces. Put

$$Z_n = \sup_{1\leq k\leq n}\frac{\left|\bar{S}_k\right|}{k}, \qquad Z_{n,i} = \sup_{1\leq k\leq n}\frac{1}{k}\left|\sum_{j\leq k, j\neq i}\bar{X}_j\right|,$$
$$\zeta_i = \mathsf{E}_i\,\|Z_n\| - \mathsf{E}_{i-1}\,\|Z_n\|,$$
$$\mathsf{E}_i\,\eta = \mathsf{E}(\eta/F_i),$$

where $F_i = \sigma(X_1,\ldots,X_i)$ is the σ-algebra generated by the random elements $X_1,\ldots,X_i$, and F_0 is the trivial σ-algebra.

Then

$$\|Z_n\| - \mathsf{E}\,\|Z_n\| = \sum_{i=1}^{n}\zeta_i. \tag{12}$$

Since the random variable ζ_i can be written as

$$\zeta_i = (\mathsf{E}_i - \mathsf{E}_{i-1})(\|Z_n\| - \|Z_{n,i}\|),$$

we apply the inequality

$$\big|\|Z_n\| - \|Z_{n,i}\|\big| \leq \left\|\frac{\bar{X}_i}{i}\right\|$$

and get

$$|\zeta_i| \leq \left\|\frac{\bar{X}_i}{i}\right\| + \mathsf{E}\left\|\frac{\bar{X}_i}{i}\right\|. \tag{13}$$

Note that (ζ_i) is a martingale difference; thus we derive from (13) that

$$\mathsf{E}\left|\sum_{i=1}^{n}\zeta_i\right|^2 \le \sum_{i=1}^{n}\mathsf{E}\,|\zeta_i|^2 \le C\sum_{i=1}^{n}\mathsf{E}\left\|\frac{\bar{X}_i}{i}\right\|^2.$$

The series on the right hand side can be estimated similarly to the proof of the Kolmogorov strong law of large numbers (see [11, Chapter VII, §8, proof of Theorem 1]). This leads to the following bound:

$$\sum_{i=1}^{n}\mathsf{E}\left\|\frac{\bar{X}_i}{i}\right\|^2 \le C\,\mathsf{E}\,\|X\|.$$

We conclude from the latter inequality and representation (12) that the sequence of random variables

$$(\|Z_n\| - \mathsf{E}\,\|Z_n\|)$$

is stochastically bounded, that is,

$$\lim_{x\to\infty}\sup_{n\ge 1}\mathsf{P}\left(\big|\|Z_n\| - \mathsf{E}\,\|Z_n\|\big| > x\right) = 0. \tag{14}$$

The conditions of the theorem imply that the Banach lattice B is separable and σ-complete, whence it follows that the lattice is ordinally continuous ([8, p. 7]). Applying bound (11) we obtain as $n\to\infty$ that

$$\|Z_n\| \uparrow \|Z\| < \infty \quad \text{a.s.}$$

By Fatou's lemma,

$$\mathsf{E}\,\|Z_n\| \uparrow \mathsf{E}\,\|Z\|.$$

Were inequality (9) wrong, we would obtain

$$\mathsf{E}\,\|Z\| = \infty.$$

Then taking into account the preceding relations we get

$$\big|\|Z_n\| - \mathsf{E}\,\|Z_n\|\big| \xrightarrow{\mathsf{P}} \infty,$$

which contradicts (14). □

3. Proofs of Theorems 1 and 2

We start with the proof of Theorem 1. According to Section 2, equalities (4) follow from the ordinal law of large numbers (7) in the space $L_p(-\infty,\infty)$ for the random element $X_n = (X_n(t), -\infty < t < \infty)$ defined by (1). The space $L_p(-\infty,\infty)$ is a separable ([9, Chapter 4, Section 3.3]), p-concave, and p-convex Banach lattice that does not uniformly contain ℓ_1^n in the case of $1 < p < \infty$ ([5, pp. 242, 248]). Thus Theorem A for $1 < p < \infty$ implies that the law of large numbers (4) is equivalent to condition (3).

Now we show that conditions (3) and (5) are equivalent. The definition of the random element X_n implies that

$$
\begin{aligned}
\mathsf{E}\,\|X_n\|_{L_p} &= \mathsf{E}\left(\int_{-\infty}^{\infty}\left(|1-F(t)|^p\,\mathbb{I}(\xi<t)+|F(t)|^p\,\mathbb{I}(\xi\ge t)\right)dt\right)^{1/p}\\
&= \mathsf{E}\left(\int_{\xi}^{\infty}|1-F(t)|^p\,dt+\int_{-\infty}^{\xi}|F(t)|^p\,dt\right)^{1/p}\\
&\le \left(\int_0^{\infty}|1-F(t)|^p\,dt+\int_{-\infty}^{0}|F(t)|^p\,dt\right)^{1/p} \qquad (15)\\
&\quad+\mathsf{E}\left(\int_{-|\xi|}^{0}|1-F(t)|^p\,dt+\int_0^{|\xi|}|F(t)|^p\,dt\right)^{1/p}\\
&\le 2\,\mathsf{E}\,|\xi|^{1/p}+\left(\int_0^{\infty}|1-F(t)|^p\,dt+\int_{-\infty}^{0}|F(t)|^p\,dt\right)^{1/p}.
\end{aligned}
$$

Since

$$\mathsf{E}\,|X_n(t)|=2(1-F(t))F(t),$$

we have for $d_\varepsilon>0$ that

$$
\begin{aligned}
\mathsf{E}\,\|X_n\|_{L_p} &\ge \left\|\mathsf{E}\,|X_n|\right\|_{L_p}=2\left(\int_{-\infty}^{\infty}|(1-F(t))F(t)|^p\,dt\right)^{1/p}\\
&\ge 2\left(|F(d_\varepsilon)|^p\int_{d_\varepsilon}^{\infty}|1-F(t)|^p\,dt+|1-F(-d_\varepsilon)|^p\int_{-\infty}^{-d_\varepsilon}|F(t)|^p\,dt\right)^{1/p}. \qquad (16)
\end{aligned}
$$

For a given number $\varepsilon>0$ we choose d_ε such that

$$1-F(-d_\varepsilon)>1-\varepsilon,\qquad F(d_\varepsilon)>1-\varepsilon.$$

Then bound (16) can be rewritten as follows:

$$\left(\int_{d_\varepsilon}^{\infty}|1-F(t)|^p\,dt+\int_{-\infty}^{-d_\varepsilon}|F(t)|^p\,dt\right)^{1/p}\le\frac{\mathsf{E}\,\|X_n\|_{L_p}}{2(1-\varepsilon)}.$$

This together with (15) implies that

$$\mathsf{E}\,\|X_n\|_{L_p}\left(1-\frac{1}{2(1-\varepsilon)}\right)\le 2\,\mathsf{E}\,|\xi|^{1/p}+\left(\int_0^{d_\varepsilon}|1-F(t)|^p\,dt+\int_{-d_\varepsilon}^{0}|F(t)|^p\,dt\right)^{1/p}.$$

The latter inequality for $\varepsilon<1/2$ proves the implication (5) $\Rightarrow$ (3).

Now we establish the converse implication. First we consider the case of $0<F(0)<1$. We again use bounds (15):

$$
\begin{aligned}
\mathsf{E}\,\|X_n\|_{L_p} &\ge \mathsf{E}\left(\mathbb{I}(\xi>0)\int_0^{\xi}|F(t)|^p\,dt+\mathbb{I}(\xi\le 0)\int_{\xi}^{0}|1-F(t)|^p\,dt\right)^{1/p}\\
&\ge \mathsf{E}\big(\mathbb{I}(\xi>0)|\xi||F(0)|^p+\mathbb{I}(\xi\le 0)|\xi||1-F(0)|^p dt\big)^{1/p}\\
&\ge \min\left(F(0),1-F(0)\right)\mathsf{E}\,|\xi|^{1/p},
\end{aligned}
$$

whence the implication (3) $\Rightarrow$ (5) follows.

If the condition $0<F(0)<1$ fails, we choose a point x_0 instead of 0 such that $0<F(x_0)<1$ and repeat the above reasoning for the point x_0.

Now we pass to the proof of Theorem 2. In view of Theorem B, the condition

$$\int_{-\infty}^{\infty} \mathfrak{S}_\psi(X_n(t))\,dt < \infty \tag{17}$$

for $\psi(t) = |t|\ln(1+|t|)$ is sufficient for the law of large numbers (4) if a Banach lattice is 1-concave.

We prove that condition (17) follows from (6). For this, we need an upper estimate for the mean deviation $\mathfrak{S}_\psi(X_n(t))$.

Since $\mathfrak{S}_\psi(\cdot)$ is a norm on the Orlicz space of random variables,

$$\begin{aligned} \mathfrak{S}_\psi(X_n(t)) &= \mathfrak{S}_\psi\big((1-F(t))\,\mathbb{I}(\xi < t) + F(t)\,\mathbb{I}(\xi \ge t)\big) \\ &\le (1-F(t))\mathfrak{S}_\psi\,\mathbb{I}(\xi < t) + F(t)\mathfrak{S}_\psi\,\mathbb{I}(\xi \ge t). \end{aligned} \tag{18}$$

It is known ([9, Chapter 4, Section 3.6]) that the norm $\mathfrak{S}_\psi(\eta)$ is equivalent to the norm

$$\|\eta\|_{\psi,1} = \inf\left(\lambda > 0\colon\ \mathsf{E}\,\psi\left(\frac{\eta}{\lambda}\right) \le 1\right),$$

which is easier to evaluate in the case under consideration.

Indeed

$$\|I(\xi < t)\|_{\psi,1} = \inf\left(\lambda > 0\colon \psi\left(\frac{1}{\lambda}\right) F(t) \le 1\right) = \frac{1}{\psi^{-1}\big(\frac{1}{F(t)}\big)}.$$

We obtain similarly that

$$\|I(\xi \ge t)\|_{\psi,1} = \frac{1}{\psi^{-1}\big(\frac{1}{1-F(t)}\big)}.$$

Taking into account (18) we get

$$\mathfrak{S}_\psi(X_n(t)) \le C\left(\frac{1-F(t)}{\psi^{-1}\big(\frac{1}{F(t)}\big)} + \frac{F(t)}{\psi^{-1}\big(\frac{1}{1-F(t)}\big)}\right). \tag{19}$$

It is easy to check that

$$\psi^{-1}(t) \sim \frac{t}{\ln t} \quad \text{as } t \to \infty.$$

Therefore inequality (19) implies that

$$\mathfrak{S}_\psi(X_n(t)) \le C(1-F(t))F(t)\left(\ln\left(\frac{1}{F(t)}\right) + \ln\left(\frac{1}{1-F(t)}\right)\right)$$

for large $|t|$. It is easy to see that the latter estimate and condition (6) imply condition (17).

4. Example

Let ξ be a random variable with the distribution function $F(t)$ being continuous and such that

$$1 - F(t) = \frac{1}{t\ln t(\ln\ln t)^2}$$

for $t \ge e^e$. If we set $F(0) = 1/2$, linearly extrapolate $F(t)$ on the interval $0 < t < e^e$, and let $F(-t) = 1 - F(t)$ for negative t, then F is well defined for all real arguments. Let (ξ_i) be a sequence of independent copies of ξ and let $F_n(t)$ be the empirical distribution function constructed from the random variables $\xi_1, \dots, \xi_n$.

It is clear that the random variable ξ satisfies conditions (3) and (5) for $p = 1$. Thus it follows from the results of Section 3 that the law of large numbers (2) holds for the empirical distribution function $F_n(t)$. Let us prove that the ordinal law of large numbers (4) does not hold in the space $L_1(-\infty, \infty)$ for the random variable ξ.

By Theorem 3, it is sufficient to show that

$$\int_{-\infty}^{\infty} \mathsf{E}\sup_{n\ge 1}\frac{1}{n}\left|\sum_{i=1}^{n} X_i(t)\,\mathbb{I}(\|X_i\|_{L_1}\le n)\right| dt=\infty,$$

where $X_i(t)$ are stochastic processes defined by equality (1). Note that condition (3) holds for the process $X_i(t)$ if $p=1$.

It is easy to see that the latter integral is unbounded if

$$\int_{-\infty}^{\infty} \mathsf{E}\sup_{n\ge 1}\frac{1}{n}\big|X_n(t)\,\mathbb{I}(\|X_n\|_{L_1}\le n)\big|\,dt=\infty. \tag{20}$$

Using the same method as in the proof of inequality (15) we write

$$\begin{aligned}\|X_n\|_{L_1}&\le \int_0^{\infty}(1-F(t))\,dt+\int_{-\infty}^{0}F(t)\,dt+\int_{\xi_n}^{0}(1-F(t))\,dt\\ &\le \mathsf{E}\,|\xi|+(1-F(0))|\xi_n|=\mathsf{E}\,|\xi|+\frac12|\xi_n|\end{aligned}$$

for $\xi_n<0$, and

$$\|X_n\|_{L_1}\le \int_0^{\infty}(1-F(t))\,dt+\int_{-\infty}^{0}F(t)\,dt+\int_0^{\xi_n}F(t)\,dt\le \mathsf{E}\,|\xi|+F(0)|\xi_n|=\mathsf{E}\,|\xi|+\frac12|\xi_n|$$

for $\xi_n\ge 0$. Therefore

$$\|X_n\|_{L_1}\le \mathsf{E}\,|\xi|+\frac12|\xi_n| \tag{21}$$

in either case. Since

$$\mathbb{I}(|\xi_n|\le n)\le \mathbb{I}\left(\mathsf{E}\,|\xi|+\frac{|\xi_n|}{2}\le n\right)$$

for $n>2\,\mathsf{E}\,|\xi|$, the latter inequality and bound (21) imply that relation (20) follows from

$$\int_{-\infty}^{\infty} \mathsf{E}\sup_{n\ge 1}\frac{1}{n}\,|X_n(t)|\,\mathbb{I}(|\xi_n|\le n)\,dt=\infty. \tag{22}$$

It is obvious that

$$|X_n(t)|\ge (1-F(t))\,\mathbb{I}(\xi_n<t),$$

whence

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathsf{E}\sup_{n\ge 1}\frac{1}{n}\,|X_n(t)|\,\mathbb{I}(|\xi_n|\le n)\,dt\\ &\qquad\ge (1-F(0))\int_{-\infty}^{0}\mathsf{E}\sup_{n\ge 1}\frac{1}{n}\,\mathbb{I}(\xi_n\le t)\,\mathbb{I}(|\xi_n|\le n)\,dt.\end{aligned} \tag{23}$$

Now we estimate the expression under the integral sign on the right hand side of (23). Put

$$A_n=\{\xi_n\le t,\ \xi_n>-n\}.$$

Then

$$\begin{aligned}\mathsf{E}\sup_{n\ge 1}\frac{I(A_n)}{n}&=\sum_{n>|t|}\frac{1}{n}\,\mathsf{P}\left(A_n\cap\left(\bigcap_{k=1}^{n-1}\bar A_k\right)\right)\\ &\ge \sum_{n>|t|}\frac{1}{n}(F(t)-F(-n))\prod_{k=1}^{n-1}(1-F(t)+F(-n))\\ &\ge \sum_{n>|t|}\frac{1}{n}(F(t)-F(-n))(1-F(t))^{n-1}\end{aligned} \tag{24}$$

for $t < 0$. It is not complicated to obtain an upper bound for the following series if $t < -e^e$, namely

$$\sum_{n>|t|} \frac{1}{n} F(-n) = \sum_{n>|t|} \frac{1}{n^2 \ln n (\ln\ln n)^2} \le \frac{1}{\ln|t|(\ln\ln|t|)^2} \sum_{n>|t|} \frac{1}{n^2} \sim F(t). \tag{25}$$

Since $\int_{-\infty}^{0} F(t)\,dt < \infty$, we collect together estimates (23)–(25) and see that relation (22) follows from

$$\int_{-\infty}^{0} \sum_{n>|t|} \frac{1}{n} F(t)(1-F(t))^{n-1}\,dt = \infty. \tag{26}$$

Now we find the asymptotic behavior of the expression under the integral sign in (26) for large $|t|$. Using the Taylor expansion

$$\ln \frac{1}{1-x} = \sum_{n\ge 1} \frac{x^n}{n}, \qquad -1 < x < 1,$$

we get for $t < -e^e$ that

$$\sum_{n\ge 1} \frac{1}{n}(1-F(t))^n = \ln \frac{1}{F(t)} = \ln|t| + \ln\ln|t| + 2\ln\ln\ln|t|. \tag{27}$$

Then we estimate the sum of the first terms of the series in (27):

$$\begin{aligned} \sum_{1\le n\le |t|} \frac{1}{n}(1-F(t))^n &= \int_1^{|t|} \frac{(1-F(t))^x}{x} + O(1) \\ &= \ln|t|(1-F(t))^{|t|} - \ln(1-F(t)) \int_1^{|t|} \ln x (1-F(t))^x\,dx + O(1) \\ &= d_1 + d_2 + O(1). \end{aligned} \tag{28}$$

Applying an elementary inequality $(1-x^{-1})^x < e^{-1}$, $x > 1$, we write a bound for the first term in (28):

$$\begin{aligned} d_1 &= \ln|t| \left(1 - \frac{1}{|t|\ln|t|(\ln\ln|t|)^2}\right)^{|t|} \\ &\le \ln|t| \exp\left\{-1/\ln|t|(\ln\ln|t|)^2\right\} = \ln|t| \left(1 - \frac{1}{\ln|t|(\ln\ln|t|)^2} + o\left(\frac{1}{\ln|t|\ln\ln|t|}\right)^2\right) \\ &= \ln|t| - \frac{1}{(\ln\ln|t|)^2} + o\left(\frac{1}{\ln\ln|t|}\right)^2. \end{aligned}$$

The second term in (28) is estimated in the following way:

$$d_2 \sim F(t) \int_1^{|t|} \ln x (1-F(t))^x\,dx \le \frac{1}{|t|\ln|t|(\ln\ln|t|)^2} \int_1^{|t|} \ln x\,dx \le \frac{1}{(\ln\ln|t|)^2}.$$

These estimates together with (28) yield

$$\sum_{1\le n\le |t|} \frac{1}{n}(1-F(t))^n \le \ln|t| + O(1),$$

whence

$$\sum_{n>|t|} \frac{1}{n}(1-F(t))^n \ge \ln\ln|t| + O(\ln\ln\ln|t|)$$

by (27). This is what had to be proved, in fact. Indeed

$$\sum_{n>|t|}\frac{1}{n}F(t)(1-F(t))^{n-1}\geq\frac{F(t)}{1-F(t)}\bigl(\ln\ln|t|+O(\ln\ln\ln|t|)\bigr)$$
$$\geq\frac{1}{|t|\ln|t|\ln\ln|t|}+O\left(\frac{\ln\ln\ln|t|}{|t|\ln|t|(\ln\ln|t|)^2}\right).$$

As a corollary of the latter inequality we prove that integral (26) is unbounded. We proved above that this means that equality (4) does not hold for the empirical distribution function $F_n(t)$ in the space $L_1(-\infty,\infty)$.

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Department of Operations Research, Faculty for Cybernetics, National Taras Shevchenko University, Academician Glushkov Avenue, 6, Kyiv 03127, Ukraine

E-mail address: mik@unicyb.kiev.ua

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