Теорія Ймовір. та Матем. Статист.
Вип. 76, 2007

Theor. Probability and Math. Statist.
No. 76, 2008, Pages 33–40
S 0094-9000(08)00729-1
Article electronically published on July 10, 2008

MARKOV RENEWAL LIMIT THEOREMS

UDC 519.21

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Abstract. We extend the fundamental results of the classical renewal theory to the so-called Markov renewal equation. We prove the Markov renewal theorems for the scheme of series.

1. Introduction

The classical renewal theorems describe the asymptotic properties of the convolution

$$U * g(t) = \int_0^t U(ds) g(t-s) \quad \text{as } t \to \infty$$

where U is the potential of a homogeneous kernel $F(du)$, which is a probability distribution in a phase space $(R_+, \mathcal{B}_+)$.

Without doubt, renewal theory arose from three basic results, namely from theorems due to Feller–Erdös–Pollard, Blackwell, and Smith. Not the last place in renewal theory is taken by the notion of direct integrability introduced by W. Feller. This notion allows one to weaken the assumption on the monotonicity of $g(t)$ in the Smith theorem which resulted in a wide area of applications of renewal theory. We consider the corresponding results for the so-called scheme of Markov renewals.

2. Main results

Let an abstract measurable space $(E, \mathcal{A})$ be given and let $G(x, dy \times dt)$ be a semi-homogeneous kernel in $(E \times R_+, \mathcal{A} \times \mathcal{B}_+)$ (see [1]), where $\mathcal{B}_+$ is the Borel σ-algebra in $R_+ = [0, \infty)$. Denote by $G(x, dy)$ the base of the kernel $G(x, dy \times dt)$, that is,

$$G(x, A) = G(x, A \times R_+), \qquad x \in E, \ A \in \mathcal{A}.$$

Assume that

(1) σ-algebra $\mathcal{A}$ is generated by a finite number of its elements,

the base $G(x, dy)$ of the kernel $G(x, dy \times dt)$ is conservative [1], and

(2) the Perron root of $G(x, dy)$ equals 1.

According to a result of [1, §1.2.1], conditions (1) and (2) imply that there exist a nontrivial σ-finite measure l and a positive $\mathcal{A}$-measurable and l-almost everywhere finite

2000 *Mathematics Subject Classification.* Primary 60K15, 60J25.
Key words and phrases. Markov renewal equation.

function h such that

$$\int_E l(dx)G(x,A) = l(A), \qquad A \in \mathcal{A}, \tag{3}$$

$$\int_E G(x,dy)h(y) = h(x), \qquad x \in E, \tag{4}$$

$$\sum_{n\geq 0} G^n(x,A) = \infty \quad \text{if} \;\; l(A) > 0. \tag{5}$$

The measure l is unique up to a multiplicative constant, while the function h is unique up to a multiplicative constant and up to the l-equivalence.

Consider the Banach space $\mathbb{B}(l,h)$ of complex $\mathcal{A}$-measurable functions f such that

$$\|f\| = \inf\{c\colon |f(x)| \leq ch(x)\}$$

is finite l-almost everywhere. Note that $\|f\|$ is a norm in the space $\mathbb{B}(l,h)$ if one identifies, as usual, the functions coinciding l-almost everywhere.

It follows from (3) and (4) that

$$Gf(x) = \int_E G(x,dy)f(y), \qquad f \in \mathbb{B}(l,h),$$

defines a linear operator acting from $\mathbb{B}(l,h)$ to $\mathbb{B}(l,h)$.

Consider a family of nonnegative semihomogeneous kernels $G_\varepsilon(x, dy \times dt)$ depending on a small parameter $\varepsilon > 0$ and assume that $G_\varepsilon(x, dy \times dt)$ converges as $\varepsilon \to 0$ to the kernel $G(x, dy \times dt)$ in the following sense:

$$\sup_{A\in\mathcal{A}} \left\| \int_0^\infty G_\varepsilon(\cdot, A \times dt)\varphi(t) - \int_0^\infty G(\cdot, A \times dt)\varphi(t) \right\| \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{6}$$

for all continuous bounded functions $\varphi(t)$, $t \geq 0$. In other words, the kernel $G_\varepsilon(x, dy \times dt)$ converges to $G(x, dy \times dt)$ "uniformly with respect to dy and weakly with respect to dt in the norm $\|\cdot\|$ evaluated with respect to x".

Denote by $G_\varepsilon(x, dy)$ the base of the kernel $G_\varepsilon(x, dy \times dt)$. It is shown in [1] that the kernels $G_\varepsilon(x, dy \times dt)$ and $G(x, dy \times dt)$ can be represented in the following form:

$$G_\varepsilon(x, dy \times dt) = G_\varepsilon(x, dy)F_\varepsilon(x, y; dt),$$
$$G(x, dy \times dt) = G(x, dy)F(x, y; dt),$$

where the probability distributions $F_\varepsilon(x, y; dt)$ and $F(x, y; dt)$ are measurable functions of the arguments $x, y \in E$. We assume that

$$G_\varepsilon(x, E) \leq 1. \tag{7}$$

Relation (6) implies that

$$\sup_{A\in\mathcal{A}} \|G_\varepsilon(x,A) - G(x,A)\| \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{8}$$

Relation (8) is equivalent to the convergence $G_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} G$ in the operator norm, where the operator G_ε is generated by the kernel $G_\varepsilon(x, A)$.

It follows from condition (2) that the norm of the operator G equals one. Moreover, the unity is an isolated simple eigenvalue of the operator G and the unity is an isolated point of the spectrum corresponding to the eigenfunction $h(x)$ in the space $\mathbb{B}(l,h)$.

Assume that

$$0 < \inf_{x\in E} h(x) < \sup_{x\in E} h(x) < \infty. \tag{9}$$

Then the same properties hold for the conjugate operators of G_ε and G. The conjugate operators are defined by the following equalities:

$$\mu G_\varepsilon(A) = \int_E \mu(dx) G_\varepsilon(x, A),$$
$$\mu G(A) = \int_E \mu(dx) G(x, A), \qquad A \in \mathcal{A}.$$

Note that the conjugate operators act in the Banach space $\mathbb{M} = \mathbb{M}(E, \mathcal{A})$ of charges μ of bounded variation. (The full variation is understood as the norm in the space $\mathbb{M}$. The convergence in this norm is equivalent to the uniform convergence for all measurable sets. The unit eigenvalue of the operator G corresponds to the measure l in $\mathbb{M}$.)

The theorem on the semicontinuity of the spectrum (Theorem 3.16 in Chapter 4 of the book [2]) implies that for all sufficiently small $\varepsilon > 0$, the operator G_ε has an isolated simple eigenvalue $\lambda_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 1$ corresponding to the eigenfunction h_ε in the space $\mathbb{B}(l, h)$ and to the eigenmeasure $\widetilde{l}_\varepsilon$ in the space $\mathbb{M}$. Recall that λ_ε is the spectral radius. Without loss of generality one can assume that

$$\int_E l(dx) h(x) = 1, \qquad l(E) = 1, \tag{10}$$
$$\|h_\varepsilon - h\| \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Obviously, the operator G_ε preserves the cone of nonnegative functions $\mathbb{B}_+$ in the space $\mathbb{B}(l, h)$. Denote by r_ε the spectral radius of G_ε and note that $r_\varepsilon \le 1$ by (7). It is clear that the cone $\mathbb{B}_+$ is normal and reproducing (see [3]). Thus r_ε is a point of the spectrum of the operator G_ε [3]. This implies that $r_\varepsilon = \lambda_\varepsilon$ and $h_\varepsilon(x) \ge 0$.

Relations (8)–(10) together with the results of [1, 4] imply that there exists a family of measures l_ε in $(E, \mathcal{A})$ that, after an appropriate normalization, are such that

$$\int_E l_\varepsilon(dx) G_\varepsilon(x, A) = \lambda_\varepsilon l_\varepsilon(A), \qquad A \in \mathcal{A}, \tag{11}$$

$$\int_E l_\varepsilon(dx) h_\varepsilon(x) = 1, \tag{12}$$

$$\sup_{A\in\mathcal{A}} |l_\varepsilon(A) - l(A)| \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{13}$$

Assume that

$$0 < m = \int_E \int_E \int_0^\infty l(dx)\, G(x, dy \times dt) h(y) t < \infty. \tag{14}$$

The aim of this paper is to investigate the asymptotic behavior of the convolution

$$U_\varepsilon * g_\varepsilon(x, t) = \int_E \int_0^t U_\varepsilon(x, dy \times ds) g_\varepsilon(y, t - s),$$

where $U_\varepsilon(x, dy \times dt)$ is the potential of the kernel $G_\varepsilon(x, dy \times dt)$, while the function $g_\varepsilon(x, t)$ is nonnegative and $\mathcal{A} \times \mathcal{B}_+$-measurable.

The following result is an analogue of the so-called elementary renewal theorem. Put $\gamma_\varepsilon = (1 - \lambda_\varepsilon)/m$.

Theorem 1. *Let conditions* (1)–(2)*,* (6)–(7)*,* (9)*, and* (14) *hold. If*

$$\sup_{\varepsilon>0} \sup_{x\in E} \int_E \int_0^\infty G_\varepsilon(x, dy \times dt) h(y) t < \infty, \tag{15}$$

$$\sup_{\varepsilon>0} \int_E \int_E \int_T^\infty l(dx)\, G_\varepsilon(x, dy \times dt) h(y) t \underset{T\to\infty}{\longrightarrow} 0, \tag{16}$$

then

$$\frac{1}{t} U_\varepsilon(x, A \times [0,t]) \underset{\substack{\gamma_\varepsilon t \to c \\ \varepsilon \to 0 \\ t \to \infty}}{\longrightarrow} \frac{1 - e^{-c}}{c} \frac{1}{m} h(x) l(A)$$

uniformly in $x \in E$ *and* $A \in \mathcal{A}$.

Proof. For $\operatorname{Im} \alpha \ge 0$, put

$$\widehat{G}_\varepsilon(x, A, \alpha) = \int_0^\infty e^{i\alpha t} G_\varepsilon(x, A \times dt)$$

and consider the operator

$$\widehat{G}_\varepsilon(\alpha) f(x) = \int_E \widehat{G}_\varepsilon(x, dy, \alpha) f(y) = \int_E \int_0^\infty G_\varepsilon(x, dy \times dt) f(y) e^{i\alpha t}.$$

It is obvious that the operators $\widehat{G}_\varepsilon(i\alpha)$ are real and nonnegative for $\alpha \ge 0$. It is easy to show that

$$\widehat{U}_\varepsilon(i\alpha) = \left[I - \widehat{G}_\varepsilon(i\alpha)\right]^{-1}$$

for $\alpha > 0$ and for sufficiently small $\varepsilon > 0$. It follows from the limit relation (6) that

$$\widehat{G}_\varepsilon(i\alpha) \underset{\substack{\varepsilon \to 0 \\ \alpha \to 0}}{\longrightarrow} G$$

in the operator norm. Let $\lambda_\varepsilon(\alpha)$ be the maximal eigenvalue (spectral radius) of the operator $\widehat{G}_\varepsilon(i\alpha)$ that corresponds to the eigenfunction $h_\varepsilon(\alpha, x)$ and to the eigenmeasure $l_\varepsilon(\alpha, dy)$. Note that

$$\lambda_\varepsilon(\alpha) \underset{\substack{\varepsilon \to 0 \\ \alpha \to 0}}{\longrightarrow} 1, \tag{17}$$

$$\|h_\varepsilon(\alpha) - h\| \underset{\substack{\varepsilon \to 0 \\ \alpha \to 0}}{\longrightarrow} 0, \qquad \sup_{A \in \mathcal{A}} |l_\varepsilon(\alpha, A) - l(A)| \underset{\substack{\varepsilon \to 0 \\ \alpha \to 0}}{\longrightarrow} 0.$$

Without loss of generality one can assume that $h_\varepsilon(0, x) = h_\varepsilon(x)$ and $l_\varepsilon(0, dx) = l_\varepsilon(dx)$, where h_ε and l_ε are defined in (10)–(13).

By C_1 we denote a smooth closed contour in the complex plane that surrounds the point $\lambda_\varepsilon(\alpha)$ and whose inner part does not contain any other point of the spectrum of the operator $\widehat{G}_\varepsilon(i\alpha)$. By C_0 we denote a smooth closed contour in the complex plane that surrounds all the points of the spectrum of the operator $\widehat{G}_\varepsilon(i\alpha)$ except the point $\lambda_\varepsilon(\alpha)$. By the Cauchy formula,

$$\begin{aligned}(I - \widehat{G}_\varepsilon(i\alpha))^{-1} &= \frac{1}{2\pi i} \oint_{C_0} \left(zI - \widehat{G}_\varepsilon(i\alpha)\right)^{-1} \frac{dz}{1-z} + \frac{1}{2\pi i} \oint_{C_1} \left(zI - \widehat{G}_\varepsilon(i\alpha)\right)^{-1} \frac{dz}{1-z} \\ &= \frac{1}{1 - \lambda_\varepsilon(\alpha)} \Pi_\varepsilon(\alpha) + \frac{1}{2\pi i} \oint_{C_0} \left(zI - \widehat{G}_\varepsilon(i\alpha)\right)^{-1} \frac{dz}{1-z},\end{aligned}$$

where $i^2 = -1$, $\pi = 3.14\ldots$, and the operator $\Pi_\varepsilon(\alpha)$ maps the function $f \in \mathbb{B}(l, h)$ to the function $\Pi_\varepsilon(\alpha) f(x) = h_\varepsilon(\alpha, x) \int_E l_\varepsilon(\alpha, dy) f(y)$, $x \in E$.

It is also obvious that

$$\left(zI - \widehat{G}_\varepsilon(i\alpha)\right)^{-1} \underset{\substack{\varepsilon \to 0 \\ \alpha \to 0}}{\longrightarrow} (zI - G)^{-1}$$

uniformly in $z \in C_0$ in the operator norm. Thus

$$\frac{1}{2\pi i} \oint_{C_0} \left(zI - \widehat{G}_\varepsilon(i\alpha)\right)^{-1} \frac{dz}{1-z} \underset{\substack{\varepsilon \to 0 \\ \alpha \to 0}}{\longrightarrow} \frac{1}{2\pi i} \oint_{C_0} (zI - G)^{-1} \frac{dz}{1-z} = I - \Pi,$$

where the operator Π maps the function $f \in \mathbb{B}(l,h)$ to the function

$$\Pi f(x) = h(x) \int_E l(dy) f(y), \qquad x \in E.$$

Hence

$$\big(I - \widehat{G}_\varepsilon(i\alpha)\big)^{-1} - \frac{1}{1-\lambda_\varepsilon(\alpha)} \Pi_\varepsilon(\alpha) \underset{\substack{\varepsilon\to 0\\ \alpha\to 0}}{\longrightarrow} I - \Pi$$

in the operator norm. Therefore

$$(1-\lambda_\varepsilon(\alpha))\big(I - \widehat{G}_\varepsilon(i\alpha)\big)^{-1} \underset{\substack{\varepsilon\to 0\\ \alpha\to 0}}{\longrightarrow} \Pi. \tag{18}$$

Further we show that

$$1 - \lambda_\varepsilon(\alpha\gamma_\varepsilon) \sim \gamma_\varepsilon(m + \alpha m) \tag{19}$$

for a fixed $\alpha > 0$. Put $\langle \mu, f\rangle = \int_E \mu(dx) f(x)$. By the definitions of $h_\varepsilon(\alpha, x)$ and $l_\varepsilon(dy)$ we have

$$\frac{1-\lambda_\varepsilon(\alpha)}{\gamma_\varepsilon} \langle l_\varepsilon, h_\varepsilon(\alpha)\rangle = m\langle l_\varepsilon, h_\varepsilon(\alpha)\rangle + \int_E l_\varepsilon(dx) \int_E G_\varepsilon(x,dy) \frac{1-\varphi_\varepsilon(x,y;i\alpha)}{\gamma_\varepsilon} h_\varepsilon(\alpha, y),$$

where $\varphi_\varepsilon(x,y;\alpha) = \int_0^\infty e^{i\alpha t} F_\varepsilon(x,y;dt)$.

This implies that the asymptotic equivalence (19) follows from

$$\lim_{\varepsilon\to 0} \int_E l_\varepsilon(dx) \int_E G_\varepsilon(x,dy) \frac{1-\varphi_\varepsilon(x,y;i\alpha\gamma_\varepsilon)}{\gamma_\varepsilon} h_\varepsilon(\alpha\gamma_\varepsilon, y) = \alpha m, \tag{20}$$

since

$$\frac{1-\varphi_\varepsilon(x,y;i\alpha\gamma_\varepsilon)}{\gamma_\varepsilon} \le \alpha \int_0^\infty t F_\varepsilon(x,y;dt)$$

by (15), (17), and

$$\sup_{\varepsilon>0} \sup_{x\in E} \int_E G_\varepsilon(x,dy) \frac{1-\varphi_\varepsilon(x,y;i\alpha\gamma_\varepsilon)}{\gamma_\varepsilon} h_\varepsilon(\alpha\gamma_\varepsilon, y) < \infty.$$

Taking into account the convergence $l_\varepsilon(dx) \to l(dx)$ in variation we see that equality (20) follows from

$$\lim_{\varepsilon\to 0} \int_E l(dx) \int_E G_\varepsilon(x,dy) \frac{1-\varphi_\varepsilon(x,y;i\alpha\gamma_\varepsilon)}{\gamma_\varepsilon} h_\varepsilon(\alpha\gamma_\varepsilon, y) = \alpha m. \tag{21}$$

Condition (16) means that

$$\sup_{\varepsilon>0} \int_E l(dx) \int_E G_\varepsilon(x,dy) \left[\mu_\varepsilon(x,y) - \frac{1-\varphi_\varepsilon(x,y;i\alpha)}{\alpha}\right] h(y) \underset{\alpha\to 0}{\longrightarrow} 0, \tag{22}$$

where $\mu_\varepsilon(x,y) = \int_0^\infty t F_\varepsilon(x,y;dt)$. Moreover the expression in the square brackets on the left hand side of (22) is nonnegative. It is easy to see that (22) and (17) imply

$$\sup_{\varepsilon>0} \int_E l(dx) \int_E G_\varepsilon(x,dy) \left[\alpha\mu_\varepsilon(x,y) - \frac{1-\varphi_\varepsilon(x,y;i\alpha\gamma_\varepsilon)}{\gamma_\varepsilon}\right] h_\varepsilon(\alpha\gamma_\varepsilon, y) \underset{\alpha\to 0}{\longrightarrow} 0.$$

Thus (21) follows from

$$\int_E l(dx) \int_E G_\varepsilon(x,dy) \mu_\varepsilon(x,y) h_\varepsilon(\alpha\gamma_\varepsilon, y) \underset{\varepsilon\to 0}{\longrightarrow} m. \tag{23}$$

Considering (17) we see that (23) follows from (16) and (6).

Relations (18) and (19) imply that

$$\gamma_\varepsilon \int_0^\infty e^{-\alpha\gamma_\varepsilon t} U_\varepsilon(x, A \times dt) \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{1+\alpha} \frac{1}{m} h(x) l(A)$$

uniformly in $x \in E$ and $A \in \mathcal{A}$.

Using the continuity theorem for the Laplace transform we get

$$\gamma_\varepsilon U_\varepsilon(x, A\times[0,t/\gamma_\varepsilon]) \underset{\varepsilon\to 0}{\longrightarrow} \left(1-e^{-t}\right)\frac{1}{m}h(x)l(A)$$

for all $t \ge 0$ uniformly in $x \in E$ and $A \in \mathcal{A}$, whence Theorem 1 follows. □

In what follows we assume that $U_\varepsilon * g_\varepsilon(x,t) = 0$ for $t < 0$. For an arbitrary integrable on $(-\infty,\infty)$ function $\omega(s)$, put

$$\omega * U_\varepsilon * g_\varepsilon(x,t) = \int_{-\infty}^{\infty} \omega(s) U_\varepsilon * g_\varepsilon(t-s)\, ds.$$

Theorem 2. *Assume conditions* (1)–(2), (6)–(7), (9), (14)–(16), *and additionally that the kernel $G(x, dy\times dt)$ is nonlattice. Let $g_\varepsilon(x,t)\ge 0$ and*

$$\sup_{\varepsilon>0}\sup_{x\in E}\int_0^\infty g_\varepsilon(x,t)\,dt < \infty, \tag{24}$$

$$\sup_{\varepsilon>0}\sup_{x\in E} U_\varepsilon * g_\varepsilon(x,t) < \infty. \tag{25}$$

If there exists a $\mathcal{B}_+$-measurable function $g(t)$ such that

$$\int_0^\infty \left| g(t) - \int_E l(dx) g_\varepsilon(x,t)\right| dt \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{26}$$

then

$$\lim_{\substack{\gamma_\varepsilon t\to c\\ \varepsilon\to 0\\ t\to\infty}} \omega * U_\varepsilon * g_\varepsilon(x,t) = e^{-c}\frac{h(x)}{m}\int_0^\infty g(s)\,ds \int_{-\infty}^{\infty}\omega(s)\,ds$$

uniformly in $x\in E$ for all absolutely integrable on $(-\infty,\infty)$ functions ω.

Proof. By $V_\varepsilon(x, dy\times dt)$ we denote the potential of the kernel $\lambda_\varepsilon^{-1}G_\varepsilon(x, dy\times dt)$. Applying the identity $\lambda_\varepsilon U_\varepsilon = V_\varepsilon + (\lambda_\varepsilon - 1)U_\varepsilon * V_\varepsilon$, we see that the theorem follows from

$$\omega * V_\varepsilon * g_\varepsilon(x,t) \underset{\substack{\varepsilon\to 0\\ t\to\infty}}{\longrightarrow} \frac{h(x)}{m}\int_0^\infty g(s)\,ds\int_{-\infty}^{\infty}\omega(s)\,ds.$$

By condition (25), the convolution $\omega * V_\varepsilon * g_\varepsilon(x,t)$ can be understood in the sense of the Abel convergence of the corresponding series, that is,

$$\omega * V_\varepsilon * g_\varepsilon(x,t) = \lim_{\delta\to 0}\sum_{k=0}^{\infty}\left(\frac{1-\delta}{\lambda_\varepsilon}\right)^k \omega * C_\varepsilon^{k*} * g_\varepsilon(x,t).$$

Consider the sequence of operators

$$\Psi_\varepsilon(\alpha) = 1 - \frac{1}{\lambda_\varepsilon}\widehat{G}_\varepsilon(\alpha) - \frac{1}{i\alpha\lambda_\varepsilon}\Pi_\varepsilon(\alpha)\big[G_\varepsilon - \widehat{G}_\varepsilon(\alpha)\big].$$

One can prove that the inverse operator of $\Psi_\varepsilon(\alpha)$ exists for all real numbers α if ε is sufficiently small. Moreover, without loss of generality, one can assume that

$$\sup_{\varepsilon>0}\sup_{|\alpha|\le a}\left\|\Psi_\varepsilon(\alpha)^{-1}\right\| < \infty$$

for all $a > 0$. Now similarly to the proof of Lemma 3 of Chapter 1 of the book [1] we obtain that

$$\begin{aligned} &\omega * V_\varepsilon * g_\varepsilon(x,t) - \omega * V_\varepsilon * g_\varepsilon(x,-t)\\ &\quad = \frac{1}{\pi i}\int_{-a}^{a}\sin\alpha t\,\widehat{\omega}(\alpha)\left[\Psi_\varepsilon(\alpha)^{-1}\widehat{g}_\varepsilon(x,\alpha) - \frac{1}{i\alpha}\Psi_\varepsilon(\alpha)^{-1}\Pi_\varepsilon\widehat{g}_\varepsilon(x,\alpha)\right]d\alpha \end{aligned} \tag{27}$$

for absolutely integrable on $(-\infty,\infty)$ functions ω such that

$$\widehat{\omega}(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha t}\omega(s)\,ds = 0 \quad \text{for } |\alpha| > a.$$

Here

$$\widehat{g}_\varepsilon(x,\alpha) = \int_0^\infty e^{i\alpha t} g_\varepsilon(x,t)\,dt$$

and $\Psi_\varepsilon(\alpha)^{-1}\widehat{g}_\varepsilon(x,\alpha)$ is understood as the action of the operator $\Psi_\varepsilon(\alpha)^{-1}$ for the function $\widehat{g}_\varepsilon(\cdot,\alpha)$.

Similarly to the proof of Lemma 3 of Chapter 4 in the book [1], one can check that there exists a number $r > 0$ such that

$$\Psi_\varepsilon(\alpha)^{-1}\widehat{g}_\varepsilon(x,\alpha) = \int_{-\infty}^{\infty} e^{i\alpha t} f_\varepsilon(x,t)\,dt$$

for $|\alpha| \le r$, where the functions $f_\varepsilon(x,t)$ are measurable with respect to the arguments $x \in E$ and $t \in \mathbf{R}$ and

$$\sup_{\varepsilon>0}\sup_{x\in E}\int_{-\infty}^{\infty} |f_\varepsilon(x,t)|\,dt < \infty.$$

Moreover relation (26) and equality (27) imply that there exists an $\mathcal{A}\times\mathcal{B}$-measurable function $f(x,t)$ such that

$$\sup_{x\in E}\int_{-\infty}^{+\infty} |f_\varepsilon(x,t) - f(x,t)|\,dt \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The rest of the proof is the same as that of the proof of Lemma 3 in [1] mentioned above. □

We deduce from Theorem 2 the following analogue of the Blackwell theorem.

Theorem 3. *Let the assumptions of Theorem 1 hold. If $G(x, dy \times dt)$ is a nonlattice kernel and there exists $\delta > 0$ such that*

$$\inf_{\varepsilon>0}\inf_{x\in E} G_\varepsilon(x, E\times[\delta,\infty)) \ge \delta, \tag{28}$$

then

$$\lim_{\substack{\gamma_\varepsilon t\to c\\ \varepsilon\to 0\\ t\to\infty}} U_\varepsilon(x, A\times[t,t+s]) = e^{-c}s\frac{h(x)}{m}l(A)$$

uniformly in $x \in E$ and $A \in \mathcal{A}$ for all $s \ge 0$.

The following result is analogous to the Smith–Feller key renewal theorem.

Theorem 4. *Suppose the assumptions of Theorem 2 hold and, in addition, inequality (28) is satisfied,*

$$\int_E l_\varepsilon(dx) U_\varepsilon(x, A\times[t,t+s]) \le \frac{s+\delta}{\delta^2} l_\varepsilon(A), \qquad A\in\mathcal{A}. \tag{29}$$

Then if the series $\sum_{k=0}^\infty \sup_{k\le t\le k+1} g_\varepsilon(x,t)$ converges uniformly in $\varepsilon > 0$ and $x \in E$ and

$$\sup_{\varepsilon>0}\delta\int_E l(dx)\sum_{k=0}^\infty\left\{\sup_{k\delta\le t\le k\delta+\delta} g_\varepsilon(x,t) - \inf_{k\delta\le t\le k\delta+\delta} g_\varepsilon(x,t)\right\} \underset{\delta\to 0}{\longrightarrow} 0,$$

it follows that

$$\lim_{\substack{\gamma_\varepsilon t\to c\\ \varepsilon\to 0\\ t\to\infty}} U_\varepsilon * g_\varepsilon(x,t) = e^{-c}\frac{h(x)}{m}\int_0^\infty g(s)\,ds$$

uniformly in $x \in E$.

3. Concluding remarks

The above renewal theorems have many applications. In particular, applications to the study of final probabilities (in other words, of the limits of transient probabilities) are collected in [1] for the processes with Markov random switchings.

We plan to use renewal theorems in forthcoming papers for studying the asymptotically degenerate sequences of Markov functionals with a countable set of values.

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Received 6/SEP/2005

Translated by O. I. KLESOV