

## ASYMPTOTIC QUANTIZATION ERRORS FOR UNBOUNDED QUANTIZERS

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ABSTRACT. We consider non-uniform scalar quantization for a wide class of unbounded random variables (or values of a random process sampled in time). Asymptotic stochastic structures for quantization errors are derived for two types of quantizers when the number of quantization levels tends to infinity. The corresponding results for bounded random variables are generalized. Some numerical examples illustrate the rate of convergence.

### 1. INTRODUCTION

Quantization of continuous random signals (or random variables and processes) is an important part of digital representation of analog signals for various coding techniques (e.g., source coding, data compression, archiving, restoration). Quantization aims at representing a realization of a continuous in value and/or time random process (or signal) by a finite-dimensional vector. The most comprehensive overview of quantization problems can be found in [6] (for later references, see [5]). A quantizer introduces some distortion into the data and the problem of evaluation of this distortion arises. A number of authors estimate distortion measures for quantizers (see, e.g., [14] and [8]). A class of asymptotically optimal quantizers with respect to an  $r$ th-mean error distortion measure is considered in [4] (see also [3, 13]). A different approach for uniform scalar quantization is developed in [12], where the correlation properties of a Gaussian process are exploited to evaluate the asymptotic behavior of the random quantization rate for uniform quantizers. General quantization problems for Gaussian processes in infinite-dimensional functional spaces are considered in [10].

Mapping of a realization of a continuous in value and/or time process into a finite-dimensional vector requires both discretization in time (or *sampling*) and amplitude (or *quantization*). The problems of optimal sampling for spline approximation are studied in [11]. We consider quantization of a random variable, which is the value of a random process at a fixed sampling point. In this paper we derive the stochastic structure of the asymptotic quantization error for a general class of quantizers when the number of quantization levels tends to infinity. Let  $X$  be a continuous random variable with a density function  $f(x)$ ,  $x \in \mathbb{R}$ . Following Bennett's notations (see [1]), the  $N$ -level *companding quantizer* (also called *compander*)  $Q_{N,G}(x)$  is defined as

$$(1) \quad Q_{N,G}(x) = G^{-1}(Q_{N,U}(G(x))),$$

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where  $G: \mathbb{R} \rightarrow (0, 1)$  is onto and increasing, and  $Q_{N,U}$  is the  $N$ -level uniform quantizer on  $(0, 1)$ ,

$$(2) \quad Q_{N,U}(y) = \begin{cases} \frac{n}{N} - \frac{1}{2N} & \text{if } y \in \left(\frac{n-1}{N}, \frac{n}{N}\right], \quad n = 1, \dots, N-1, \\ 1 - \frac{1}{2N} & \text{if } y \in \left(\frac{N-1}{N}, 1\right). \end{cases}$$

Note that the inverse function  $S := G^{-1}$  of  $G$  provides a transformation from a uniform quantization to a non-uniform one. The quantization intervals of the companding quantizer  $Q_{N,G}$ ,

$$I_{1,N} = \left(-\infty, S\left(\frac{1}{N}\right)\right], \quad I_{n,N} = \left(S\left(\frac{n-1}{N}\right), S\left(\frac{n}{N}\right)\right], \quad n = 2, \dots, N-1,$$

and

$$I_{N,N} = \left(S\left(\frac{N-1}{N}\right), \infty\right),$$

and the corresponding quantization levels,

$$u_{n,N} = S\left(\frac{n}{N} - \frac{1}{2N}\right), \quad n = 1, \dots, N,$$

set up a design  $D_N = D_{N,G}$ ,  $N \geq 1$ . Let  $g(x)$  define the probability density function corresponding to  $G(x)$ , i.e.,  $G(x) = \int_{-\infty}^x g(v) dv$ ,  $x \in \mathbb{R}$ . We call  $g(x)$  the *quantization density function*. In what follows we consider a sequence of *regular* designs

$$D_N(g) = D_{N,G}(g), \quad N \geq 1,$$

generated by a positive continuous  $g(x)$ ,  $x \in \mathbb{R}$ ,

$$\int_{-\infty}^{S(n/N)} g(x) dx = \frac{n}{N}, \quad n = 1, \dots, N-1,$$

and

$$\int_{-\infty}^{S(n/N - 1/(2N))} g(x) dx = \frac{n}{N} - \frac{1}{2N}, \quad n = 1, \dots, N.$$

Note that  $g(x)$  characterizes the distribution of quantization levels. It follows from (1) that  $G(x)$  provides a transformation from a non-uniform quantization of the random variable  $X$  to a uniform quantization of a random variable  $G(X)$  with values in  $(0, 1)$ , i.e.,  $G(Q_{N,G}(x)) = Q_{N,U}(G(x))$ . Henceforth, let  $\xrightarrow{d}$  denote convergence in distribution.

Some problems on asymptotically optimal companding quantization are discussed in [1]. For example, for the  $r$ th-mean error distortion measure

$$D(Q_{N,G}(X)) = \mathbb{E} |X - Q_{N,G}(X)|^r, \quad r > 0,$$

it is shown in [1] that

$$(3) \quad \lim_{N \rightarrow \infty} N^r D(Q_{N,G}(X)) = \frac{1}{(r+1)2^r} \int_{\mathbb{R}} \frac{f(x)}{g(x)^r} dx = C_r > 0.$$

It follows by Hölder's inequality that the right-hand side of (3) is minimal if and only if  $g(x) = f(x)^{1/(r+1)} / \int_{\mathbb{R}} f(z)^{1/(r+1)} dz$  (cf. [3]). For a random variable  $X$  with values in a *finite* interval, the stochastic structure of the asymptotic normalized quantization error and asymptotic properties of the corresponding additive noise non-uniform quantization model are derived in [13]. Probability densities of the asymptotic vector quantization errors are studied in [7]. In our paper we obtain the stochastic structure of the asymptotic non-uniform scalar quantization errors for a random variable when the support of the density is not necessarily finite (unbounded quantizer).

The paper is organized as follows. In Section 2 we consider two types of non-uniform scalar quantizers. For these quantizers we find the stochastic structures of asymptotic normalized errors for non-uniform quantization of random variables. Section 3 provides some examples and numerical experiments to illustrate the rate of convergence in the obtained results. In Section 4 proofs of the statements in the previous sections are given.

## 2. RESULTS

In this section we consider the stochastic structure of the asymptotic quantization error for two non-uniform companding quantizers. For a random variable with values in a finite interval the stochastic structure of non-uniform quantization error is derived in [13] asymptotically when the number of quantization levels tends to infinity. Henceforth, let  $X$  be a continuous random variable with density function  $f(x)$ ,  $x \in \mathbb{R}$ , not necessarily with bounded support. Let  $g(x)$  be a quantization density function. From now on we assume that the density functions  $f(x)$  and  $g(x)$  have the same support. Various conditions on the density functions of quantized random variables and quantization density functions are considered in a number of quantization papers, mostly in the applied literature (see, e.g., [7]). We introduce the following conditions on  $f(x)$  and  $g(x)$ :

- (A<sub>1</sub>) The quantization density function  $g(x)$ ,  $x \in \mathbb{R}$ , is positive and uniformly continuous.
- (A<sub>2</sub>) There exists  $C > 0$  such that  $f(x) \leq Cg(x)$  for every  $x \in \mathbb{R}$ .
- (A<sub>3</sub>) The density function  $f(x)$ ,  $x \in \mathbb{R}$ , is continuous.

For example, the assumptions (A<sub>1</sub>)–(A<sub>3</sub>) hold if the density functions  $f(x)$  and  $g(x)$  are Gaussian with identical parameters.

Let  $G(x)$  be a distribution function corresponding to  $g(x)$ . For  $X$  and  $G(x)$  recall that  $Q_{N,G}(X)$  denotes the  $N$ -level companding quantizer with quantization design

$$D_N = D_N(g), \quad N \geq 1$$

(cf. (1)). In the following theorem, the stochastic structure of the normalized non-uniform quantization error  $Z_{N,G}(X) = N(X - Q_{N,G}(X))$  is derived asymptotically as  $N \rightarrow \infty$ . Let  $U$  be a random variable uniformly distributed on  $[-1/2, 1/2]$ .

**Theorem 1.** *Let  $X$  be a continuous random variable with density function  $f(x)$ ,  $x \in \mathbb{R}$ . Let  $g(x)$  be a quantization density function and  $Q_{N,G}(X)$  the non-uniform companding quantizer. If (A<sub>1</sub>) and (A<sub>2</sub>) hold, then*

- (i)  $g(X)Z_{N,G}(X) \xrightarrow{d} U$  as  $N \rightarrow \infty$ ;
- (ii)  $Z_{N,G}(X) \xrightarrow{d} U/g(X)$  as  $N \rightarrow \infty$ , where  $X$  and  $U$  are independent random variables.

*Remark 1.* In particular, this result is valid for random variables with values in a finite interval (cf. [13]).

Various quantization methods are used in quantization applications and theory. Now we consider a quantizer  $\overline{Q}_{N,G}$  with the same quantization intervals as for  $Q_{N,G}$ ,  $I_{n,N}$ ,  $n = 1, \dots, N$ , but with quantization levels in the middle of the intervals, except for the intervals  $I_{1,N}$  and  $I_{N,N}$ , where the levels do not change, i.e.,

$$\overline{Q}_{N,G}(x) = \begin{cases} S(\frac{1}{2N}) & \text{if } x \in I_{1,N}, \\ \frac{1}{2}(S(\frac{n}{N}) + S(\frac{n-1}{N})) & \text{if } x \in I_{n,N}, n = 2, \dots, N-1, \\ S(\frac{2N-1}{2N}) & \text{if } x \in I_{N,N} \end{cases}$$

(cf. [3]). Further, for a random variable  $X$  we use the result from [9] demonstrating the asymptotic equivalence of the non-uniform quantizers  $Q_{N,G}(X)$  and  $\overline{Q}_{N,G}(X)$  with

respect to the  $r$ th-mean error distortion measure  $D(Q_{N,G}) = E |X - Q_{N,G}(X)|^r$ ,  $r > 0$ ,  $N \geq 1$ . For  $X$  with density function  $f(x)$  let  $p(y)$  denote the probability density function of the random variable  $Y := G(X)$ ,  $p(y) = f(S(y))/g(S(y))$ ,  $y \in [0, 1]$ . Recall that  $S$  is the inverse function of  $G$ . In particular, the assumption  $(A_2)$  corresponds to boundedness of the density  $p(y)$ .

As in [9] we introduce the following sufficient conditions on  $f(x)$  and  $g(x)$  for the validity of (3):

- $(B_1)$   $g(x)$  is positive and continuous.
- $(B_2)$  There exists  $M > 0$  such that  $g(x)$  is increasing if  $x < -M$  and  $g(x)$  is decreasing if  $x > M$ .
- $(B_3)$   $\int_{\mathbb{R}} f(x)/g(x)^r dx < \infty$  for some  $r > 0$ .
- $(B_4)$  For some  $\varepsilon > 0$

$$\int_0^\varepsilon s(x/2)^r p(x) dx < \infty$$

and

$$\int_{1-\varepsilon}^1 s((x+1)/2)^r p(x) dx < \infty,$$

where, by definition,  $s(y) = S^{(1)}(y)$ ,  $y \in (0, 1)$  (see also [2] for some different sufficient conditions on  $f(x)$  and  $g(x)$ ).

The assumption  $(B_1)$  together with the tail conditions in  $(B_4)$  is somewhat similar to  $(A_1)$ . Under the assumptions  $(B_1)$ – $(B_4)$ , it is shown in [9] that the  $r$ th-mean quantization errors for  $N$ -level non-uniform quantizers  $Q_{N,G}$  and  $\bar{Q}_{N,G}$  have the same behavior asymptotically as  $N \rightarrow \infty$ ,

$$(4) \quad \lim_{N \rightarrow \infty} N^r D(Q_{N,G}) = \lim_{N \rightarrow \infty} N^r D(\bar{Q}_{N,G}) = C_r > 0$$

(cf. (3)). This result justifies the application of the quantizer  $Q_{N,G}$ , at least with respect to the  $r$ th-mean distortion measure. Moreover, in the following theorem we show that for a random variable  $X$  the asymptotic normalized quantization errors

$$Z_{N,G}(X) = N(X - Q_{N,G}(X)) \quad \text{and} \quad \bar{Z}_{N,G}(X) = N(X - \bar{Q}_{N,G}(X))$$

have identical stochastic structure as  $N \rightarrow \infty$  under the assumptions  $(A_1)$ – $(A_3)$ .

**Theorem 2.** *Let  $X$  be a continuous random variable with density function  $f(x)$ ,  $x \in \mathbb{R}$ . Let  $g(x)$  be a quantization density function and  $\bar{Q}_{N,G}(X)$  the non-uniform quantizer. If  $(A_1)$ – $(A_3)$  hold, then*

- (i)  $g(X)\bar{Z}_{N,G}(X) \xrightarrow{d} U$  as  $N \rightarrow \infty$ .
- (ii)  $\bar{Z}_{N,G}(X) \xrightarrow{d} U/g(X)$  as  $N \rightarrow \infty$ , where  $X$  and  $U$  are independent random variables.

*Remark 2.* Theorems 1(ii) and 2(ii) correspond to the asymptotic result (3). In fact,  $E(U/g(X))^r = C_r$ . These results can be applied when various asymptotic properties of quantization errors are investigated.

### 3. NUMERICAL EXAMPLES AND SIMULATIONS

The following numerical examples illustrate the rate of convergence in distribution for Theorems 1 and 2 from Section 2. In this section, the quantizers  $Q_{N,G}$  and  $\bar{Q}_{N,G}$  are said to be Quantizer 1 and 2, respectively. We approximate the distribution functions in the examples below by the corresponding empirical distribution functions. Denote by  $F_V(x)$  the distribution function of a random variable  $V$ . Recall that  $N$  denotes the number of quantization levels. Let  $N_s$  denote the number of simulations.

**Example 1.** Let  $X_1$  be a standard Gaussian variable. Let  $g_1(x)$  be a standard Gaussian quantization density function. The assumptions  $(A_1)$ – $(A_3)$  hold and we rewrite Theorems 1(i) and 2(i) in the following form:

$$(5) \quad F_{g_1(X_1)Z_{N,G_1}(X_1)}(x) \rightarrow x + 1/2 \quad \text{as } N \rightarrow \infty$$

for Quantizer 1 and

$$(6) \quad F_{g_1(X_1)\bar{Z}_{N,G_1}(X_1)}(x) \rightarrow x + 1/2 \quad \text{as } N \rightarrow \infty$$

for Quantizer 2, where  $G_1(x)$  is the distribution function corresponding to the density  $g_1(x)$ . Figures 1 and 2 illustrate the rate of convergence in (5) and (6), respectively.

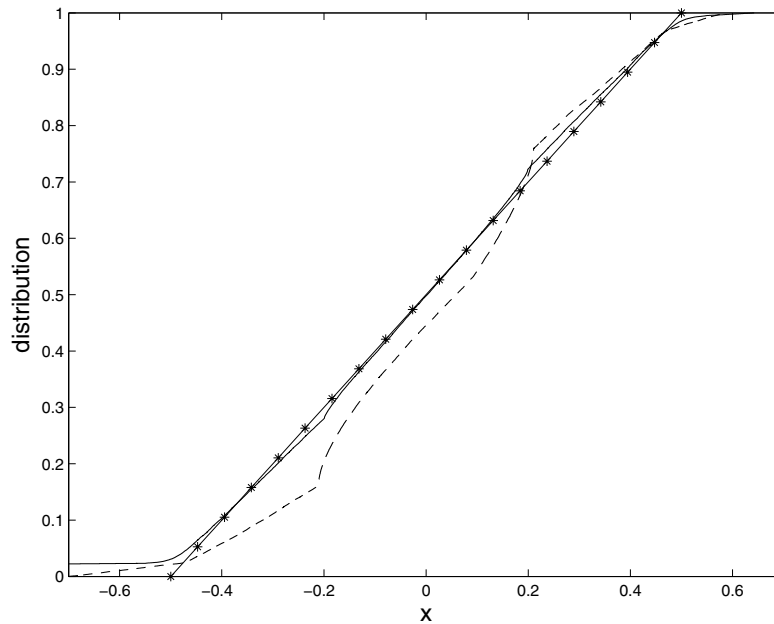


FIGURE 1. The empirical distribution functions  $\hat{F}_{g_1(X_1)Z_{N,G_1}(X_1)}(x)$  (Quantizer 1) for  $N = 3$  (*dashed*) and  $N = 20$  (*solid*), and the uniform distribution function  $F_U(x) = x + 1/2$  (*stars*);  $N_s = 50000$ .

**Example 2.** Let  $X_2$  be a Gaussian variable with zero mean and standard deviation 3. Let  $g_2(x)$  be a quantization density function of the Cauchy distribution with parameters 0 and 1. The assumptions  $(A_1)$ – $(A_3)$  hold and we rewrite Theorems 1(ii) and 2(ii) in the following form:

$$(7) \quad F_{Z_{N,G_2}(X_2)}(x) \rightarrow F_{U/g_2(X_2)}(x) \quad \text{as } N \rightarrow \infty$$

for Quantizer 1 and

$$(8) \quad 1F_{\bar{Z}_{N,G_2}(X_2)}(x) \rightarrow F_{U/g_2(X_2)}(x) \quad \text{as } N \rightarrow \infty$$

for Quantizer 2, where  $G_2(x)$  corresponds to  $g_2(x)$ . Figures 3 and 4 illustrate the rate of convergence in (7) and (8), respectively.

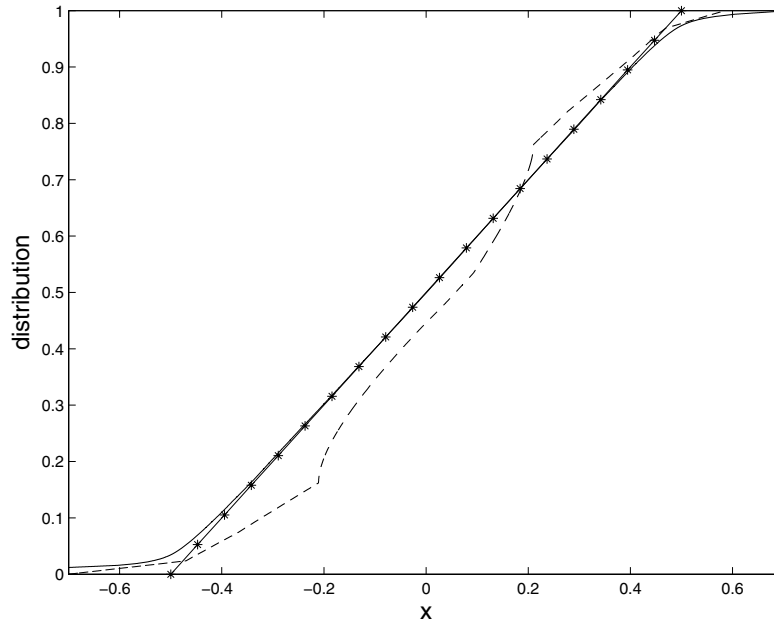


FIGURE 2. The empirical distribution functions  $\widehat{F}_{g_1(X_1)\overline{Z}_{N,G_1}(X_1)}(x)$  (Quantizer 2) for  $N = 3$  (*dashed*) and  $N = 20$  (*solid*), and the uniform distribution function  $F_U(x) = x + 1/2$  (*stars*);  $N_s = 50000$ .

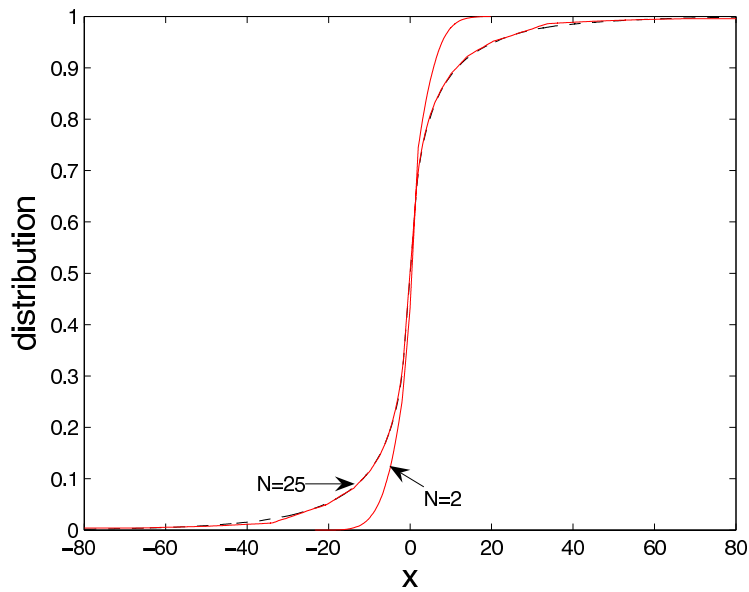


FIGURE 3. The empirical distribution functions  $\widehat{F}_{Z_{N,G_2}(X_2)}(x)$  (Quantizer 1) for  $N = 2.25$  (*solid*), and  $\widehat{F}_{U/g_2(X_2)}(x)$  (*dashed*);  $N_s = 50000$ .

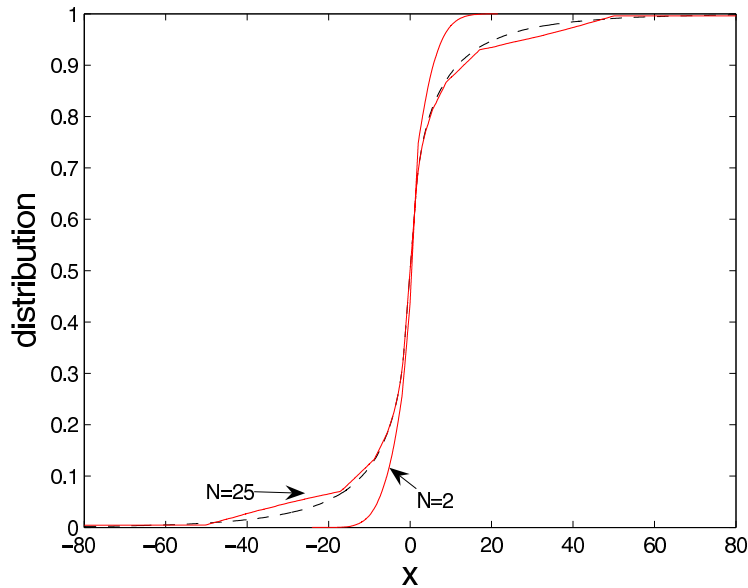


FIGURE 4. The empirical distribution functions  $\widehat{F}_{Z_{N,G_2}(X_2)}(x)$  (Quantizer 2) for  $N = 2.25$  (solid), and  $\widehat{F}_{U/g_2(X_2)}(x)$  (dashed);  $N_s = 50000$ .

4. PROOFS

Henceforth, let  $\phi_V(v)$  denote the characteristic function of a random variable  $V$ .

*Proof of Theorem 1.* We begin with (ii). For the random variable  $Y = G(X)$  with values in  $(0, 1)$ , we have the uniform  $N$ -level scalar quantizer

$$Q_{N,U}(Y) = Q_{N,U}(G(X)) = G(Q_{N,G}(X))$$

(cf. (1) and (2)), and the normalized quantization error  $Z_{N,U}(Y) = N(Y - Q_{N,U}(Y))$ . Note that the function  $G(x)$  is increasing and continuously differentiable,  $G^{(1)}(x) = g(x)$ ,  $x \in \mathbb{R}$ . Thus the inverse function  $S$  of  $G$  is also continuously differentiable and

$$(9) \quad S^{(1)}(y) = \frac{1}{g(S(y))}, \quad y \in (0, 1).$$

By the mean value theorem, there exists  $X_N$  between  $X$  and  $Q_{N,G}(X)$  such that

$$Z_{N,U}(Y) = N(G(X) - G(Q_{N,G}(X))) = g(X_N)Z_{N,G}(X).$$

We get

$$Z_{N,G}(X) = \left(1 + \frac{g(X_N) - g(X)}{g(X)}\right)^{-1} \frac{Z_{N,U}(Y)}{g(X)}.$$

Further, uniform continuity of  $g(x)$  (the assumption  $(A_1)$ ) implies that

$$(g(X_N) - g(X))/g(X) \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad (\text{a.s.}).$$

Thus it follows by the Cramér–Slutsky Theorem that

$$Z_{N,G}(X) \xrightarrow{d} U/g(X) \quad \text{as } N \rightarrow \infty$$

since

$$(10) \quad Z_{N,U}(Y)/g(X) \xrightarrow{d} U/g(X) \quad \text{as } N \rightarrow \infty,$$

$X = S(Y)$ . In order to prove (10), we show that

$$(11) \quad \phi_{Z_{N,U}(Y)/g(S(Y))}(v) \rightarrow \phi_{U/g(S(Y))}(v) \quad \text{as } N \rightarrow \infty, \quad v \in \mathbb{R}.$$

By the definition (2), for the density function  $p(y)$  of the random variable  $Y$ , we have

$$(12) \quad \begin{aligned} \phi_{Z_{N,U}(Y)/g(S(Y))}(v) &= \int_0^1 \exp\{ivN(y - Q_{N,U}(y))/g(S(y))\} p(y) dy \\ &= \sum_{n=1}^N \int_{(n-1)/N}^{n/N} \exp\left\{ivN\left(y - \left(\frac{n}{N} - \frac{1}{2N}\right)\right) / g(S(y))\right\} p(y) dy \\ &= \sum_{n=1}^N \int_{(n-1)/N}^{n/N-1/(2N)} \exp\left\{ivN\left(y - \left(\frac{n}{N} - \frac{1}{2N}\right)\right) / g(S(y))\right\} p(y) dy \\ &\quad + \sum_{n=1}^N \int_{n/N-1/(2N)}^{n/N} \exp\left\{ivN\left(y - \left(\frac{n}{N} - \frac{1}{2N}\right)\right) / g(S(y))\right\} p(y) dy \\ &= S_1(N) + S_2(N). \end{aligned}$$

Consider the sum  $S_1(N)$ . We get

$$\begin{aligned} S_1(N) &= \sum_{n=1}^N \int_{(n-1)/N}^{n/N-1/(2N)} \exp\left\{ivN\left(y - \left(\frac{n}{N} - \frac{1}{2N}\right)\right) / g(S(y))\right\} p(y) dy \\ &= \int_{-1/2}^0 \left( \sum_{n=1}^N \exp\left\{ivz / g\left(S\left(\frac{1}{N}\left(n - \frac{1}{2}\right) + \frac{z}{N}\right)\right)\right\} \right. \\ &\quad \left. \times p\left(\left(n - \frac{1}{2}\right)N^{-1} + zN^{-1}\right) N^{-1} \right) dz, \end{aligned}$$

by changing the variable,  $z = N(y - N^{-1}(n - 1/2))$ . It follows from  $(A_2)$  that there exists  $C > 0$  such that

$$p(y) = f(S(y))/g(S(y)) \leq C,$$

and we get

$$\left| \sum_{n=1}^N \exp\left\{ivz / g\left(S\left(\frac{1}{N}\left(n - \frac{1}{2}\right) + \frac{z}{N}\right)\right)\right\} p\left(\left(n - \frac{1}{2}\right)N^{-1} + zN^{-1}\right) N^{-1} \right| \leq C.$$

Hence, by the dominated convergence theorem and elementary properties of the Riemann integral,

$$(13) \quad S_1(N) \rightarrow \int_{-1/2}^0 \int_0^1 e^{ivz/g(S(y))} p(y) dy dz \quad \text{as } N \rightarrow \infty.$$

Similarly as for  $S_1(N)$ , we have

$$(14) \quad S_2(N) \rightarrow \int_0^{1/2} \int_0^1 e^{ivz/g(S(y))} p(y) dy dz \quad \text{as } N \rightarrow \infty.$$

Now (12) together with (13) and (14) implies

$$\phi_{Z_{N,U}(Y)/g(S(Y))}(v) \rightarrow \int_{-1/2}^{1/2} \int_0^1 e^{ivz/g(S(y))} p(y) dy dz = \phi_{U/g(S(Y))}(v) \quad \text{as } N \rightarrow \infty$$

and  $S(Y) = X$  and  $U$  are independent random variables. Hence (11) holds. This completes the proof of the assertion (ii).

The idea of the proof of (i) is the same as in the proof of (ii). Notice that

$$Q_{N,G}(X) = S(Q_{N,U}(Y)), \quad X = S(Y).$$



Thus

$$(15) \quad \begin{aligned} g(X)Z_{N,G}(X) &= Ng(X)(X - Q_{N,G}(X)) \\ &= Ng(S(Y))(S(Y) - S(Q_{N,U}(Y))). \end{aligned}$$

Furthermore, it follows by the mean value theorem and (9) that there exists  $Y_N$  between  $Y$  and  $Q_{N,U}(Y)$  such that  $S(Y) - S(Q_{N,U}(Y)) = (Y - Q_{N,U}(Y))/g(S(Y_N))$ . Therefore, combining  $(A_1)$  with the Cramér–Slutsky Theorem applied to (15) and using that  $g(S(Y_N)) \rightarrow g(S(Y))$  (a.s.) as  $N \rightarrow \infty$  for  $|Y_N - Y| \leq |Q_{N,U}(Y) - Y| \leq 1/N$ , we get

$$g(X)Z_{N,G}(X) = \frac{g(S(Y))}{g(S(Y_N))} Z_{N,U}(Y) \xrightarrow{d} U \quad \text{as } N \rightarrow \infty.$$

In order to prove the assertion (i), it is enough to show that

$$Z_{N,U}(Y) \xrightarrow{d} U \quad \text{as } N \rightarrow \infty$$

or equivalently

$$(16) \quad \phi_{Z_{N,U}(Y)}(v) \rightarrow \phi_U(v) \quad \text{as } N \rightarrow \infty, \quad v \in \mathbb{R}.$$

But (16) can be proved in the same way as in the proof of (11). Hence (i) holds. This completes the proof of the theorem.  $\square$

*Proof of Theorem 2.* First we prove (ii). By the definition we have

$$(17) \quad \bar{Z}_{N,G}(X) = Z_{N,G}(X) + d_N(X),$$

where  $d_N(X) = d_{N,G}(X) := N(Q_{N,G}(X) - \bar{Q}_{N,G}(X))$ ,  $N \geq 1$ . Therefore, in view of the Cramér–Slutsky Theorem, to prove the assertion of the theorem it is enough to show that  $d_N(X) \xrightarrow{d} 0$  or, equivalently,

$$(18) \quad \phi_{d_N(X)}(v) \rightarrow 1 \quad \text{as } N \rightarrow \infty$$

since, by Theorem 1,  $Z_{N,G}(X) \xrightarrow{d} U/g(X)$  as  $N \rightarrow \infty$ . Observe that by the definitions

$$(19) \quad \begin{aligned} \phi_{d_N(X)}(v) &= \mathbf{E} e^{ivd_N(X)} \\ &= \sum_{n=2}^{N-1} \int_{S((n-1)/N)}^{S(n/N)} \exp \left\{ ivN \left( S \left( \frac{n}{N} - \frac{1}{2N} \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \left( S \left( \frac{n}{N} \right) + S \left( \frac{n-1}{N} \right) \right) \right) \right\} f(x) dx. \end{aligned}$$

To prove (18) we estimate  $|\phi_{d_N(X)}(v) - 1|$  by splitting up the sum in (19). For a given  $\delta > 0$ , select a sufficiently small  $a_0 = a_0(\delta) > 0$  such that

$$(20) \quad I_2(a_0) := \int_0^{a_0} p(y) dy + \int_{1-a_0}^1 p(y) dy < \delta.$$

We represent  $\phi_{d_N(X)}(v)$  as follows:

$$\begin{aligned} \phi_{d_N(X)}(v) &= S_{1,N}(a_0) + S_{2,N}(a_0), \\ 1 &= I_1(a_0) + I_2(a_0), \end{aligned}$$

where

$$S_{1,N}(a_0) := \sum_{\substack{2 \leq n \leq N-1: \\ n/N \in [a_0, 1-a_0]}} \exp \left\{ iv \frac{N}{2} \left( \left( S \left( \frac{n}{N} - \frac{1}{2N} \right) - S \left( \frac{n-1}{N} \right) \right) + \left( S \left( \frac{n}{N} - \frac{1}{2N} \right) - S \left( \frac{n}{N} \right) \right) \right) \right\} \\ \times \int_{S((n-1)/N)}^{S(n/N)} f(x) dx.$$

Further, combining the mean value theorem with (A<sub>3</sub>) and (9), we get

$$(21) \quad S_{1,N}(a_0) = \sum_{\substack{2 \leq n \leq N-1: \\ n/N \in [a_0, 1-a_0]}} \exp \left\{ i \frac{v}{4} \left( \frac{1}{g(S(\nu'_n))} - \frac{1}{g(S(\nu''_n))} \right) \right\} \\ \times f(S(\omega'_n)) \frac{1}{g(S(\omega''_n))} \frac{1}{N}$$

for some

$$\nu'_n, \nu''_n, \omega'_n, \omega''_n \in [(n-1)/N, n/N], \quad n = 2, \dots, N-1.$$

We use the following elementary property of the Riemann integral: if two functions  $h(x)$  and  $\varphi(x)$  are Riemann integrable on  $[a, b] \subset \mathbb{R}$ , then

$$(22) \quad \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} h(\xi_i) \varphi(\theta_i) \Delta x_i = \int_a^b h(x) \varphi(x) dx,$$

where

$$x_i \leq \xi_i, \quad \theta_i \leq x_{i+1}, \quad \Delta x_i = x_{i+1} - x_i, \quad i = 0, 1, \dots, n-1$$

( $a = x_0 < x_1 < x_2 < \dots < x_n = b$ ),  $n \geq 1$ . This property can be generalized to any finite number of Riemann integrable functions. Thus (22) applied to functions

$$\exp \left\{ i \frac{v}{4} \left( \frac{1}{g(S(y))} - \frac{1}{g(S(y))} \right) \right\},$$

$f(S(y))$ , and  $1/g(S(y))$  in (21) implies that

$$S_{1,N}(a_0) \rightarrow \int_{a_0}^{1-a_0} \frac{f(S(y))}{g(S(y))} dy = I_1(a_0) \quad \text{as } N \rightarrow \infty.$$

Hence

$$(23) \quad |S_{1,N}(a_0) - I_1(a_0)| < \delta$$

for a sufficiently large  $N$ . We have also that

$$(24) \quad |S_{2,N}(a_0)| \leq \sum_{\substack{2 \leq n \leq N-1: \\ n/N \in [0,1] \setminus [a_0, 1-a_0]}} \int_{S((n-1)/N)}^{S(n/N)} f(x) dx \\ \leq \int_{-\infty}^{S(a_0)} f(x) dx + \int_{S(1-a_0-1/N)}^{\infty} f(x) dx \\ = \int_0^{a_0} p(y) dy + \int_{1-a_0-1/N}^1 p(y) dy < 2\delta$$

for a sufficiently large  $N$ , by changing the variable  $y = G(x)$ . Applying (20), (23), and (24), we obtain

$$|\phi_{d_N(X)}(v) - 1| \leq |S_{1,N}(a_0) - I_1(a_0)| + |S_{2,N}(a_0)| + |I_2(a_0)| < 4\delta$$

for a sufficiently large  $N$ . Since  $\delta > 0$  is arbitrary, (18) holds. Using a similar technique as in the proof of (ii) and taking into account (17), Theorem 1(i), and (18), we immediately get the assertion (i). This completes the proof.  $\square$

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