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Abstract. We consider stochastic Cox processes governed by a random intensity. Namely we consider the case where the logarithm of intensity is a separable stationary Gaussian stochastic process. We construct models approximating log Gaussian Cox processes with a given reliability and accuracy.

1. Introduction

The paper is devoted to the construction of models for the so-called doubly stochastic Poisson processes or, in other words, Cox processes governed by a random intensity. In particular, we consider the case where the intensity is a log Gaussian stochastic process. Log Gaussian Cox processes and their models are studied in the papers [1, 2] for the case where the intensity is a random field. In contrast to the papers [1, 2], we propose an approach to model stochastic processes with a given reliability and accuracy.

Below we consider necessary definitions and the setting of the modelling problem. Section 2 is devoted to the problem of choosing a partition used in the model. The model itself is constructed in Section 3. Sufficient conditions for the approximation with a given reliability and accuracy are also given in Section 3.

Let \( \{ \Omega, \mathcal{F}, P \} \) be a standard probability space, \( \mathcal{B} \) the \( \sigma \)-algebra of Borel sets in \( \mathbb{R} \), \( \{ Y(t), t \in T \} \) a stationary mean square continuous Gaussian stochastic process such that \( \mathbb{E} Y(t) = 0 \) and \( \mathbb{E} Y(t) Y(s) = B(t-s) \).

Definition 1.1. We say that \( \{ \nu(B), B \in \mathcal{B} \} \) is a log Gaussian Cox process or a Cox process governed by a log Gaussian process \( \exp \{ Y(t) \} \) if

1) the random variables \( \nu(B_1) \) and \( \nu(B_2) \) are independent for all \( B_1, B_2 \in \mathcal{B} \) such that \( B_1 \cap B_2 = \emptyset \);
2) \( \mathbb{P} \{ \nu(B) = k / Y(t), t \in T \} = \exp \{ -\mu(B) \} (\mu(B))^k / k! \), \( k = 0, 1, 2, \ldots \), where
\[
\mu(B) = \int_B \exp \{ Y(t, \cdot) \} \, dt,
\]
and \( Y(t, \cdot), t \in T \), is a realization of the process \( \{ Y(t), t \in T \} \).

A model for the log Gaussian Cox process is constructed in the following way: consider a partition of the interval \( T = [0, T] \) substituted by \( k \) subintervals of length \( d = T/k \). Let \( 0 = t_0 < t_1 < \cdots < t_k = T \) and \( t_{i+1} - t_i = d \) for all \( i = 0, \ldots, k - 1 \). Put \( B_i = [t_i, t_{i+1}] \).

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and denote by $\tilde{Y}(t)$ a model of the process $Y(t)$. Let $\tilde{\mu}(B_i) = \int_{B_i} \exp\{\tilde{Y}(t)\} \, dt$. For every $i = 0, \ldots, k - 1$, we construct a model of the log Gaussian Cox process $\tilde{\nu}(B_i)$, that is, a model for a Poisson random variable with the expectation $\tilde{\mu}(B_i)$.

Since $\tilde{\nu}(B_i)$ means the number of points belonging to the set $B_i$, we choose these points in the interval $B_i$ in an arbitrary way. However if $\tilde{\nu}(B_i) = 1$, then we choose the middle point of the interval.

We treat a model as acceptable if the conditional probabilities

$$p_{kY}(B_i) = \mathbb{P}\{\nu(B_i) = k / Y(t), t \in T\}$$

and

$$\tilde{p}_{kY}(B_i) = \mathbb{P}\{\tilde{\nu}(B_i) = k / \tilde{Y}(t), t \in T\}$$

are close to each other and the probability of the event that the number of points $\nu(B_i)$ (as well as $\tilde{\nu}(B_i)$) exceeds 1 is small.

Thus the problem of modelling for a log Gaussian Cox process consists of two steps. Namely, a partition of the interval $T$ is chosen at the first step, while a model for the process $Y(t)$ is constructed at the second step.

2. The partition of the domain $T$

A partition of the domain $T$ (in other words, a number $d$ or, equivalently, a number $k$) should be such that

$$\mathbb{P}\{\nu(B_i) > 1\} < \delta,$$

where $\delta$ is a given number (say, $\delta = 0.01$).

**Theorem 2.1.** The inequality $\mathbb{P}\{\nu(B_i) > 1\} < \delta$ holds if the number $d = T/k$ is chosen such that

$$d \leq [2\delta \exp\{\lambda \sigma^2\}]^{1/2}.$$

**Proof.** Since

$$\mathbb{P}\{\nu(B_i) > 1\} = \mathbb{E}(1 - \exp\{-\nu(B_i)\} - \mu(B_i) \exp\{-\mu(B_i)\}),$$

a partition should be such that

$$\mathbb{E}(1 - \exp\{-\nu(B_i)\} - \mu(B_i) \exp\{-\mu(B_i)\}) < \delta.$$ 

Using the inequality $1 - \exp\{-x\}(1 + x) \leq x^2/2$ for all $x > 0$, the latter bound follows from

$$\mathbb{E}\nu(B_i)^2 < 2\delta.$$

We know for Gaussian random variables $\xi \sim N(0, \sigma^2)$ that

$$\mathbb{E}\exp\{\lambda \xi\} = \exp\left\{\frac{\lambda^2 \sigma^2}{2}\right\},$$
whence
\[
\mathbb{E} \left[ \mu(B_i) \right]^2 = \mathbb{E} \left[ \int_{B_i} \exp(Y(t)) \, dt \right]^2 = \mathbb{E} \int_{B_i} \exp(Y(t)) \, dt \int_{B_i} \exp(Y(s)) \, ds \\
= \iint_{B_i \times B_i} \mathbb{E} \exp(Y(t) + Y(s)) \, dt \, ds \\
= \iint_{B_i \times B_i} \exp \left( \frac{\mathbb{E}(Y(t) + Y(s))^2}{2} \right) \, dt \, ds \\
= \iint_{B_i \times B_i} \exp \left( \frac{\mathbb{E}(Y(t))^2}{2} + \mathbb{E}Y(t)Y(s) + \frac{\mathbb{E}(Y(s))^2}{2} \right) \, dt \, ds \\
= \iint_{B_i \times B_i} \exp \{B(0) + B(t - s)\} \, dt \, ds \\
= \exp \{B(0)\} \iint_{B_i \times B_i} \exp \{B(t - s)\} \, dt \, ds \\
\leq d^2 \exp(2B(0)).
\]

Therefore inequality (3) follows from assumption (2). Thus inequality (1) also follows from (2). \(\square\)

3. Construction of a model for the process \(Y(t)\)

Recall that our aim is to construct a model for the process \(Y(t)\) such that the conditional probabilities
\[
p_b(Y, B_i) = \mathbb{P}\{Y(B_i) = k / Y(t), t \in T\} \quad \text{and} \quad \bar{p}_b(Y, B_i) = \mathbb{P}\{\nu(B_i) = k / \bar{Y}(t), t \in T\},
\]
i = 0, \ldots, k - 1,

are close to each other with probability approaching one. The following definition is natural for our setting.

**Definition 3.1.** We say that a model \(\nu(B_i)\) of a log Gaussian Cox stochastic process \(\{\nu(B), B \in \mathcal{B}\}\) approximates it with an accuracy \(\alpha, 0 < \alpha < 1\), and reliability \(1 - \gamma, 0 < \gamma < 1\), if
\[
\max_{B_i \in \mathcal{B}, i = 0, \ldots, k - 1} \mathbb{P}\{\left| p_{bY}(B_i) - \bar{p}_{bY}(B_i) \right| > \alpha\} < \gamma.
\]

Now we estimate the difference \(\left| p_{bY}(B_i) - \bar{p}_{bY}(B_i) \right|\) by using the mean value theorem for derivatives:
\[
\left| p_{bY}(B_i) - \bar{p}_{bY}(B_i) \right| = \left| \frac{\exp \{-\mu(B_i)\} (\mu(B_i))^k}{k!} - \frac{\exp \{-\bar{\mu}(B_i)\} (\bar{\mu}(B_i))^k}{k!} \right| \\
= \left| \mu(B_i) - \bar{\mu}(B_i) \right| \frac{1}{k!} \exp \{\left| \frac{1}{k!} \exp \{-\mu(B_i)\} (\mu(B_i))^k \right| \}
\]
\[
= \begin{cases} \\
\left| \mu(B_i) - \bar{\mu}(B_i) \right| \frac{1}{k!} \exp \{-\mu(B_i)\} (\mu(B_i))^k, & k \geq \hat{\mu}(B_i), \\
\left| \mu(B_i) - \bar{\mu}(B_i) \right| \frac{1}{k!} \exp \{-\bar{\mu}(B_i)\} (\bar{\mu}(B_i))^k, & k < \hat{\mu}(B_i). \\
\end{cases}
\]

If \(k = 0\), then
\[
\left| p_{bY}(B_i) - \bar{p}_{bY}(B_i) \right| = \left| \exp \{-\mu(B_i)\} - \exp \{-\bar{\mu}(B_i)\} \right| \\
\leq \left| \mu(B_i) - \bar{\mu}(B_i) \right| \left| \exp \{-\mu(B_i)\} \right| \leq \left| \mu(B_i) - \bar{\mu}(B_i) \right|.
\]
Therefore the difference of probabilities $|p_{kY}(B_i) - \tilde{p}_{kY}(B_i)|$ is estimated in terms of $|\mu(B_i) - \tilde{\mu}(B_i)|$, whence we obtain
\begin{equation}
P\{|p_{kY}(B_i) - \tilde{p}_{kY}(B_i)| > \alpha\} \leq P\{|\mu(B_i) - \tilde{\mu}(B_i)| > \alpha\}, \quad i = 0, \ldots, k - 1.
\end{equation}

It is easy to check that
\begin{equation}
\max_{B_i \in \mathcal{B}, i = 0, \ldots, k - 1} P\{|\mu(B_i) - \tilde{\mu}(B_i)| > \alpha\}
= \max_{B_i \in \mathcal{B}, i = 0, \ldots, k - 1} P\left\{\left|\int_{B_i} \exp \{Y(t)\} dt - \int_{B_i} \exp \{\tilde{Y}(t)\} dt\right| > \alpha\right\}
\leq \max_{B_i \in \mathcal{B}, i = 0, \ldots, k - 1} P\left\{\sup_{t \in B_i} |\exp\{Y(t)\} - \exp\{\tilde{Y}(t)\}| dt > \alpha\right\}
= \max_{B_i \in \mathcal{B}, i = 0, \ldots, k - 1} P\left\{\int_{B_i} dt \cdot \sup_{t \in B_i} |\exp\{Y(t)\} - \exp\{\tilde{Y}(t)\}| > \alpha\right\}
= P\left\{\sup_{t \in B_i} |\exp\{Y(t)\} - \exp\{\tilde{Y}(t)\}| > \frac{\alpha}{d}\right\}.
\end{equation}

Thus the problem of estimating $|\mu(B_i) - \tilde{\mu}(B_i)|$ is reduced to that for
\begin{equation*}
\sup_{t \in B_i} |\exp\{Y(t)\} - \exp\{\tilde{Y}(t)\}|.
\end{equation*}

In what follows we use the following result.

**Theorem 3.1.** Let $X = \{X(t), t \in \mathcal{T}\}$ be a stochastic process in the space $L_p(\Omega)$ such that $\sup_{t \in \mathcal{T}} \|X(t)\|_{L_p} < +\infty$. Assume that the space $(\mathcal{T}, \rho_X)$ as well as the pseudometric $\rho_X(t, s)$ generated by the process $X(t)$ and the process itself are separable. Let
\begin{equation*}
\int_0^\infty N^{1/p}(\varepsilon) d\varepsilon < +\infty,
\end{equation*}

where $N(\varepsilon)$ is the metric capacity of the set $\mathcal{T}$ and $\varepsilon_0 = \sup_{t, s \in \mathcal{T}} \rho_X(t, s)$. Then
\begin{equation*}
\left(E \left[\sup_{t \in \mathcal{T}} |X(t)|^p\right]\right)^{1/p} \leq V_p.
\end{equation*}

Moreover
\begin{equation*}
P\left\{\sup_{t \in \mathcal{T}} |X(t)| \geq x\right\} \leq \frac{V_p}{x^p}
\end{equation*}

for all $x > 0$, where
\begin{equation*}
V_p = \inf_{t \in \mathcal{T}} (E |X(t)|^p)^{1/p} + \inf_{0 < \theta < 1} \frac{1}{\theta (1 - \theta)} \int_0^{\theta \varepsilon_0} N^{1/p}(\varepsilon) d\varepsilon.
\end{equation*}

Let $\mathcal{T} = [0, T]$ and $\sup_{t \in \mathcal{T}} \|X(t + \varepsilon) - X(t)\|_{L_p} \leq \varphi(\varepsilon)$, where $\varphi(\varepsilon)$, $\varepsilon > 0$, is a nondecreasing function such that $\varphi(0) = 0$. Then
\begin{equation*}
N(\varepsilon) \leq \frac{T}{2\varphi(-1)(\varepsilon)} + 1 \leq \frac{T}{\varphi(-1)(\varepsilon)}.
\end{equation*}

The proof of the latter theorem and the above inequality can be found in [3].

We consider a centered mean square continuous and wide sense stationary stochastic process. The covariance function $B(\tau)$ of such a process can be represented as follows:
\begin{equation*}
B(\tau) = \int_0^\infty \cos \lambda \tau \ dF(\lambda),
\end{equation*}
where \( F(\lambda), \lambda \in [0, +\infty], \) is a nondecreasing left continuous function such that \( F(0) = 0 \) and \( F(+\infty) = B(0) \). By the Karhunen theorem the centered stationary process \( \tilde{Y}(t) \) can be represented in the following form:

\[
\tilde{Y}(t) = \int_0^\infty \cos \lambda t \, d\xi(\lambda) + \int_0^\infty \sin \lambda t \, d\eta(\lambda),
\]

where \( \xi(\lambda) \) and \( \eta(\lambda) \) are centered stochastic processes with uncorrelated increments and such that

\[
E(\xi(\lambda_2) - \xi(\lambda_1))^2 = E(\eta(\lambda_2) - \eta(\lambda_1))^2 = F(\lambda_2) - F(\lambda_1)
\]

for all \( \lambda_1 < \lambda_2 \). Here \( F(\lambda) \) is the spectral function defined in the representation of the covariance function.

We say that a finite sum

\[
(6) \quad \tilde{Y}(t) = \sum_{k=0}^{N-1} \left( \cos \lambda_k t \Delta_k \xi(\lambda) + \sin \lambda_k t \Delta_k \eta(\lambda) \right)
\]

is a model of the process \( \tilde{Y}(t) \), where

\[
\Delta_k \xi(\lambda) = \int_{\lambda_k}^{\lambda_{k+1}} d\xi(\lambda), \quad \Delta_k \eta(\lambda) = \int_{\lambda_k}^{\lambda_{k+1}} d\eta(\lambda),
\]

and \( \lambda_k \) are the points of the partition \( D_\Lambda \), that is, \( 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N = \Lambda \).

Remark 3.1. Since the process \( Y(t) \) is Gaussian, the processes \( \xi(\lambda) \) and \( \eta(\lambda) \) also are Gaussian (this follows from the Karhunen theorem).

**Theorem 3.2.** Let \( Y(t) \) be a separable centered mean square continuous stationary Gaussian stochastic process with the spectral function \( F(\lambda) \). Let the assumptions of Theorem 3.1 hold for the process \( Y(t) \) and assume that the spectral moment

\[
\int_0^\infty \lambda^b \, dF(\lambda)
\]

exists for some \( 0 < b \leq 1 \). Consider a partition \( D_\Lambda: \)

\[
\lambda_{k-1} - \lambda_k = \Lambda/N, \quad \Lambda \in \mathbb{R}, \quad N \in \mathbb{N}.
\]

Then a model \( \{\tilde{\nu}(B), B \in \mathcal{B} \} \) with \( \tilde{Y}(t) \) defined by (6) approximates the log Gaussian Cox process with accuracy \( \alpha \) and reliability \( 1 - \gamma \) if

\[
(7) \quad \left( \frac{dG}{\alpha} \right)^{\ln \frac{\ln \frac{\ln \frac{\lambda}{M \alpha N d}}{2Z}}{Z}} \exp \left\{ \frac{\nu_2 B(0) \ln^2 \frac{\alpha}{M \alpha N d}}{2Z^2} \right\}
\]

\[
+ D \left( \ln \frac{\ln \frac{\ln \frac{\lambda}{M \alpha N d}}{2Z}}{Z} \right)^{\ln \frac{\ln \frac{\ln \frac{\lambda}{M \alpha N d}}{2Z}}{2Z}} \exp \left\{ -\frac{\ln^2 \frac{\ln \frac{\ln \frac{\lambda}{M \alpha N d}}{2Z}}{2Z} \right\} \left( 1 - \frac{Z}{b \ln \frac{\ln \frac{\ln \frac{\lambda}{M \alpha N d}}{2Z}}{2Z} \right)^{\ln \frac{\ln \frac{\ln \frac{\lambda}{M \alpha N d}}{2Z}}{2Z}} < \gamma,
\]
where
\[ G = 2(B(0) - F(\Lambda))^{1/2}(v_1 + 1)^{1/2}, \quad Z = r_2(f_1A_N + f_2B(0)), \]
\[ M_N = 2d^b(\max(r_1, s_1, s_2) + 1)K_{N,b}, \quad D = b^1/b \exp \left\{ \frac{1}{b} + \frac{1}{b^2} \right\}, \]
\[ K_{N,b} = k^{1/2}_{N,b} + \left( 2^{3-2b}A^{2b}F(\Lambda)A_N \right)^{1/2}, \]
\[ A_N = B(0) - F(\Lambda) + 2^{2-2b}d_{2b}^{2b} \left( \frac{\Lambda}{N} \right)^{2b} F(\Lambda), \]
\[ k_{N,b} = 2^{3-4b} \left( \left( \frac{\Lambda}{N} \right)^b + \left( \frac{\Lambda}{N} \right)^a \left( \frac{d}{2} \right)^a (2\Lambda)^b \right)^2 F(\Lambda) + 2^{3-2b}C_b, \]
\[ C_b = \int_{\Lambda}^{\infty} \lambda^{2b} dF(\lambda), \quad 0 < a \leq 1, \quad 0 < b \leq 1. \]

Here the numbers \( v_1, v_2, f_1, f_2, s_1, s_2, s_3, r_1, \) and \( r_2 \) are such that
\[ \frac{1}{v_1} + \frac{1}{v_2} = 1, \quad \frac{1}{f_1} + \frac{1}{f_2} = 1, \quad \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1, \quad \frac{1}{r_1} + \frac{1}{r_2} = 1, \quad r_2 = s_3 \]
and \( B(t) \) is the covariance function of the process \( Y(t) \).

**Proof.** To estimate the moment of the difference \( \{E[\exp\{Y(t)\} - \exp\{\tilde{Y}(t)\}]\}^{1/p} \), we use the inequality \( |\exp\{x\} - \exp\{y\}| \leq |x-y|\exp\{\max(x, y)\} \) and then the Hölder inequality. As a result we obtain
\[
\left( E\left| \exp\{Y(t)\} - \exp\{\tilde{Y}(t)\} \right|^{p} \right)^{1/p} \\
\leq \left( E\left| Y(t) - \tilde{Y}(t) \right|^{p} \exp\left\{ p \max \left( Y(t), \tilde{Y}(t) \right) \right\} \right)^{1/p} \\
\leq \left( E\left| Y(t) - \tilde{Y}(t) \right|^{pv_1} \left( \exp\left\{ pv_2 \max \left( Y(t), \tilde{Y}(t) \right) \right\} \right)^{1/(pv_2)} \right)^{1/(pv_1)},
\]
where \( v_1 \) and \( v_2 \) are real numbers such that \( v_1^{-1} + v_2^{-1} = 1 \).

If a random variable \( \xi \) is Gaussian with parameters 0 and \( \sigma^2 \), then its moments are such that
\[
E|\xi|^p = c_p (\sigma^2)^{p/2}
\]
for \( c_p = 2^{p/2} \pi^{-1/2} \Gamma((p + 1)/2) \). Using Stirling’s formula we get
\[
c_p \leq \frac{2^{p/2}}{\sqrt{\pi}} \sqrt{\frac{p + 1}{2 \pi}} \left( \frac{p + 1}{2e} \right)^{(p+1)/2} \exp \left\{ \frac{1}{12(2p+1)} \right\} \\
\leq 2^{(p+1)/2} \left( \frac{p + 1}{2} \right)^{p/2} \frac{\exp\{1/12\}}{\exp\{p/2\}} \leq \sqrt{2} (p + 1)^{p/2}.
\]
Considering equality (9) we obtain
\[
E\left| Y(t) - \tilde{Y}(t) \right|^{pv_1} = \left( E\left| Y(t) - \tilde{Y}(t) \right|^{2} \right)^{pv_1/2} c_{pv_1}.
\]
Since \( E(\xi(t))^2 = B(0) \) and \( E(\tilde{\xi}(t))^2 = F(\Lambda) \) for centered stationary Gaussian stochastic processes, we have
\[
E\left| Y(t) - \tilde{Y}(t) \right|^{2} = B(0) + F(\Lambda) - 2E Y(t) \tilde{Y}(t).
\]
Furthermore,

\[ E \left( Y(t) \tilde{Y}(t) \right) = \mathbb{E} \left( \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \cos \lambda t \, d\xi(\lambda) + \int_{\lambda_k}^{\lambda_{k+1}} \sin \lambda t \, d\eta(\lambda) \right. \]

\[ \left. + \int_{\lambda_k}^{\infty} \cos \lambda t \, d\xi(\lambda) + \int_{\lambda_k}^{\infty} \sin \lambda t \, d\eta(\lambda) \right) \]

\[ \times \left( \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \cos \lambda_k t \, d\xi(\lambda) + \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \sin \lambda_k t \, d\eta(\lambda) \right) \]

\[ = \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \cos \lambda t \cos \lambda_k t \, dF(\lambda) + \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \sin \lambda t \sin \lambda_k t \, dF(\lambda) \]

\[ = \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \cos t(\lambda - \lambda_k) \, dF(\lambda), \]

\[ E \left| Y(t) - \tilde{Y}(t) \right|^2 = B(0) - F(\Lambda) + 2F(\Lambda) - 2 \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \cos t(\lambda - \lambda_k) \, dF(\lambda) \]

\[ = B(0) - F(\Lambda) + 2 \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 2 \sin^2 \left( \frac{t(\lambda - \lambda_k)}{2} \right) \, dF(\lambda) \]

\[ \leq B(0) - F(\Lambda) + 4 \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} e^{2b(\lambda - \lambda_k)^2b} \frac{2b}{2^{2b}} \, dF(\lambda). \]

Since \( \lambda - \lambda_k \leq \lambda_{k+1} - \lambda_k = \Lambda/N \) for \( k = 0, \ldots, N-1 \), we get

\[ \mathbb{E} \left| Y(t) - \tilde{Y}(t) \right|^2 \leq A_N, \]

(11)

\[ A_N = B(0) - F(\Lambda) + 2^{2-2b}d^{2b} \left( \frac{\Lambda}{N} \right)^{2b} F(\Lambda). \]

Thus

(12) \[ \left( \mathbb{E} \left| Y(t) - \tilde{Y}(t) \right|^2 \right)^{1/(2\nu_1)} \leq A_N^{1/2} \left( 2^{\nu_1} \right). \]

Now we estimate the expectation of the exponent \( \exp \{ p\nu_2 \max(Y(t), \tilde{Y}(t)) \} \). Note that

\[ \mathbb{E} \exp \{ \lambda \xi \} = \exp \{ \lambda^2 \sigma^2/2 \} \]

for a random variable \( \xi \sim N(0, \sigma^2) \). Hence

\[ \mathbb{E} \exp \left\{ p\nu_2 \max \left( Y(t), \tilde{Y}(t) \right) \right\} \leq \mathbb{E} \exp \{ p\nu_2 Y(t) \} + \mathbb{E} \exp \left\{ p\nu_2 \tilde{Y}(t) \right\} \]

\[ = \exp \left\{ \frac{(p\nu_2)^2}{2} B(0) \right\} + \exp \left\{ \frac{(p\nu_2)^2}{2} F(\Lambda) \right\} \]

\[ \leq 2 \exp \left\{ \frac{(p\nu_2)^2}{2} B(0) \right\}. \]
The latter inequality and (12) imply that
\[
\left( E \left| \exp\{Y(t)\} - \exp\{\bar{Y}(t)\} \right|^p \right)^{1/p} \\
\leq A_{N_1}^{1/2} c_{p_1}^{1/(p_1)} 2^{1/(p_2)} \exp \left\{ \frac{p_{p_2}}{2} B(0) \right\} \\
= A_{N_1}^{1/2} \left( \sqrt{2} (p_{p_1} + 1)^{p_{p_1}/2} \right)^{1/(p_1)} 2^{1/(p_2)} \exp \left\{ \frac{p_{p_2}}{2} B(0) \right\} \\
\leq 2^{1/p} A_{N_1}^{1/2} (p_{p_1} + 1)^{1/2} \exp \left\{ \frac{p_{p_2}}{2} B(0) \right\}
\]
by bound (8).

Then
\[
\left( E \left| \exp\{Y(t+h)\} - \exp\{\bar{Y}(t+h)\} - \left( \exp\{Y(t)\} - \exp\{\bar{Y}(t)\} \right) \right|^p \right)^{1/p} \\
= \left( E \left| \left( e^{Y(t+h)} - \bar{Y}(t+h) \right) - e^{\bar{Y}(t)} \right|^p \right)^{1/p} \\
\leq \left( E \left| \left( e^{Y(t+h)} - \bar{Y}(t+h) \right) - e^{\bar{Y}(t)} \right|^p \right)^{1/p} \\
+ \left( E \left| \bar{Y}(t+h) - e^{\bar{Y}(t)} \right|^p \right)^{1/p} \\
\leq (E|\Delta_1(Y)V_1|^p)^{1/p} + (E|\Delta_2(Y)\Delta_3(Y)V_2|^p)^{1/p},
\]
where
\[
\Delta_1(Y) = \left| Y(t+h) - \bar{Y}(t+h) - Y(t) + \bar{Y}(t) \right|, \\
V_1 = \exp \left\{ \max \left( Y(t+h) - \bar{Y}(t+h), Y(t) - \bar{Y}(t) \right) \right\} \exp \left\{ \bar{Y}(t+h) \right\}, \\
\Delta_2(Y) = \left| \bar{Y}(t+h) - \bar{Y}(t) \right|, \\
\Delta_3(Y) = \left| Y(t) - \bar{Y}(t) \right|, \\
V_2 = \exp \left\{ \max \left( \bar{Y}(t+h), \bar{Y}(t) \right) \right\} \exp \left\{ \left| Y(t) - \bar{Y}(t) \right| \right\}.
\]

Let
\[
r_1^{-1} + r_2^{-1} = 1 \quad \text{and} \quad s_1^{-1} + s_2^{-1} + s_3^{-1} = 1.
\]
An application of the Hölder inequality yields
\[
(E|\Delta_1(Y)V_1|^p)^{1/p} \leq (E|\Delta_1(Y)|^{pr_1})^{1/r_1} (E|V_1|^{pr_2})^{1/r_2}, \\
(E|\Delta_2(Y)\Delta_3(Y)V_2|^p)^{1/p} \leq (E|\Delta_2(Y)|^{ps_1})^{1/s_1} (E|\Delta_3(Y)|^{ps_2})^{1/s_2} (E|V_2|^{ps_3})^{1/s_3}.
\]
Note that
\[
(E|\Delta_1(Y)|^{pr_1}) = (\sigma^2)^{pr_1/2} c_{pr_1},
\]
by (9), where \( \sigma^2 = E|\Delta_1(Y)|^2 \).
Now we estimate $\sigma^2$:

\[
\sigma^2 = E \left[ Y(t+h) - \bar{Y}(t+h) - Y(t) + \bar{Y}(t) \right]^2
\]

\[
= E \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} (\cos \lambda (t+h) - \cos \lambda_k (t+h) - \cos \lambda t + \cos \lambda_k t) \, d\xi(\lambda)
\]

\[
+ \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} (\sin \lambda (t+h) - \sin \lambda_k (t+h) - \sin \lambda t + \sin \lambda_k t) \, d\eta(\lambda)
\]

\[
+ \int_{\Lambda}^{\infty} (\cos \lambda (t+h) - \cos \lambda t) \, d\xi(\lambda) + \int_{\Lambda}^{\infty} (\sin \lambda (t+h) - \sin \lambda t) \, d\eta(\lambda)
\]

\[
= \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} (\cos \lambda (t+h) - \cos \lambda_k (t+h) - \cos \lambda t + \cos \lambda_k t)^2 \, dF(\lambda)
\]

\[
+ \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} (\sin \lambda (t+h) - \sin \lambda_k (t+h) - \sin \lambda t + \sin \lambda_k t)^2 \, dF(\lambda)
\]

\[
+ \int_{\Lambda}^{\infty} (\cos \lambda (t+h) - \cos \lambda t)^2 \, dF(\lambda) + \int_{\Lambda}^{\infty} (\sin \lambda (t+h) - \sin \lambda t)^2 \, dF(\lambda)
\]

\[
= \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \left[ \sin \frac{(t+h)(\lambda + \lambda_k)}{2} \sin \frac{(t+h)(\lambda - \lambda_k)}{2} \right] \left[ -\sin \frac{(\lambda + \lambda_k) t}{2} \sin \frac{(\lambda - \lambda_k) t}{2} \right] \, dF(\lambda)
\]

\[
+ \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \left[ \cos \frac{(t+h)(\lambda + \lambda_k)}{2} \sin \frac{(t+h)(\lambda - \lambda_k)}{2} \right] \left[ -\cos \frac{(\lambda + \lambda_k) t}{2} \sin \frac{(\lambda - \lambda_k) t}{2} \right] \, dF(\lambda)
\]

\[
+ \int_{\Lambda}^{\infty} 4 \left[ \sin \frac{\lambda (2t+h)}{2} \right]^2 \left[ \sin \frac{\lambda h}{2} \right]^2 \, dF(\lambda) + \int_{\Lambda}^{\infty} 4 \left[ \cos \frac{\lambda (2t+h)}{2} \right]^2 \left[ \sin \frac{\lambda h}{2} \right]^2 \, dF(\lambda)
\]

\[
\leq \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \left( \frac{\sin (t+h)(\lambda + \lambda_k)}{2} \frac{\sin (t+h)(\lambda - \lambda_k)}{2} \right) \left( \frac{\sin (\lambda - \lambda_k) t}{2} \frac{\sin (\lambda + \lambda_k) t}{2} \right) \, dF(\lambda)
\]

\[
+ \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \left( \frac{\sin (t+h)(\lambda + \lambda_k)}{2} \frac{\cos (t+h)(\lambda + \lambda_k)}{2} \frac{\cos (\lambda + \lambda_k) t}{2} \frac{-\cos (\lambda - \lambda_k) t}{2} \right) \, dF(\lambda)
\]

\[
+ \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \left( \frac{\sin (t+h)(\lambda - \lambda_k)}{2} \frac{\cos (t+h)(\lambda + \lambda_k)}{2} \frac{\cos (\lambda + \lambda_k) t}{2} \frac{-\cos (\lambda - \lambda_k) t}{2} \right) \, dF(\lambda)
\]

\[
+ 8 \int_{\Lambda}^{\infty} \left( \frac{\lambda h}{2} \right)^{2b} \, dF(\lambda)
\]
\[
\sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \left( 2 \cos \frac{(2t+h)(\lambda - \lambda_k)}{4} \sin \frac{\lambda - \lambda_k}{4} \right. \\
\left. + \left| \sin \frac{\lambda - \lambda_k}{2} \right| 2 \cos \frac{(2t+h)(\lambda + \lambda_k)}{4} \sin \frac{\lambda + \lambda_k}{4} \right) \right)^2 dF(\lambda) \\
+ \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \left( \left| \sin \frac{(t+h)(\lambda - \lambda_k)}{2} \right| 2 \sin \frac{(2t+h)(\lambda + \lambda_k)}{4} \sin \frac{\lambda + \lambda_k}{4} \right) \\
\left. + \left| 2 \cos \frac{(2t+h)(\lambda - \lambda_k)}{4} \sin \frac{\lambda - \lambda_k}{4} \right) \right)^2 dF(\lambda) \\
+ h^{2b} 2^{3-2b} \int_{\lambda}^{\infty} \lambda^{2b} dF(\lambda) \\
\leq \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 16 \left( \frac{(\lambda - \lambda_k)^b h^b}{4^b} + \frac{(\lambda - \lambda_k)^a 4^a (\lambda + \lambda_k)^b h^b}{4^b} \right)^2 dF(\lambda) \\
+ \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 16 \left( \frac{(t+h)^a (\lambda - \lambda_k)^a (\lambda + \lambda_k)^b h^b}{2^a} + \frac{(\lambda - \lambda_k)^b h^b}{4^b} \right)^2 dF(\lambda) \\
+ h^{2b} 2^{3-2b} C_b \\
= 32 \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \left( \left( \frac{\Lambda}{N} \right)^b \frac{h^b}{4^b} + \left( \frac{\Lambda}{N} \right)^a \frac{d^a}{2^a} (2\Lambda)^b \frac{h^b}{4^b} \right)^2 dF(\lambda) + h^{2b} 2^{3-2b} C_b \\
= h^{2b} \left( 2^{5-4b} \left( \frac{\Lambda}{N} \right)^b + \left( \frac{\Lambda}{N} \right)^a \frac{d^a}{2^a} (2\Lambda)^b \right)^2 F(\Lambda) + h^{2b} 2^{3-2b} C_b,
\]
where \(C_b = \int_{\lambda}^{\infty} \lambda^{2b} dF(\lambda), 0 < a \leq 1, \) and \(0 < b \leq 1.\) Thus
\[
E |\Delta_1(Y)|^2 \leq k_{N,b} h^{2b}
\]
for
\[
k_{N,b} = \left( 2^{5-4b} \left( \frac{\Lambda}{N} \right)^b + \left( \frac{\Lambda}{N} \right)^a \frac{d^a}{2^a} (2\Lambda)^b \right)^2 F(\Lambda) + 2^{3-2b} C_b.
\]

Therefore
\[
(15) \quad E |\Delta_1(Y)|^{p_{r_1}} \leq h^{p_{r_1} b} h_{N,b}^{p_{r_1}/2} c_{p_{r_1}}.
\]

Now we estimate \(E |V_1|^{p_{r_2}}.\) Let \(f_1^{-1} + f_2^{-1} = 1.\) It follows from the Hölder inequality that
\[
E |V_1|^{p_{r_2}} = E \left| \exp \left\{ \max \left( Y(t + h) - \bar{Y}(t + h), Y(t) - \bar{Y}(t) \right) \right\} \exp \left\{ \bar{Y}(t + h) \right\} \right|^{p_{r_2}} \\
\leq \left( E \exp \left\{ p_{r_2} f_1 \max \left( Y(t + h) - \bar{Y}(t + h), Y(t) - \bar{Y}(t) \right) \right\} \right)^{1/f_1} \\
\times \left( E \exp \left\{ p_{r_2} f_2 \bar{Y}(t + h) \right\} \right)^{1/f_2}.
\]
We have
\[
E \exp \left\{ p_{r_2} f_1 \max \left( Y(t + h) - \bar{Y}(t + h), Y(t) - \bar{Y}(t) \right) \right\} \\
\leq \exp \left\{ \left( p_{r_2} f_1 \right)^2 E \left| Y(t + h) - \bar{Y}(t + h) \right|^2 \right\} + \exp \left\{ \left( p_{r_2} f_1 \right)^2 E \left| Y(t) - \bar{Y}(t) \right|^2 \right\}.
\]
Applying the same argument as that used in the proof of formula (11) we easily see that
\[ E[Y(t+h) - \bar{Y}(t+h)]^2 \leq A_N. \]
Thus
\[ \left( E \exp \left\{ (pr_2 f_1 \max (Y(t+h) - \bar{Y}(t+h), Y(t) - \bar{Y}(t)) \right\} \right)^{1/f_1} \leq 2^{1/f_1} \exp \left\{ \frac{(pr_2 f_1 A_N}{2} \right\}, \]
\[ \left( E \exp \left\{ (pr_2 f_2 \bar{Y}(t+h)) \right\} \right)^{1/f_2} \leq \left( \exp \left\{ \frac{(pr_2 f_2)^2 F(\lambda)}{2} \right\} \right)^{1/f_2} = \exp \left\{ \frac{(pr_2 f_2 F(\lambda)}{2} \right\}. \]
Combining the two preceding inequalities we derive from (16) that
\[ E[V_1]^{pr_2} \leq 2^{1/f_1} \exp \left\{ \frac{(pr_2)^2}{2} (f_1 A_N + f_2 B(0)) \right\}. \]

Now we estimate \( E[\Delta_2(Y)]^{ps_1}. \) By equality (19),
\[ |\Delta_2(Y)|^{ps_1} = \left( E[\Delta_2(Y)]^2 \right)^{ps_1/2} c_{ps_1}. \]
Then
\[ E[\Delta_2(Y)]^2 = E[\bar{Y}(t+h) - \bar{Y}(t)]^2 \]
\[ = E \left( \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \cos \lambda_k (t+h) d\xi(\lambda) + \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \sin \lambda_k (t+h) d\eta(\lambda) \right) \]
\[ - \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \cos \lambda_k t d\xi(\lambda) - \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \sin \lambda_k t d\eta(\lambda) \right)^2 \]
\[ = E \left( \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} -2 \sin \frac{\lambda_k (2t+h)}{2} \sin \frac{\lambda_k h}{2} d\xi(\lambda) \right) \]
\[ + \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 2 \sin \frac{\lambda_k h}{2} \cos \frac{\lambda_k (2t+h)}{2} d\eta(\lambda) \right)^2 \]
\[ = \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \sin^2 \frac{\lambda_k (2t+h)}{2} \sin^2 \frac{\lambda_k h}{2} dF(\lambda) \]
\[ + \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} 4 \sin^2 \frac{\lambda_k h}{2} \cos^2 \frac{\lambda_k (2t+h)}{2} dF(\lambda) \]
\[ \leq 8 \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \sin^2 \frac{\lambda_k h}{2} dF(\lambda) \leq 8 \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \left( \frac{\lambda_k h}{2} \right)^{2b} dF(\lambda) \]
\[ \leq 2^{3-2b} \Lambda^{2b} h^{2b} F(\Lambda), \quad 0 < b \leq 1. \]
Therefore
\[ E[\Delta_2(Y)]^{ps_1} \leq (2^{3-2b} \Lambda^{2b} h^{2b} F(\Lambda))^{ps_1/2} c_{ps_1}. \]

Since
\[ E[\Delta_3(Y)]^{ps_2} = \left( E[\Delta_3(Y)]^2 \right)^{ps_2/2} c_{ps_2}, \]
\[ E[\Delta_3(Y)]^2 = E[Y(t) - \bar{Y}(t)]^2 \leq A_N, \]
we have

\[(19) \quad E |\Delta_3(Y)|^p s_2 = A_N^{p s_2 / 2} c_{p s_2}.\]

To estimate \(E |V_2|^p s_3\), let \(c_1^{-1} + c_2^{-1} = 1\). We use the Hölder inequality again and find that

\[
E |V_2|^p s_3 = E \left[ \exp \left\{ \max \left( \hat{Y}(t + h), \hat{Y}(t) \right) \right\} \exp \left\{ \left| Y(t) - \hat{Y}(t) \right| \right\} \right]^{p s_3}
\]

\[
\leq \left( E \exp \left\{ p s_3 c_1 \max \left( \hat{Y}(t + h), \hat{Y}(t) \right) \right\} \right)^{1 / c_1}
\times \left( E \exp \left\{ p s_3 c_2 \left| Y(t) - \hat{Y}(t) \right| \right\} \right)^{1 / c_2}.
\]

Hence

\[
E \exp \left\{ p s_3 c_1 \max \left( \hat{Y}(t + h), \hat{Y}(t) \right) \right\} \leq 2 \exp \left\{ \frac{(p s_3 c_1)^2}{2} F(\Lambda) \right\}.
\]

It is known for a Gaussian random variable \(\xi\) with parameters 0 and \(\sigma^2\) that

\[
E \exp \{ b |\xi| \} \leq E \exp \{ b \xi \} + E \exp \{ -b \xi \} = 2 \exp \left\{ \frac{b^2 \sigma^2}{2} \right\},
\]

whence

\[
E \exp \left\{ p s_3 c_2 \left| Y(t) - \hat{Y}(t) \right| \right\} \leq 2 \exp \left\{ \frac{(p s_3 c_2)^2}{2} A_N \right\}.
\]

Therefore

\[(20) \quad E |V_2|^p s_3 \leq 2 \exp \left\{ \frac{(p s_3)^2}{2} \left( e_2 A_N + e_1 B(0) \right) \right\}.
\]

Applying \((15), (17)–(20)\), and \((10)\), we obtain

\[
\left( E \left| \exp \left\{ Y(t + h) \right\} - \exp \left\{ \hat{Y}(t + h) \right\} \right| - \left( \exp\{Y(t)\} - \exp\{\hat{Y}(t)\} \right) \right)^{1 / p}
\]

\[
\leq \left( E |\Delta_1(Y)|^{p r_1} \left( 1 / (p r_1) \right) (E |V_1|^{p r_2}) \left( 1 / (p r_2) \right)
\]

\[
+ \left( E |\Delta_2(Y)|^{p s_1} \left( 1 / (p s_1) \right) (E |\Delta_3(Y)|^{p s_2}) \left( 1 / (p s_2) \right) (E |V_2|^p s_3) \left( 1 / (p s_3) \right)
\]

\[
\leq h b h_{p s_2}^{1 / 2} \left( 1 / (p r_1) \right) 2^{1 / (p r_2 f_1)} \exp \left\{ \frac{p r_2}{2} \left( f_1 A_N + f_2 B(0) \right) \right\}
\]

\[
+ \left( 2^{3 - 2b} F(\Lambda) A_N \right)^{1 / 2} \left( 1 / p s_1 \right) A_N \left( 1 / p s_2 \right) A_N \exp \left\{ \frac{p s_3}{2} \left( e_2 A_N + e_1 B(0) \right) \right\}
\]

\[
= h b h_{p s_2}^{1 / 2} \left( 1 / (2 p r_1) \right) \left( 1 / (2 p r_2 f_1) \right) \exp \left\{ \frac{p r_2}{2} \left( f_1 A_N + f_2 B(0) \right) \right\}
\]

\[
+ \left( 2^{3 - 2b} F(\Lambda) A_N \right)^{1 / 2} 2^{1 / (2 p s_1)} \left( 1 / p s_1 \right) 2^{1 / (2 p s_2)} \left( 1 / p s_2 \right) \exp \left\{ \frac{p s_3}{2} \left( e_2 A_N + e_1 B(0) \right) \right\}.
\]

Put \(f_1 = e_2, f_2 = e_1,\) and \(r_2 = s_3\). Taking into account the following bounds,

\[
2^{1 / (2 p r_1)} 2^{1 / (2 p r_2 f_1)} \leq 2^{1 / p}, \quad 2^{1 / (2 p s_1)} 2^{1 / (2 p s_2)} \leq 2^{1 / p},
\]

we have

\[
\frac{(p s_3)^2}{2} (e_2 A_N + e_1 B(0)) \leq 2^{1 / p}.
\]
we get
\[
\left( \mathbb{E} \left| \exp \{ Y(t + h) \} - \exp \{ \widetilde{Y}(t + h) \} \right| - \left( \exp \{ Y(t) \} - \exp \{ \widetilde{Y}(t) \} \right) \right)^{1/p} 
\leq h^b \left[ k_{N,b}^{1/2} p^{1/2} (r_1 + 1)^{1/2} \exp \left\{ \frac{p r_2}{2} (f_1 A_N + f_2 B(0)) \right\} \right. 
\left. + \left( 2^{3-2b} \Lambda^2 F(A_N) \right) \frac{1}{2} \right] \left( \begin{array}{c} 1 \end{array} \right) 
\times \exp \left\{ \frac{p r_2}{2} (f_1 A_N + f_2 B(0)) \right\} \right]
\]
\leq h^b K_{N,b,p},
\]
where
\[
K_{N,b,p} = 2^{1/p} (\max (r_1, s_1, s_2) + 1) \exp \left\{ \frac{p r_2}{2} (f_1 A_N + f_2 B(0)) \right\} K_{N,b},
\]
(21)
\[
K_{N,b} = k_{N,b}^{1/2} + (2^{3-2b} \Lambda^2 F(A_N))^{1/2}.
\]
We evaluate the integral used in the definition of the constant $V_p$ in Theorem 3.1:
\[
\int_0^{\theta_0} \left( \frac{d}{\theta^{(p-1)}} (h) \right)^{1/p} dh = \int_0^{\theta_0} \frac{d^{1/p} K_{N,b,p}^{1/2}(h)}{h^{1/p}} dh = \frac{d^b K_{N,b,p}^{1/2} \theta^{1-1/(pb)}}{1 - \frac{1}{pb}}, \quad 1 - \frac{1}{pb} > 0.
\]
The function
\[
f(\theta) = \frac{\theta^{1-1/(pb)}}{\theta(1-\theta)}
\]
attains its minimal value
\[
f \left( \frac{1}{1+pb} \right) = \frac{(1+pb)^{1+1/(pb)}}{pb}
\]
at the point $\theta_0 = (1+pb)^{-1}$. One can easily derive from inequality (13) that
\[
\inf_{t \in B_t} \left( \mathbb{E} \left| \exp \{ Y(t) \} - \exp \{ \widetilde{Y}(t) \} \right| \right)^{1/p} 
\leq (B(0) - F(A))^{1/2} \frac{1}{2} \right] \left( \begin{array}{c} 1 \end{array} \right) \left( v_1 + 1 \right)^{1/2} \exp \left\{ \frac{p r_2}{2} B(0) \right\}.\]
Using the notation of Theorem 3.1 we get
\[
\frac{V_p}{x^p} \leq \left[ (B(0) - F(A))^{1/2} \frac{1}{2} \right] \left( v_1 + 1 \right)^{1/2} \exp \left\{ \frac{p r_2}{2} B(0) \right\} + d^b K_{N,b,p}^{(1+pb)^{1+1/(pb)}} \right].
\]
Applying the elementary inequality $(a+b)^p \leq 2^{p-1} (a^p + b^p)$ we obtain
\[
\frac{V_p}{x^p} \leq \left[ \frac{2^p (B(0) - F(A))^{P/2} \left( v_1 + 1 \right)^{p/2}}{x^p} \exp \left\{ \frac{p r_2}{2} B(0) \right\} \right]
\leq \left[ \frac{2^{p-1} d^b K_{N,b,p}^{(1+pb)^{1+1/(pb)}}}{x^p} \right].
\]
(22)
\[
\frac{(1+pb)^{p+1/b}}{(pb-1)^p} \leq p^{1/b} \frac{1}{b} \exp \left\{ \frac{1}{b} \right\} \left( 1 - \frac{1}{pb} \right)^p
\]
Table 1

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<th>c₂</th>
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Table 2

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and definitions (21) to estimate from above the second term of the expression on the right hand side of (22):

\[
(23) \quad \frac{2^{p-1}d^b K_{N,b,p}^p}{x^p} \frac{(1+pb)^{1+(1/b)}}{p^{b-1}} \leq \frac{D M_{N}^{p} p^{p+1/p} \exp \left\{ \frac{x^p}{2} (f_1 A_N + f_2 B(0)) \right\}}{x^p \left(1 - \frac{1}{p^b}\right)^p},
\]

where

\[
D = b^{1/b} \exp \left\{ \frac{1}{b} + \frac{1}{b^2} \right\},
\]

\[
M_{N} = 2d^b (\max (r_1, s_1, s_2) + 1) K_{N,b}.
\]

Let \( r_2(f_1 A_N + f_2 B(0)) = Z \) and \( 2(B(0) - F(\Lambda))^{1/2}(v_1 + 1)^{1/2} = G \). Taking into account (23) we evaluate the right hand side of (22) at the point

\[
p_0 = \ln \frac{\sigma}{Z},
\]

which is close to the point of minimum of the function

\[
\frac{D M_{N}^{p} p^{p+1/p} \exp \left\{ \frac{x^p}{2} Z \right\}}{x^p \left(1 - \frac{1}{p^b}\right)^p}.
\]

Since \(1 - (p_0 b)^{-1} > 0\), we apply Theorem 3.1 to obtain bounds (7), whence Theorem 3.2 follows.

**Example 3.1.** Let a stochastic process \(Y(t)\) satisfy the assumptions of Theorem 3.2 and let its spectral density be \(f(\lambda) = c_1/(1 + c_2 \lambda^k)\). Table 1 contains the values of \(d\) and \(N\) for this process, given the accuracy \(\alpha\) and reliability \(1 - \gamma\).

Table 2 contains the values of \(d\) and \(N\) evaluated for a stochastic process with the spectral density \(f(\lambda) = \exp\{-c\lambda^k\} \).

**Remark 3.2.** One can use the following condition,

\[
P \left\{ \max_{B_i \in B, i=0, \ldots, k-1} |p_{kY}(B_i) - \bar{p}_{kY}(B_i)| > \alpha \right\} < \gamma,
\]

in Definition 3.1 instead of

\[
\max_{B_i \in B, i=0, \ldots, k-1} P \{ |p_{kY}(B_i) - \bar{p}_{kY}(B_i)| > \alpha \} < \gamma.
\]
Then we obtain
\[
\mathbb{P}\left\{ \max_{B_i \in \mathcal{B}, i=0, \ldots, k-1} |\mu(B_i) - \bar{\mu}(B_i)| > \alpha \right\} \leq \sum_{i=1}^{k} \mathbb{P}\left\{ |\mu(B_i) - \bar{\mu}(B_i)| > \alpha \right\}
\]
\[
\leq k \cdot \max_{B_i \in \mathcal{B}, i=0, \ldots, k-1} \mathbb{P}\left\{ |\mu(B_i) - \bar{\mu}(B_i)| > \alpha \right\}
\]
\[
\leq k \cdot \mathbb{P}\left\{ \sup_{t \in B_i} \left\| \exp\{Y(t)\} - \exp\{\tilde{Y}(t)\} \right\| > \frac{\alpha}{d} \right\}
\]
using estimates (4) and (5). This result is analogous to Theorem 3.2 where $\gamma/k$ is substituted for $\gamma$ in formula (7).

The latter setting is more involved, since the model approximates the process only for sufficiently large $N$.

4. CONCLUDING REMARKS

A method for modelling a log Gaussian Cox stochastic process with a given accuracy and reliability is presented in this paper for the case where the logarithm of the intensity is a separable stationary Gaussian stochastic process. The main result of the paper, Theorem 3.2, contains sufficient conditions that a log Gaussian Cox stochastic process is approximated by its model.

BIBLIOGRAPHY