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TWO-PARAMETER GARSIA–RODEMICH–RUMSEY INEQUALITY AND ITS APPLICATION TO FRACTIONAL BROWNIAN FIELDS

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Abstract. The paper contains a generalization of the Garsia–Rodemich–Rumsey inequality to the case of a function of two arguments. Based on this result, we obtain two other inequalities for the fractional Brownian field on the plane, namely an inequality for the Hölder constant of the field and similar bounds for its fractional derivatives.

1. Introduction

We generalize the Garsia–Rodemich–Rumsey inequality to the two-parameter case; some applications of this result are considered for fractional Brownian fields.

The Garsia–Rodemich–Rumsey inequality is proved in the paper [1]. The following partial case of this result is used in [2, Lemma 7.3] to obtain some properties of a fractional Brownian field.

Lemma 1. *Let $p \geq 1$ and $\alpha > p^{-1}$. Then there exists a constant $C_{\alpha,p} > 0$ such that*

$$|f(t) - f(s)|^p \leq C_{\alpha,p} |t-s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha+1}} \, dx \, dy \tag{1}$$

for all continuous functions f on $[0,T]$ and for all $t, s \in [0,T]$.

As a corollary of Lemma 1, the following result is also proved for a fractional Brownian motion in [2].

Lemma 2 ([2, Lemma 7.4])**.** *Let $\{B_t, t \geq 0\}$ be a fractional Brownian motion with the Hurst index $H \in (0,1)$. Then for all $0 < \varepsilon < H$ and $T > 0$ there exists a positive random variable $\eta_{\varepsilon,T}$ such that $\mathsf{E}(|\eta_{\varepsilon,T}|^p) < \infty$ for all $p \in [1, \infty)$, and*

$$|B(t) - B(s)| \leq \eta_{\varepsilon,T} |t-s|^{H-\varepsilon} \quad a.s.$$

for all $s, t \in [0,T]$.

Here "a.s." stands for "almost surely".

Definition 1. Let $f \colon \mathbb{R}_+ \to \mathbb{R}$ and $0 < \alpha < 1$. The function

$$D_{b-}^{\alpha} f = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{f(x)}{(b-x)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_x^b \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} \, dt$$

is called the right Marchaud derivative of order α on the interval (a,b).

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Lemma 3 ([2, Lemma 7.5]). *Let $\{B_t, t \geq 0\}$ be a fractional Brownian motion with the Hurst index $H \in (\frac{1}{2}, 1)$. If $1 - H < \alpha < \frac{1}{2}$, then*

$$\mathsf{E} \sup_{0 \leq s \leq t \leq T} \left| D_{t-}^{1-\alpha} B_{t-}(s) \right|^p < \infty$$

for all $T > 0$ and $p \in [1, \infty)$.

This paper is devoted to a generalization of the Garsia–Rodemich–Rumsey inequality (1) to the two-parameter case and to those of Lemmas 2 and 3 for fractional Brownian fields on the plane.

Section 2 contains the proof of the two-parameter Garsia–Rodemich–Rumsey inequality. Some of its applications to fractional Brownian fields are given in Section 3. The two-parameter analogs of the above-mentioned results due to Nualart and Răşcanu [2] are also given.

2. The Garsia–Rodemich–Rumsey inequality in the plane

Let $s = (s_1, s_2)$ and $t = (t_1, t_2)$, that is $s, t \in \mathbb{R}^2$. We write $t > s$ if $t_1 > s_1$ and $t_2 > s_2$.

Lemma 4. *Let $p \geq 1$, $\alpha_1 > p^{-1}$, $\alpha_2 > p^{-1}$, and $f \in C([0,1]^2)$. Then there exists a constant $C_{\alpha_1,\alpha_2,p} > 0$ such that*

$$|\square f(s,t)|^p \leq C_{\alpha_1,\alpha_2,p} B \, |s_1 - t_1|^{\alpha_1 p - 1} \, |s_2 - t_2|^{\alpha_2 p - 1} \tag{2}$$

for all $s, t \in [0,1]^2$ where $\square f(s,t) = f(s_1,s_2) - f(s_1,t_2) - f(t_1,s_2) + f(t_1,t_2)$ and

$$B = \int_{[0,1]^4} \frac{|\square f(x,y)|^p}{|x_1 - y_1|^{\alpha_1 p + 1} \, |x_2 - y_2|^{\alpha_2 p + 1}} \, dx_1 \, dx_2 \, dy_1 \, dy_2. \tag{3}$$

Proof. I. First we consider $|\square f((0,0),(1,1))|$. Put

$$I(t) := \int_{[0,1]^2} \frac{|\square f(s,t)|^p}{|s_1 - t_1|^{\alpha_1 p + 1} \, |s_2 - t_2|^{\alpha_2 p + 1}} \, ds_1 \, ds_2.$$

According to (3),

$$\int_{[0,1]^2} I(t) \, dt_1 \, dt_2 = B. \tag{4}$$

Let

$$J(t_1,t_2) := I(t_1,t_2) + \int_0^1 I(t_1,u) \, du + \int_0^1 I(u,t_2) \, du.$$

Using (4), we get

$$\int_{[0,1]^2} J(t_1,t_2) \, dt_1 \, dt_2 \leq 3B.$$

Thus there exists $t^{(0)} \in (0,1)^2$ such that $J(t^{(0)}) \leq 3B$, whence

$$I\big(t^{(0)}\big) \leq 3B, \qquad \int_0^1 I\big(t_1^{(0)}, u\big) \, du \leq 3B, \qquad \int_0^1 I\big(u, t_2^{(0)}\big) \, du \leq 3B. \tag{5}$$

II. Choose a sequence

$$\big\{t^{(n)} \colon n \geq 1\big\}, \qquad t^{(n)} \xrightarrow[n \to \infty]{} (0,0)$$

such that $t^{(0)} > t^{(1)} > t^{(2)} > \dots$. The other defining properties of this sequence are as follows. Assume that $t^{(n-1)}$ is already chosen. Let

$$d_1^{(n-1)} := \left(\frac{1}{2}\right)^{\frac{1}{\alpha_1 + 1/p}} t_1^{(n-1)}, \qquad d_2^{(n-1)} := \left(\frac{1}{2}\right)^{\frac{1}{\alpha_2 + 1/p}} t_2^{(n-1)}. \tag{6}$$

Then we choose $t^{(n)}$ such that

$$t^{(n)} \le d^{(n-1)} \tag{7}$$

and

$$\text{a)}\quad I\big(t^{(n)}\big) \le \frac{7B}{d_1^{(n-1)} d_2^{(n-1)}}, \tag{8}$$

$$\text{b)}\quad \frac{\left|\Box f\left(t^{(n-1)}, t^{(n)}\right)\right|^p}{\left|t_1^{(n-1)} - t_1^{(n)}\right|^{\alpha_1 p+1} \left|t_2^{(n-1)} - t_2^{(n)}\right|^{\alpha_2 p+1}} \le \frac{7I\left(t^{(n-1)}\right)}{d_1^{(n-1)} d_2^{(n-1)}}, \tag{9}$$

$$\text{c)}\quad I\left(t_1^{(n)}, t_2^{(0)}\right) \le \frac{21B}{d_1^{(n-1)}}, \tag{10}$$

$$I\left(t_1^{(0)}, t_2^{(n)}\right) \le \frac{21B}{d_2^{(n-1)}}, \tag{11}$$

$$\text{d)}\quad \frac{\left|\Box f\left(\left(t_1^{(n-1)}, t_2^{(0)}\right), t^{(n)}\right)\right|^p}{\left|t_1^{(n-1)} - t_1^{(n)}\right|^{\alpha_1 p+1} \left|t_2^{(0)} - t_2^{(n)}\right|^{\alpha_2 p+1}} \le \frac{7I\left(t_1^{(n-1)}, t_2^{(0)}\right)}{d_1^{(n-1)} d_2^{(n-1)}}, \tag{12}$$

$$\text{e)}\quad \frac{\left|\Box f\left(\left(t_1^{(0)}, t_2^{(n-1)}\right), t^{(n)}\right)\right|^p}{\left|t_1^{(0)} - t_1^{(n)}\right|^{\alpha_1 p+1} \left|t_2^{(n-1)} - t_2^{(n)}\right|^{\alpha_2 p+1}} \le \frac{7I\left(t_1^{(0)}, t_2^{(n-1)}\right)}{d_1^{(n-1)} d_2^{(n-1)}}. \tag{13}$$

We show that such a point $t^{(n)}$ always exists. We prove that any of the inequalities (8), (9), (12), and (13) may fail on a set whose measure is less than $\frac{1}{7} d_1^{(n-1)} d_2^{(n-1)}$, while the system (10), (11) does not hold on a set whose measure is less than $\frac{2}{7} d_1^{(n-1)} d_2^{(n-1)}$. This means that the measure of a set where at least one of the inequalities (8)–(13) fails is less than $\frac{6}{7} d_1^{(n-1)} d_2^{(n-1)}$ and this implies that a point $t^{(n)}$ with the indicated properties exists.

a) Assume that inequality (8) fails on the set X, that is,

$$I(z) > \frac{7B}{d_1^{(n-1)} d_2^{(n-1)}} \qquad \text{for all } z \in X.$$

Thus if $m(X) \ge \frac{1}{7} d_1^{(n-1)} d_2^{(n-1)}$, then

$$\int_{[0,1]^2} I(z)\, dz_1\, dz_2 > \frac{7B}{d_1^{(n-1)} d_2^{(n-1)}} m(X) \ge B,$$

which contradicts (4).

b) Assume that inequality (9) fails on the set Y, that is,

$$\frac{\left|\Box f\left(t^{(n-1)}, z\right)\right|^p}{\left|t_1^{(n-1)} - z_1\right|^{\alpha_1 p+1} \left|t_2^{(n-1)} - z_2\right|^{\alpha_2 p+1}} > \frac{7I\left(t^{(n-1)}\right)}{d_1^{(n-1)} d_2^{(n-1)}} \qquad \text{for all } z \in Y.$$

Thus if $m(Y) \ge \frac{1}{7} d_1^{(n-1)} d_2^{(n-1)}$, then

$$I\big(t^{(n-1)}\big) > \frac{7I\big(t^{(n-1)}\big)}{d_1^{(n-1)} d_2^{(n-1)}} m(Y) \ge I\big(t^{(n-1)}\big).$$

We get a contradiction.

c) Assume that inequality (10) fails on the set X_1 and that inequality (11) fails on the set X_2, that is,

$$I\left(z_1, t_2^{(0)}\right) > \frac{21B}{d_1^{(n-1)}} \qquad \text{for all } z_1 \in X_1,$$

$$I\left(t_1^{(0)}, z_2\right) > \frac{21B}{d_2^{(n-1)}} \qquad \text{for all } z_2 \in X_2.$$

Thus if $m(X_1) \geq \frac{1}{7} d_1^{(n-1)}$, then

$$\int_0^1 I\left(z_1, t_2^{(0)}\right)\, dz_1 > \frac{21B}{d_1^{(n-1)}} m(X_1) \geq 3B,$$

which contradicts (5). Hence $m(X_1) < \frac{1}{7} d_1^{(n-1)}$. Similarly, $m(X_2) < \frac{1}{7} d_2^{(n-1)}$.

Let A be the set where system (10), (11) fails. Then

$$m(A) = m\left(\left(X_1 \times \left[0, d_2^{(n-1)}\right]\right) \cup \left(\left[0, d_1^{(n-1)}\right] \times X_2\right)\right) \leq \frac{2}{7} d_1^{(n-1)} d_2^{(n-1)}.$$

Relations d) and e) are proved similarly to b).

III. Since $d^{(n)} \leq d^{(n-1)}$ for $n \geq 1$, we derive from inequalities (8), (10), and (11) that

$$I\left(t^{(n)}\right) \leq \frac{7B}{d_1^{(n)} d_2^{(n)}}, \tag{14}$$

$$I\left(t_1^{(n)}, t_2^{(0)}\right) \leq \frac{21B}{d_1^{(n)}}, \tag{15}$$

$$I\left(t_1^{(0)}, t_2^{(n)}\right) \leq \frac{21B}{d_2^{(n)}} \tag{16}$$

for all $n \geq 1$.

Inequality (14) holds for $n = 0$, too.

IV. We prove that

$$1) \quad \left|t_1^{(n-1)} - t_1^{(n)}\right|^{\alpha_1+1/p} \leq 4\left(\left(d_1^{(n-1)}\right)^{\alpha_1+1/p} - \left(d_1^{(n)}\right)^{\alpha_1+1/p}\right), \tag{17}$$

$$2) \quad \left|t_2^{(n-1)} - t_2^{(n)}\right|^{\alpha_2+1/p} \leq 4\left(\left(d_2^{(n-1)}\right)^{\alpha_2+1/p} - \left(d_2^{(n)}\right)^{\alpha_2+1/p}\right) \tag{18}$$

for all $n \geq 1$.

1) Relations (6) and (7) imply that

$$2\left(d_1^{(n)}\right)^{\alpha_1+1/p} = \left(t_1^{(n)}\right)^{\alpha_1+1/p} \leq \left(d_1^{(n-1)}\right)^{\alpha_1+1/p}.$$

Using (6), we get

$$\begin{aligned} \left|t_1^{(n-1)} - t_1^{(n)}\right|^{\alpha_1+1/p} &\leq \left(t_1^{(n-1)}\right)^{\alpha_1+1/p} = 2\left(d_1^{(n-1)}\right)^{\alpha_1+1/p} \\ &\leq 4\left(\left(d_1^{(n-1)}\right)^{\alpha_1+1/p} - \left(d_1^{(n)}\right)^{\alpha_1+1/p}\right). \end{aligned}$$

2) Inequality (18) is proved analogously.

V. Now we prove that

1) for all $n \geq 1$,

$$\left|\Box f\left(t^{(n-1)}, t^{(n)}\right)\right| \leq G_p B^{1/p} \int_{d_1^{(n)}}^{d_1^{(n-1)}} \int_{d_2^{(n)}}^{d_2^{(n-1)}} \frac{du_1^{\alpha_1+1/p} du_2^{\alpha_2+1/p}}{u_1^{2/p} u_2^{2/p}}, \tag{19}$$

2) for all $n \geq 2$,

$$\begin{aligned} &\left|\Box f\left(\left(t_1^{(n-1)}, t_2^{(0)}\right), t^{(n)}\right)\right| \\ &\qquad \leq H_p B^{1/p} \int_{d_1^{(n)}}^{d_1^{(n-1)}} \int_0^{\left(d_2^{(n-1)}\right)^{\frac{1}{\alpha_2 p+1}}} \frac{du_1^{\alpha_1+1/p} du_2^{\alpha_2+1/p}}{u_1^{2/p} u_2^{2/p}}, \end{aligned} \tag{20}$$

3) for all $n \geq 2$,

$$\begin{aligned} &\left|\Box f\left(\left(t_1^{(0)}, t_2^{(n-1)}\right), t^{(n)}\right)\right| \\ &\qquad \leq H_p B^{1/p} \int_0^{\left(d_1^{(n-1)}\right)^{\frac{1}{\alpha_1 p+1}}} \int_{d_2^{(n)}}^{d_2^{(n-1)}} \frac{du_1^{\alpha_1+1/p} du_2^{\alpha_2+1/p}}{u_1^{2/p} u_2^{2/p}} \end{aligned} \tag{21}$$

where $G_p = 16 \cdot 49^{1/p}$ and $H_p = 4 \cdot 147^{1/p}$.

1) Inequality (9) can be rewritten for all $n \geq 1$ as

$$\left|\Box f\left(t^{(n-1)}, t^{(n)}\right)\right| \leq \left|t_1^{(n-1)} - t_1^{(n)}\right|^{\alpha_1+1/p} \left|t_2^{(n-1)} - t_2^{(n)}\right|^{\alpha_2+1/p} \left(\frac{7I\left(t^{(n-1)}\right)}{d_1^{(n-1)} d_2^{(n-1)}}\right)^{1/p}.$$

Taking into account inequality (17) together with bounds (18) and (14), we obtain

$$\begin{aligned} \left|\Box f\big(t^{(n-1)}, t^{(n)}\big)\right| &\leq 16\left(\left(d_1^{(n-1)}\right)^{\alpha_1+1/p} - \left(d_1^{(n)}\right)^{\alpha_1+1/p}\right) \\ &\qquad \times \left(\left(d_2^{(n-1)}\right)^{\alpha_2+1/p} - \left(d_2^{(n)}\right)^{\alpha_2+1/p}\right) \left(\frac{49B}{\left(d_1^{(n-1)} d_2^{(n-1)}\right)^2}\right)^{1/p} \\ &\leq 16(49B)^{1/p} \left(\int_{d_1^{(n)}}^{d_1^{(n-1)}} \frac{du_1^{\alpha_1+1/p}}{u_1^{2/p}}\right) \left(\int_{d_2^{(n)}}^{d_2^{(n-1)}} \frac{du_2^{\alpha_2+1/p}}{u_2^{2/p}}\right) \\ &= G_p B^{1/p} \int_{d_1^{(n)}}^{d_1^{(n-1)}} \int_{d_2^{(n)}}^{d_2^{(n-1)}} \frac{du_1^{\alpha_1+1/p}\, du_2^{\alpha_2+1/p}}{u_1^{2/p} u_2^{2/p}}. \end{aligned}$$

2) We rewrite inequality (12) for all $n \geq 2$ in the following form:

$$\begin{aligned} &\left|\Box f\left(\left(t_1^{(n-1)}, t_2^{(0)}\right), t^{(n)}\right)\right| \\ &\quad \leq \left|t_1^{(n-1)} - t_1^{(n)}\right|^{\alpha_1+1/p} \left|t_2^{(0)} - t_2^{(n)}\right|^{\alpha_2+1/p} \left(\frac{7I\left(t_1^{(n-1)}, t_2^{(0)}\right)}{d_1^{(n-1)} d_2^{(n-1)}}\right)^{1/p}. \end{aligned}$$

Applying inequalities (17) and (15), we get

$$
\begin{aligned}
&\left|\Box f\left(\left(t_1^{(n-1)}, t_2^{(0)}\right), t^{(n)}\right)\right| \\
&\qquad \le 4\left(\left(d_1^{(n-1)}\right)^{\alpha_1+1/p} - \left(d_1^{(n)}\right)^{\alpha_1+1/p}\right)\left(\frac{147B}{\left(d_1^{(n-1)}\right)^2 d_2^{(n-1)}}\right)^{1/p} \\
&\qquad = 4(147B)^{1/p}\left(\left(d_1^{(n-1)}\right)^{\alpha_1+1/p} - \left(d_1^{(n)}\right)^{\alpha_1+1/p}\right)\frac{1}{\left(d_1^{(n-1)}\right)^{2/p}} \\
&\qquad\quad \times \left(\left(d_2^{(n-1)}\right)^{\frac{1}{\alpha_2 p+1}}\right)^{\alpha_2+1/p}\frac{1}{\left(d_2^{(n-1)}\right)^{2/p}} \\
&\qquad \le 4(147B)^{1/p}\left(\int_{d_1^{(n)}}^{d_1^{(n-1)}}\frac{du_1^{\alpha_1+1/p}}{u_1^{2/p}}\right)\left(\int_0^{\left(d_2^{(n-1)}\right)^{\frac{1}{\alpha_2 p+1}}}\frac{du_2^{\alpha_2+1/p}}{u_2^{2/p}}\right) \\
&\qquad = H_p B^{1/p}\int_{d_1^{(n)}}^{d_1^{(n-1)}}\int_0^{\left(d_2^{(n-1)}\right)^{\frac{1}{\alpha_2 p+1}}}\frac{du_1^{\alpha_1+1/p}\,du_2^{\alpha_2+1/p}}{u_1^{2/p}u_2^{2/p}}
\end{aligned}
$$

in view of $\left|t_2^{(0)} - t_2^{(n)}\right| \le t_2^{(0)} \le 1$.

3) The proof of inequality (21) is analogous to that of inequality (20). The difference is that inequalities (13), (16), and (18) are used instead of inequalities (12), (15), and (17), respectively.

VI. It is easy to see that

$$
\begin{aligned}
\left|\Box f\left(t^{(0)}, (0,0)\right)\right| \le{}& \left|\Box f\left(t^{(0)}, t^{(1)}\right)\right| \\
&+ \left|f\left(t^{(1)}\right) - f\left(t_1^{(0)}, t_2^{(1)}\right) - f\left(t_1^{(1)}, t_2^{(0)}\right) + f\left(t_1^{(0)}, 0\right)\right. \\
&\left. + f\left(0, t_2^{(0)}\right) - f(0,0)\right|.
\end{aligned}
\tag{22}
$$

Since

$$
\begin{aligned}
&\left|f\left(t^{(N-1)}\right) - f\left(t_1^{(0)}, t_2^{(N-1)}\right) - f\left(t_1^{(N-1)}, t_2^{(0)}\right) + f\left(t_1^{(0)}, 0\right) + f\left(0, t_2^{(0)}\right) - f(0,0)\right| \\
&\quad \le \left|\Box f\left(t^{(N-1)}, t^{(N)}\right)\right| + \left|\Box f\left(\left(t_1^{(N-1)}, t_2^{(0)}\right), t^{(N)}\right)\right| + \left|\Box f\left(\left(t_1^{(0)}, t_2^{(N-1)}\right), t^{(N)}\right)\right| \\
&\quad\quad + \left|f\left(t^{(N)}\right) - f\left(t_1^{(0)}, t_2^{(N)}\right) - f\left(t_1^{(N)}, t_2^{(0)}\right) + f\left(t_1^{(0)}, 0\right) + f\left(0, t_2^{(0)}\right) - f(0,0)\right|
\end{aligned}
$$

for all $N \ge 2$, relation (22) implies that

$$
\begin{aligned}
\left|\Box f\left(t^{(0)}, (0,0)\right)\right| \le{}& \sum_{n=1}^{N}\left|\Box f\left(t^{(n-1)}, t^{(n)}\right)\right| + \sum_{n=2}^{N}\left|\Box f\left(\left(t_1^{(n-1)}, t_2^{(0)}\right), t^{(n)}\right)\right| \\
&+ \sum_{n=2}^{N}\left|\Box f\left(\left(t_1^{(0)}, t_2^{(n-1)}\right), t^{(n)}\right)\right| \\
&+ \left|f\left(t^{(N)}\right) - f\left(t_1^{(0)}, t_2^{(N)}\right) - f\left(t_1^{(N)}, t_2^{(0)}\right) + f\left(t_1^{(0)}, 0\right)\right. \\
&\left. + f\left(0, t_2^{(0)}\right) - f(0,0)\right|
\end{aligned}
$$

for all $N \ge 2$.

VII. Approaching the limit as $N \to +\infty$, the latter inequality implies that

$$\left|\Box f\left(t^{(0)},(0,0)\right)\right| \le \sum_{n=1}^{\infty}\left|\Box f\left(t^{(n-1)},t^{(n)}\right)\right| + \sum_{n=2}^{\infty}\left|\Box f\left(\left(t_1^{(n-1)},t_2^{(0)}\right),t^{(n)}\right)\right| + \sum_{n=2}^{\infty}\left|\Box f\left(\left(t_1^{(0)},t_2^{(n-1)}\right),t^{(n)}\right)\right|,$$

whence

$$\left|\Box f\left(t^{(0)},(0,0)\right)\right| \le B^{1/p}\left(G_p\sum_{n=1}^{\infty}\int_{d_1^{(n)}}^{d_1^{(n-1)}}\int_{d_2^{(n)}}^{d_2^{(n-1)}}\frac{du_1^{\alpha_1+1/p}\,du_2^{\alpha_2+1/p}}{u_1^{2/p}u_2^{2/p}} + H_p\sum_{n=2}^{\infty}\int_{d_1^{(n)}}^{d_1^{(n-1)}}\int_0^{\left(d_2^{(n-1)}\right)^{\frac{1}{\alpha_2 p+1}}}\frac{du_1^{\alpha_1+1/p}\,du_2^{\alpha_2+1/p}}{u_1^{2/p}u_2^{2/p}} + H_p\sum_{n=2}^{\infty}\int_0^{\left(d_1^{(n-1)}\right)^{\frac{1}{\alpha_1 p+1}}}\int_{d_2^{(n)}}^{d_2^{(n-1)}}\frac{du_1^{\alpha_1+1/p}\,du_2^{\alpha_2+1/p}}{u_1^{2/p}u_2^{2/p}}\right)$$

by (19)–(21). Since any of the three sums contains integrals whose areas of integration are disjoint rectangles inside $[0,1]^2$,

$$\left|\Box f\left(t^{(0)},(0,0)\right)\right| \le B^{1/p}\left(G_p+2H_p\right)\int_{[0,1]^2}\frac{du_1^{\alpha_1+1/p}\,du_2^{\alpha_2+1/p}}{u_1^{2/p}u_2^{2/p}} = B^{1/p}\left(G_p+2H_p\right)\frac{\alpha_1 p+1}{\alpha_1 p-1}\cdot\frac{\alpha_2 p+1}{\alpha_2 p-1}.$$

VIII. Repeating the reasoning of steps II–VII for functions $f(t_1,1-t_2)$, $f(1-t_1,t_2)$, and $f(1-t_1,1-t_2)$ instead of $f(t_1,t_2)$, one can get the same bounds for $\left|\Box f\left(t^{(0)},(0,1)\right)\right|$, $\left|\Box f\left(t^{(0)},(1,0)\right)\right|$, and $\left|\Box f\left(t^{(0)},(1,1)\right)\right|$, respectively. Thus

$$|\Box f((0,0),(1,1))| \le 4B^{1/p}\left(G_p+2H_p\right)\frac{\alpha_1 p+1}{\alpha_1 p-1}\cdot\frac{\alpha_2 p+1}{\alpha_2 p-1}.$$

Set

$$C_{\alpha_1,\alpha_2,p} := 4^p\left(G_p+2H_p\right)^p\left(\frac{\alpha_1 p+1}{\alpha_1 p-1}\right)^p\left(\frac{\alpha_2 p+1}{\alpha_2 p-1}\right)^p.$$

Then

$$|\Box f((0,0),(1,1))|^p \le C_{\alpha_1,\alpha_2,p}\,B. \tag{23}$$

IX. If $s_1=t_1$ or $s_2=t_2$, then inequality (2) holds. Thus we assume in what follows that $s_1 \ne t_1$ and $s_2 \ne t_2$.

Consider the function $\tilde f(x') = f(s_1+x_1'(t_1-s_1), s_2+x_2'(t_2-s_2))$, $x' \in [0,1]^2$. By equality (3),

$$\int_{[s,t]^2}\frac{|\Box f(x,y)|^p}{|x_1-y_1|^{\alpha_1 p+1}|x_2-y_2|^{\alpha_2 p+1}}\,dx_1\,dx_2\,dy_1\,dy_2 \le B$$

where $[s,t]=[s_1,t_1]\times[s_2,t_2]$. We change the variables in the latter integral as follows: $x_1=s_1+x_1'(t_1-s_1)$, $x_2=s_2+x_2'(t_2-s_2)$, $y_1=s_1+y_1'(t_1-s_1)$, and $y_2=s_2+y_2'(t_2-s_2)$. Therefore

$$\int_{[0,1]^4}\frac{\left|\Box \tilde f(x',y')\right|^p}{|x_1'-y_1'|^{\alpha_1 p+1}|x_2'-y_2'|^{\alpha_2 p+1}}\,dx_1'\,dx_2'\,dy_1'\,dy_2' \le B\,|s_1-t_1|^{\alpha_1 p-1}\,|s_2-t_2|^{\alpha_2 p-1}.$$

Using inequality (23), we get

$$|\square f(s,t)|^p = \left|\square \tilde{f}((0,0),(1,1))\right|^p \le C_{\alpha_1,\alpha_2,p}\, B\, |s_1 - t_1|^{\alpha_1 p-1}\, |s_2 - t_2|^{\alpha_2 p-1}. \qquad \square$$

Now we turn to the Garsia–Rodemich–Rumsey inequality .

Theorem 1. *Let $p \ge 1$, $\alpha_1 > p^{-1}$, and $\alpha_2 > p^{-1}$. Then there exists a constant $C_{\alpha_1,\alpha_2,p} > 0$ such that*

$$\begin{aligned} |\square f(s,t)|^p \le C_{\alpha_1,\alpha_2,p}\ & |s_1 - t_1|^{\alpha_1 p-1}\, |s_2 - t_2|^{\alpha_2 p-1} \\ & \times \int_{[0,T]^2} \frac{|\square f(x,y)|^p}{|x_1 - y_1|^{\alpha_1 p+1}\, |x_2 - y_2|^{\alpha_2 p+1}}\, dx_1\, dx_2\, dy_1\, dy_2 \end{aligned} \tag{24}$$

for all continuous functions f defined on $[0,T] := [0,T_1] \times [0,T_2]$ and for all $s,t \in [0,T]$.

Proof. Put $f_0(z) := f(z_1 T_1, z_2 T_2)$, $z \in [0,1]^2$. Then $f_0 \in C([0,1]^2)$.

If the integral on the right hand side is infinite, then the inequality holds. Otherwise put

$$B := \int_{[0,1]^4} \frac{|\square f_0(\tilde{x},\tilde{y})|^p}{|\tilde{x}_1 - \tilde{y}_1|^{\alpha_1 p+1}\, |\tilde{x}_2 - \tilde{y}_2|^{\alpha_2 p+1}}\, d\tilde{x}_1\, d\tilde{x}_2\, d\tilde{y}_1\, d\tilde{y}_2.$$

Changing the variables $\tilde{x}_1 = x_1/T_1$, $\tilde{x}_2 = x_2/T_2$, $\tilde{y}_1 = y_1/T_1$, and $\tilde{y}_2 = y_2/T_2$ in the integral, we obtain

$$B = T_1^{\alpha_1 p-1} T_2^{\alpha_2 p-1} \int_{[0,T]^2} \frac{|\square f(x,y)|^p}{|x_1 - y_1|^{\alpha_1 p+1}\, |x_2 - y_2|^{\alpha_2 p+1}}\, dx_1\, dx_2\, dy_1\, dy_2 < +\infty.$$

By Lemma 4, there exists a constant $C_{\alpha_1,\alpha_2,p} > 0$ such that

$$\left|\square f_0(\tilde{s},\tilde{t})\right|^p \le C_{\alpha_1,\alpha_2,p}\, B \left|\tilde{s}_1 - \tilde{t}_1\right|^{\alpha_1 p-1} \left|\tilde{s}_2 - \tilde{t}_2\right|^{\alpha_2 p-1}$$

for all $\tilde{s},\tilde{t} \in [0,1]^2$.

Consider $\tilde{s}_1 = s_1/T_1$, $\tilde{s}_2 = s_2/T_2$, $\tilde{t}_1 = t_1/T_1$, and $\tilde{t}_2 = t_2/T_2$. Then

$$\begin{aligned} |\square f(s,t)|^p &= \left|\square f_0\left(\left(\frac{s_1}{T_1}, \frac{s_2}{T_2}\right), \left(\frac{t_1}{T_1}, \frac{t_2}{T_2}\right)\right)\right|^p \\ &\le C_{\alpha_1,\alpha_2,p}\, B \left|\frac{s_1}{T_1} - \frac{t_1}{T_1}\right|^{\alpha_1 p-1} \left|\frac{s_2}{T_2} - \frac{t_2}{T_2}\right|^{\alpha_2 p-1} \\ &= C_{\alpha_1,\alpha_2,p}\ |s_1 - t_1|^{\alpha_1 p-1}\, |s_2 - t_2|^{\alpha_2 p-1} \\ &\quad \times \int_{[0,T]^2} \frac{|\square f(x,y)|^p}{|x_1 - y_1|^{\alpha_1 p+1}\, |x_2 - y_2|^{\alpha_2 p+1}}\, dx_1\, dx_2\, dy_1\, dy_2. \qquad \square \end{aligned}$$

3. Inequalities for fractional Brownian fields

Fractional Brownian fields in the plane can be defined in various ways. Following the paper [3], we consider the random fields that possess the fractional Brownian property coordinatewise.

Definition 2. A random field $\{B_t, t \in \mathbb{R}_+^2\}$ is called a fractional Brownian field with Hurst indices H_1 and H_2, $H_i \in (0,1)$, if

1) B_t is a Gaussian field such that $B_t = 0$, $t \in \partial\mathbb{R}_+^2$,
2) $\mathsf{E}\, B_t = 0$, $\mathsf{E}\, B_t B_s = \frac{1}{4} \prod_{i=1,2} \big(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}\big)$,
3) the trajectories of B_t are continuous with probability one.

Theorem 2. *Let* $\{B(t), t \in \mathbb{R}_+^2\}$ *be a fractional Brownian field with Hurst indices*

$$H_i \in (0,1), \qquad i = 1, 2.$$

Then for all $0 < \varepsilon < \min\{H_1, H_2\}$, $T_1 > 0$, *and* $T_2 > 0$ *there exists a positive random variable* $\eta_{\varepsilon,T_1,T_2}$ *such that* $\mathsf{E}(|\eta_{\varepsilon,T_1,T_2}|^p) < \infty$ *for all* $p \in [1,\infty)$, *and*

$$|\square B(s,t)| \le \eta_{\varepsilon,T_1,T_2} |s_1 - t_1|^{H_1-\varepsilon} |s_2 - t_2|^{H_2-\varepsilon} \quad a.s.,$$

$$|B(s_1,s_2) - B(t_1,s_2)| \le \eta_{\varepsilon,T_1,T_2} |s_1 - t_1|^{H_1-\varepsilon} \quad a.s.,$$

$$|B(s_1,s_2) - B(s_1,t_2)| \le \eta_{\varepsilon,T_1,T_2} |s_2 - t_2|^{H_2-\varepsilon} \quad a.s.$$

for all $s, t \in [0,T]$.

Proof. Given a fixed $t_2 > 0$, the process $B(\cdot, t_2)$ is a fractional Brownian motion with the Hurst index H_1, while given a fixed $t_1 > 0$, the process $B(t_1, \cdot)$ is a fractional Brownian motion with the Hurst index H_2. According to Lemma 2, there are positive random variables $\eta^{(1)}_{\varepsilon,T_1}$ and $\eta^{(2)}_{\varepsilon,T_2}$ such that $\mathsf{E}\big(|\eta^{(i)}_{\varepsilon,T_i}|^p\big) < \infty$, $i = 1, 2$, for all $p \in [1,\infty)$ and

$$|B(s_1,s_2) - B(t_1,s_2)| \le \eta^{(1)}_{\varepsilon,T_1} |s_1 - t_1|^{H_1-\varepsilon} \quad \text{a.s.},$$

$$|B(s_1,s_2) - B(s_1,t_2)| \le \eta^{(2)}_{\varepsilon,T_2} |s_2 - t_2|^{H_2-\varepsilon} \quad \text{a.s.}$$

for all $s, t \in [0,T]$. It remains to prove that there exists a positive random variable $\eta^{(12)}_{\varepsilon,T_1,T_2}$ such that

$$\mathsf{E}\left(\left|\eta^{(12)}_{\varepsilon,T_1,T_2}\right|^p\right) < \infty$$

for all $p \in [1,\infty)$ and

$$|\square B(s,t)| \le \eta^{(12)}_{\varepsilon,T_1,T_2} |s_1 - t_1|^{H_1-\varepsilon} |s_2 - t_2|^{H_2-\varepsilon} \quad \text{a.s.}$$

for all $s, t \in [0,T]$.

Applying inequality (24) for $\alpha_1 = H_1 - \varepsilon/2$, $\alpha_2 = H_2 - \varepsilon/2$, and $p = 2/\varepsilon$, we obtain for all $s, t \in [0,T]$ that

$$|\square B(s,t)| \le C_{H_1,H_2,\varepsilon} |s_1 - t_1|^{H_1-\varepsilon} |s_2 - t_2|^{H_2-\varepsilon} \xi$$

where

$$\xi = \left(\int_{[0,T]^2} \frac{|\square B(r,\theta)|^{2/\varepsilon}}{|r_1 - \theta_1|^{2H_1/\varepsilon} |r_2 - \theta_2|^{2H_2/\varepsilon}} \, dr_1\, dr_2\, d\theta_1\, d\theta_2\right)^{\varepsilon/2}.$$

Let $q \ge 2/\varepsilon$. Then

$$\|\xi\|_q^q = \mathsf{E}(|\xi|^q) \le \left(\int_{[0,T]^2} \frac{\left\||\square B(r,\theta)|^{2/\varepsilon}\right\|_{q\varepsilon/2}}{|r_1 - \theta_1|^{2H_1/\varepsilon} |r_2 - \theta_2|^{2H_2/\varepsilon}} \, dr_1\, dr_2\, d\theta_1\, d\theta_2\right)^{q\varepsilon/2}. \tag{25}$$

The definition of a fractional Brownian field implies that

$$\mathsf{E}\left(|\square B(r,\theta)|^2\right) = |r_1 - \theta_1|^{2H_1} |r_2 - \theta_2|^{2H_2},$$

whence

$$\|\square B(r,\theta)\|_q = \big(\mathsf{E}\big(|\square B(r,\theta)|^q\big)\big)^{1/q} = c_q |r_1 - \theta_1|^{H_1} |r_2 - \theta_2|^{H_2}$$

for all $q \ge 1$. Thus estimate (25) implies that

$$\|\xi\|_q^q \le c_q^q T_1^{q\varepsilon} T_2^{q\varepsilon}.$$

Choosing $\eta^{(12)}_{\varepsilon,T_1,T_2} = C_{H_1,H_2,\varepsilon}\xi$, we complete the proof. ∎

Let $f\colon \mathbb{R}_+^2 \to \mathbb{R}$, $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$. We need the following definition of the forward and backward fractional Marchaud derivative on the rectangle

$$P = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}_+^2$$

introduced in the paper [3].

Definition 3. The functions

$$\begin{aligned}
D_{a+}^{\alpha_1,\alpha_2} f(x,y) := {} & \frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \\
& \times \Biggl(\frac{f(x,y)}{(x-a_1)^{\alpha_1}(y-a_2)^{\alpha_2}} + \frac{\alpha_1}{(y-a_2)^{\alpha_2}} \int_{a_1}^{x} \frac{f(x,y)-f(s,y)}{(x-s)^{1+\alpha_1}}\, ds \\
& \quad + \frac{\alpha_2}{(x-a_1)^{\alpha_1}} \int_{a_2}^{y} \frac{f(x,y)-f(x,t)}{(y-t)^{1+\alpha_2}}\, dt \\
& \quad + \alpha_1\alpha_2 \int_{a_1}^{x}\int_{a_2}^{y} \frac{\square f((x,y),(s,t))}{(x-s)^{1+\alpha_1}(y-t)^{1+\alpha_2}}\, ds\, dt \Biggr) \cdot 1_P(x,y), \\
D_{b-}^{\alpha_1,\alpha_2} f(x,y) := {} & \frac{(-1)^{\alpha_1\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \\
& \times \Biggl(\frac{f(x,y)}{(b_1-x)^{\alpha_1}(b_2-y)^{\alpha_2}} + \frac{\alpha_1}{(b_2-y)^{\alpha_2}} \int_{x}^{b_1} \frac{f(x,y)-f(s,y)}{(s-x)^{1+\alpha_1}}\, ds \\
& \quad + \frac{\alpha_2}{(b_1-x)^{\alpha_1}} \int_{y}^{b_2} \frac{f(x,y)-f(x,t)}{(t-y)^{1+\alpha_2}}\, dt \\
& \quad + \alpha_1\alpha_2 \int_{x}^{b_1}\int_{y}^{b_2} \frac{\square f((x,y),(s,t))}{(s-x)^{1+\alpha_1}(t-y)^{1+\alpha_2}}\, ds\, dt \Biggr) \cdot 1_P(x,y)
\end{aligned}$$

are called the forward and backward fractional Marchaud derivatives of orders α_1, α_2 on the rectangle P where $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

In what follows, we use the notation

$$\begin{gathered}
f_{a+}(x,y) := \square f((x,y),a) \cdot 1_P(x,y), \\
f_{b-}(x,y) := (f(x,y) - f(x,b_2-) - f(b_1-,y) + f(b-)) \cdot 1_P(x,y)
\end{gathered}$$

understood in the sense of the paper [3].

Now we are ready to state and prove the following result.

Theorem 3. *Let $\{B(t), t \in \mathbb{R}_+^2\}$ be a fractional Brownian field with the Hurst indices $H_i \in (\frac{1}{2}, 1)$, $i = 1, 2$. If $1 - H_i < \alpha_i < \frac{1}{2}$, $i = 1, 2$, then*

$$\mathsf{E} \sup_{0 \le t \le s \le T} \left| D_{t+}^{1-\alpha_1,1-\alpha_2} B_{t+}(s) \right|^p < \infty, \tag{26}$$

$$\mathsf{E} \sup_{0 \le s \le t \le T} \left| D_{t-}^{1-\alpha_1,1-\alpha_2} B_{t-}(s) \right|^p < \infty \tag{27}$$

for all $T_1 > 0$, $T_2 > 0$, and $p \in [1, \infty)$.

Proof. We prove inequality (27). The proof of inequality (26) is analogous.

It follows from the definition of the backward fractional Marchaud derivative that

$$\begin{aligned}
&\left|D_{t-}^{1-\alpha_1,1-\alpha_2} B_{t-}(s)\right| \\
&\quad\le \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \\
&\quad\quad\times\left(\frac{|\square\, B(s,t)|}{(t_1-s_1)^{1-\alpha_1}(t_2-s_2)^{1-\alpha_2}} + \frac{1-\alpha_1}{(t_2-s_2)^{1-\alpha_2}}\int_{s_1}^{t_1}\frac{|\square\, B(s,(y_1,t_2))|}{(y_1-s_1)^{2-\alpha_1}}\,dy_1\right. \\
&\quad\quad\quad+\frac{1-\alpha_2}{(t_1-s_1)^{1-\alpha_1}}\int_{s_2}^{t_2}\frac{|\square\, B(s,(t_1,y_2))|}{(y_2-s_2)^{2-\alpha_2}}\,dy_2 \\
&\quad\quad\quad\left.+(1-\alpha_1)(1-\alpha_2)\int_{s_1}^{t_1}\int_{s_2}^{t_2}\frac{|\square\, B(s,y)|}{(y_1-s_1)^{2-\alpha_1}(y_2-s_2)^{2-\alpha_2}}\,dy_1\,dy_2\right).
\end{aligned}$$

According to Theorem 2 for $\varepsilon < \min\{\alpha_1-(1-H_1),\alpha_2-(1-H_2)\}$, there exists a positive random variable $\eta_{\varepsilon,T_1,T_2}$ with finite moments of all orders such that

$$\begin{aligned}
&\left|D_{t-}^{1-\alpha_1,1-\alpha_2} B_{t-}(s)\right| \\
&\quad\le C_\alpha \eta_{\varepsilon,T_1,T_2}\Bigg((t_1-s_1)^{H_1-\varepsilon-1+\alpha_1}(t_2-s_2)^{H_2-\varepsilon-1+\alpha_2} \\
&\quad\quad+\int_{s_1}^{t_1}(y_1-s_1)^{H_1-\varepsilon-2+\alpha_1}\,dy_1+\int_{s_2}^{t_2}(y_2-s_2)^{H_2-\varepsilon-2+\alpha_2}\,dy_2 \\
&\quad\quad+\int_{s_1}^{t_1}\int_{s_2}^{t_2}(y_1-s_1)^{H_1-\varepsilon-2+\alpha_1}(y_2-s_2)^{H_2-\varepsilon-2+\alpha_2}\,dy_1\,dy_2\Bigg) \\
&\quad\le C_\alpha \eta_{\varepsilon,T_1,T_2} T_1^{H_1-\varepsilon-1+\alpha_1} T_2^{H_2-\varepsilon-1+\alpha_2}\Bigg(1+\frac{1}{H_1-\varepsilon-1+\alpha_1}+\frac{1}{H_2-\varepsilon-1+\alpha_2} \\
&\quad\quad+\frac{1}{(H_1-\varepsilon-1+\alpha_1)(H_2-\varepsilon-1+\alpha_2)}\Bigg),
\end{aligned}$$

whence (27) follows. □

4. Concluding remarks

The Garsia–Rodemich–Rumsey inequality is proved for functions of two arguments. Some applications of this result for fractional Brownian fields in the plane are considered; namely we obtained moment bounds for the Hölder constant of a fractional Brownian field and similar bounds for fractional Marchaud derivatives.

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