ON THE RATE OF CONVERGENCE TO THE NORMAL DISTRIBUTION OF THE NUMBER OF FALSE SOLUTIONS OF A SYSTEM OF NONLINEAR RANDOM BOOLEAN EQUATIONS

UDC 519.21

V. I. MASOL AND S. YA. SLOBODYAN

Abstract. We prove a theorem on the limit normal distribution (as $n \to \infty$) of the number of false solutions of a system of nonlinear equations with independent random coefficients belonging to the field GF(2). We assume that every equation contains at least one coefficient for which the probability that it attains the value 1 is close to $\frac{1}{2}$; the number of equations $N$ and the number of unknowns $n$ are such that $n - N \to \infty$ as $n \to \infty$; the system has a solution containing $\rho(n)$ units and $\rho(n) \to \infty$ as $n \to \infty$.

1. Setting of the problem. Statement of the theorem

Consider the following system of equations over the field GF(2) containing only two elements:

$$
\sum_{k=1}^{g_i(n)} \sum_{1 \leq j_1 < \cdots < j_k \leq n} a_{j_1 \ldots j_k}^{(i)} x_{j_1} \cdots x_{j_k} = b_i, \quad i = 1, \ldots, N.
$$

We assume that

Condition (A).

1) The coefficients $a_{j_1 \ldots j_k}^{(i)}$, $1 \leq j_1 < \cdots < j_k \leq n$, $k = 1, \ldots, g_i(n)$, $i = 1, \ldots, N$, are independent random variables assuming the value 1 with probability $P\{a_{j_1 \ldots j_k}^{(i)} = 1\} = p_{ik}$ and the value 0 with probability $P\{a_{j_1 \ldots j_k}^{(i)} = 0\} = 1 - p_{ik}$.

2) The elements $b_i$, $i = 1, \ldots, N$, are equal to the left hand side of system (1), where a fixed $n$-dimensional vector $\bar{x}_0$ is substituted for unknowns (the vector $\bar{x}_0$ consists of $\rho(n)$ units and $n - \rho(n)$ zeros).

3) The functions $g_i(n)$, $i = 1, \ldots, N$, are nonrandom and such that $g_i(n) \in \{2, \ldots, n\}$ for all $i = 1, \ldots, N$. 

2000 Mathematics Subject Classification. Primary 60C05, 15A52, 15A03.

Key words and phrases. Nonlinear random Boolean equations, the limit normal distribution, the number of false solutions.
Denote by \( \nu_n \) the number of false solutions of system (1), that is, the number of solutions of system (1) that do not coincide with the vector \( \overline{x}^0 \).

We are interested in conditions under which the random variable \( \nu_n \) has the limit normal distribution as \( n \to \infty \).

Let \( m = n - N \), \( \lfloor \lambda \rfloor = 2^m \), and let \( [\cdot] \) denote the integer part.

**Theorem.** Let condition (A) hold. Let

\[
\lambda = \frac{1}{14} \log_2 \frac{\rho(n)}{\varphi(n) \ln n}, \quad \varphi(n) > 0,
\]

(2)

\[
\lambda \to \infty, \quad n \to \infty;
\]

(3)

and for an arbitrary \( i = 1, \ldots, N \), there exists a set \( T_i \) such that

\[
T_i \subseteq \{2, 3, \ldots, g_i(n)\}, \quad T_i \neq \emptyset,
\]

(4)

\[
0 \leq \delta_i(n) \leq p_i \leq 1 - \delta_i(n), \quad t \in T_i;
\]

(5)

\[
\lim_{n \to \infty} \lambda B(\rho(n) - 1, 1) < \infty
\]

for all sufficiently large \( n \), where

\[
B(X, Y) = \sum_{i=1}^{N} \exp \left\{ -2 \sum_{t \in T_i} \delta_i(n) C_{X}^{t} \right\}.
\]

If

\[
(2 + 7 \ln 2) \lambda - \frac{\ln \lambda}{2} + \ln B(\varepsilon \varphi(n), 0) \to -\infty, \quad n \to \infty,
\]

(6)

for \( \varepsilon = \text{const}, \ 0 < \varepsilon < 1 \), and

\[
\lim_{n \to \infty} \left( -\ln N + \ln B(\varepsilon \varphi(n), 1) \right) < 0, \quad n \to \infty,
\]

(7)

then the distribution function of the random variable \( (\nu_n - \lambda)/\sqrt{\lambda} \) tends to the standard normal distribution function as \( n \to \infty \).

**Remark 1.1.** The assumptions of the theorem hold if, for example, \( \rho(n) = \rho n \) for some constant \( \rho \) such that \( 0 < \rho < 1 \); \( \delta \leq p_i \leq 1 - \delta \) for some constant \( \delta, \ \delta > 0; T_i = \{2\}, \ i = 1, \ldots, N \); and

\[
\ln n \leq \varphi(n) \leq \varepsilon n^{1-p} \ln n,
\]

where \( \varepsilon \) and \( p \) are some constants such that \( 0 < \varepsilon < 1 \) and \( 0 < p < 1 \).

2. Auxiliary results

**Proposition 2.1** ([1]). Let \( X \) and \( Y \) be random variables assuming nonnegative integer values and let \( \lambda^* = E X \). Assume that the distributions of the random variables \( X \) and \( Y \) are such that

\[
\sup_{1 \leq r \leq 7 \lambda^*} \left| E(X)^r (E(Y)^r)^{-1} - 1 \right| \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0
\]

(8)

and

\[
E(Y)^r \leq C (\lambda^*)^r
\]

(9)

for all \( r \leq 7 \lambda^* \) and some constant \( C \). (Here \( E(\xi)_r \) denotes the factorial moment of order \( r \) of the random variable \( \xi \) for \( r \geq 1 \).) Then

\[
\max_{1 \leq t \leq 2 \lambda^*} \left| P\{X \geq t\} - P\{Y \geq t\} \right| \to 0.
\]
Proposition 2.2 (2). If condition (A) holds, then the mathematical expectation of the random variable \( \nu_n \) is given by

\[
E \nu_n = \sum_{i=0}^{n-\rho(n)} C_{n-\rho(n)}^i \sum_{j=0}^{\rho(n)} C_{\rho(n)}^j N \left( \frac{1}{2} + \frac{1}{2} \prod_{q=1}^N (1 - 2p_{qt}) \Gamma_t(i,j) \right),
\]

where

\[
\Gamma_t(i,j) = C_{i+\rho(n)-j}^t + C_{\rho(n)}^t - 2C_{\rho(n)-j}^t, \quad t = 1, \ldots, g_q(n), \quad q = 1, \ldots, N.
\]

Proposition 2.3 (3). If condition (A) holds, then

\[
E(\nu_n)_r = 2^{-r} N S(n,r;Q)
\]

for \( r \geq 1 \), where

\[
S(n,r;Q) = \sum_{s=0}^{n-\rho(n)} \sum (n-\rho(n))! \left( (n-\rho(n)-s)! \prod_{i\in I} d! \right)^{-1} \times \sum_{s'=0}^{\rho(n)} \sum'(\rho(n))! \left( (\rho(n)-s')! \prod_{j\in J} j! \right)^{-1} Q,
\]

\[
Q = \prod_{i=1}^{N} \left( 1 + \sum_{v=1}^r \sum_{1 \leq u_1 < \cdots < u_v \leq r} \prod_{t=1}^{\rho(n)} (1 - 2p_{tt}) \Gamma_t^{(u_1,\ldots,u_v)} \right).
\]

The symbol \( \sum (\sum') \) means the summation over all \( i \in I \) (\( j \in J \)) such that \( \sum_{i\in I} i = s \) (\( \sum_{j\in J} j = s' \)), where

\[
I = \{i_{u_1,\ldots,u_v} : 1 \leq u_1 < \cdots < u_v \leq r, \nu = 1, \ldots, \nu\},
\]

\[
J = \{j_{u_1,\ldots,u_v} : 1 \leq u_1 < \cdots < u_v \leq r, \nu = 1, \ldots, \nu\}.
\]

The elements of the set \( I \) (\( J \)) are defined in [3]. In equality (13), the numbers \( i, i \in I \), and \( j, j \in J \), are such that

\[
\sum_{i\in I(u), j\in J(u)} (i+j) \geq 1, \quad u = 1, \ldots, r,
\]

\[
\sum_{i=0}^{r-2} \sum_{1 \leq \mu_1 < \cdots < \mu_t \leq r} (i_{u_1,\mu_1,\ldots,\mu_t} + j_{u_1,\mu_1,\ldots,\mu_t} + j_{u_2,\mu_1,\ldots,\mu_t} + j_{u_2,\mu_1,\ldots,\mu_t}) \geq 1,
\]

\[
1 \leq u_1 < u_2 \leq r.
\]

Furthermore if \( 1 \leq u_1 < \cdots < u_v \leq r, \nu \in \{1, \ldots, r\}, \) and \( t \in \{1, \ldots, n\}, \) then

\[
\Gamma_t^{(u_1,\ldots,u_v)} \geq \sum_{(i,j)\in T} (C_i^t + C_j^t),
\]

where \( T = I_{\{u_1,\ldots,u_v\}} \times J_{\{u_1,\ldots,u_v\}} \).

Finally if

\[
\rho(n) - s' \geq t,
\]

then

\[
\Gamma_t^{(u_1,\ldots,u_v)} \geq C_{\rho(n)-s'}^t \sum_{(i,j)\in T} (i+j).
\]
Here
\[ I_{\{u_r, \ldots, u_\nu\}} = \{i(\sigma_1, \ldots, \sigma_p, \mu_1, \ldots, \mu_l) : A(\psi, l, r)\}, \]
\[ J_{\{u_r, \ldots, u_\nu\}} = \{j(\sigma_1, \ldots, \sigma_p, \mu_1, \ldots, \mu_l) : A(\psi, l, r)\} \]
and \( A(\psi, l, r) \) stands for the following set of restrictions:
\[ 1 \leq \sigma_1 < \cdots < \sigma_\psi \leq r, \]
\[ \sigma_z \in \{u_1, \ldots, u_\nu\}, \ z = 1, \ldots, \psi, \ \psi = 1, \ldots, \nu, \ \psi \equiv 1 \pmod{2}, \ 1 \leq \mu_1 < \cdots < \mu_l \leq r, \]
and \( \mu_1, \ldots, \mu_l \notin \{u_1, \ldots, u_\nu\}, \ l = 0, \ldots, r - \nu. \)

Remark 2.1. The explicit expression for \( \Gamma_t^{\{u_1, \ldots, u_\nu\}} \) is given by equality (2.11) in \([3]\) for
\[ 1 \leq u_1 < \cdots < u_\nu \leq r, \quad \nu \in \{1, \ldots, r\}, \]
and \( t = 1, 2, \ldots, g_t(n), \ i = 1, \ldots, N. \)

3. PROOF OF THE THEOREM

We show that the assumptions of the theorem allow one to apply Proposition 2.1. Let the random variable \( Y \) in Proposition 2.1 have the Poisson distribution with parameter \( 2^m, \) while the distribution of the random variable \( X \) coincides with the distribution of the random variable \( \nu_n. \)

Applying relation (10) we obtain an upper bound for the mathematical expectation of the random variable \( \nu_n. \) First we use condition (4) and the inequality \( 1 - t < e^{-t} \) to get
\[ \exp \{ -B(\rho(n) - 1, 1) \} \leq \prod_{q=1}^{N} \left( 1 + \prod_{i=1}^{g_t(n)} (1 - 2p_{qt})^{\Gamma_t(i,j)} \right) \leq \exp \{ B(\rho(n) - 1, 1) \}. \]

In the proof of (18) we make use of the inequality
\[ \Gamma_t(i,j) \geq C_{\rho(n) - 1}^t \]
for all \( i = 0, 1, \ldots, n - \rho(n) \) and \( j = 0, 1, \ldots, \rho(n) \) (in turn, the latter inequality is easy to check with the help of equality (11)).

Substituting (18) into (10) we get
\[ \exp \{ -B(\rho(n) - 1, 1) \} \leq M \leq \exp \{ B(\rho(n) - 1, 1) \}. \]

The equalities \( \lambda^* = \mathbb{E} \nu_n \) and (19) yield
\[ \lambda^* = 2^m \left( 1 - \frac{1}{2^n} \right) M, \]
where \( \exp \{ -B(\rho(n) - 1, 1) \} \leq M \leq \exp \{ B(\rho(n) - 1, 1) \}. \)

or
\[ \lambda^* = \lambda(1 + \gamma(n)), \]
where \( \gamma(n) = O(B(\rho(n) - 1, 1)) + O(2^{-n}) \) and \( \gamma(n) \to 0 \) as \( n \to \infty, \) since
\[ B(\rho(n) - 1, 1) \to 0, \quad n \to \infty, \]
by (3) and (5).

With the help of (18) and condition (5) we check that relation (9) holds for \( r \leq 7\lambda^* \) if the constant \( C \) is such that
\[ C \geq \left( 1 - \frac{1}{2^n} \right)^{-7 \mathbb{E} \nu_n} \exp \{ 7B(\rho(n) - 1, 1) \mathbb{E} \nu_n \}. \]
Now we check condition (5). Equality (12) can be rewritten in the following form:

\begin{equation}
E(u_n) = \frac{1}{2^r N} \sum_{\Delta = 0}^{2^{r-1}} S^{(\Delta)}(n, r; Q),
\end{equation}

where \( S^{(\Delta)}(n, r; Q) \) is defined similarly to \( S(n, r; Q) \) (see definition (8)) with the additional restriction that the indices \( i, i \in I, \) and \( j, j \in J, \) are such that there are exactly \( \Delta \) different collections

\( \omega_{\alpha} = \{ u^{(\alpha)}_1, \ldots, u^{(\alpha)}_{\xi_{\alpha}} \}, \quad 1 \leq u^{(\alpha)}_1 < \cdots < u^{(\alpha)}_{\xi_{\alpha}} \leq r, \)

\( \xi_{\alpha} \in \{1, 2, \ldots, r\}, \quad \alpha = 1, 2, \ldots, \Delta, \)

and for each of them there exists a number \( t^{(\alpha)} \in \{2, \ldots, k\} \) with the properties that

\begin{equation}
\Gamma_{t^{(\alpha)}, r} \omega_{\alpha} < C_k^{(\alpha)}, \quad k = [\varepsilon n],
\end{equation}

and

\begin{equation}
\Gamma_{t, r}^{(t_1, \ldots, t_{\gamma})} \geq C_k^{t}
\end{equation}

for all \( t \in \{2, \ldots, k\} \) and all collections \( \{ t_1, \ldots, t_{\gamma} \}, \) \( 1 \leq t_1 < \cdots < t_{\gamma} \leq r, \gamma = 1, \ldots, r, \) such that

\( \{ t_1, \ldots, t_{\gamma} \} \neq \omega_{\alpha}, \quad \alpha = 1, 2, \ldots, \Delta. \)

We show that

\begin{equation}
\sup_{1 \leq r \leq 7\lambda^*} \frac{|S^{(0)}(n, r; Q)|}{2^r N E(Y)_r} - 1 \bigg| \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0
\end{equation}

as \( n \to \infty. \)

Similarly to the paper (3) we check that the equality \( \Delta = 0 \) can be achieved. Let \( u = (2^r - 1) B(\varepsilon n, 0). \) Then the product \( Q \) can be rewritten as follows:

\[ Q = 1 + \zeta(n)u + O\left(u^2\right), \quad |\zeta(n)| \leq 1, \]

if \( \Delta = 0. \) This result follows from inequalities (23) and (4) and in view of the asymptotics \( u \to 0 \) as \( n \to \infty \) (the latter relation can be obtained from (6)). Thus

\begin{equation}
S^{(0)}(n, r; Q) = (2^n - \sigma_0) \left(1 + \zeta(n)u + O\left(u^2\right)\right)
\end{equation}

by the polynomial theorem and by equality (8), where

\begin{equation}
\sigma_0 = 1 + \sum_{q=1}^{2^r-1} S_q^{(0)}(n, r; 1).
\end{equation}

The term \( S_q^{(0)}(n, r; 1) \) is defined similarly to \( S(n, r; 1) \) with an extra restriction that the indices \( i, i \in I, \) and \( j, j \in J \) on the right hand side of (8) are such that there are exactly \( q \) expressions \( \Gamma_{t, r}^{\{u_1, \ldots, u_{\nu}\}} \) for which

\begin{equation}
\Gamma_{t, r}^{\{u_1, \ldots, u_{\nu}\}} < C_k^{t},
\end{equation}

where \( q = 1, 2, 3, \ldots, 2^r - 1. \)

To each

\[ \Gamma_{t, r}^{\{u_1, \ldots, u_{\nu}\}}, \quad 1 \leq u_1 < \cdots < u_{\nu} \leq r, \quad \nu \in \{1, \ldots, r\}, \]

we assign a number \( 1, 2, 3, \ldots, 2^r - 1. \) Then the sum \( S_q^{(0)}(n, r; 1) \) can be rewritten as follows:

\begin{equation}
S_q^{(0)}(n, r; 1) = \sum_{1 \leq \gamma_1 < \cdots < \gamma_q \leq 2^r - 1} S_{(\gamma_1, \ldots, \gamma_q)}^{(0)}(n, r; 1),
\end{equation}

THE NORMAL DISTRIBUTION OF THE NUMBER OF FALSE SOLUTIONS 121
q = 1, 2, 3, \ldots, 2^r - 1, where \( S_{(\gamma_1, \ldots, \gamma_q)}^{(0)} (n, r; 1) \) is defined similarly to \( S_q^{(0)} (n, r; 1) \) but bound (27) holds only for those \( \Gamma_{(u_1, \ldots, u_r)} \) that correspond to the numbers \( \gamma_1, \gamma_2, \ldots, \gamma_q \) in the case of \( S_{(\gamma_1, \ldots, \gamma_q)}^{(0)} (n, r; 1) \). Denote by \( A(\gamma_1, \ldots, \gamma_q) \) \( (B(\gamma_1, \ldots, \gamma_q)) \) the set of those indices \( i \in I \ (j \in J) \) that are used in the bound (16) for all \( \gamma_1, \gamma_2, \ldots, \gamma_q \). By inequalities (27) and (16), the number of elements of the set \( A(\gamma_1, \ldots, \gamma_q) \) \( (B(\gamma_1, \ldots, \gamma_q)) \) is not less than \( 2^{r-1} \):

\[
|A(\gamma_1, \ldots, \gamma_q)| \geq 2^{r-1},
\]

\[
|B(\gamma_1, \ldots, \gamma_q)| \geq 2^{r-1}.
\]

Now the sum \( S_q^{(0)} (n, r; 1) \) can be written as follows:

\[
S_q^{(0)} (n, r; 1) = \sum_{1 \leq \gamma_1 < \cdots < \gamma_q \leq 2^r - 1} \sum_{s=0}^{n-\rho(n)} C_{n-\rho(n)}^{s} \times \sum_{s_1 + s_2 = s} C_{s_1}^{s_1} \left( \sum_{1 \leq i \in A(\gamma_1, \ldots, \gamma_q)} \frac{s_1!}{i!} \right) \left( \sum_{1 \leq i \in I \setminus A(\gamma_1, \ldots, \gamma_q)} \frac{s_2!}{i!} \right) \times \sum_{s'=0}^{\rho(n)} \sum_{s_1' + s_2' = s'} C_{s'}^{s_1'} \left( \sum_{1 \leq j \in B(\gamma_1, \ldots, \gamma_q)} \frac{s_1'!}{j!} \right) \left( \sum_{1 \leq j \in J \setminus B(\gamma_1, \ldots, \gamma_q)} \frac{s_2'!}{j!} \right),
\]

where \( \sum_1 \) and \( \sum_2 \) mean the sums over all indices \( i \) such that \( i \in A(\gamma_1, \ldots, \gamma_q) \) and \( \sum i = s_1 \) or \( i \in I \setminus A(\gamma_1, \ldots, \gamma_q) \) and \( \sum i = s_2 \), respectively, while \( \sum_3 \) and \( \sum_4 \) mean the sums over all indices \( j \) such that \( j \in B(\gamma_1, \ldots, \gamma_q) \) and \( \sum j = s_1' \) or \( j \in J \setminus B(\gamma_1, \ldots, \gamma_q) \) and \( \sum j = s_2' \), respectively.

Relations (26), (28)–(31), and the polynomial formula imply the following estimate for \( \sigma_0 \):

\[
\sigma_0 \leq 2^{2^r - 1} 2^{(r-1)n} \left( \sum_{s_1} C_{n-\rho(n)}^{s_1} (2^{r-1})^{s_1} \right) \left( \sum_{s_1'} C_{\rho(n)}^{s_1'} (2^{r-1})^{s_1'} \right),
\]

where the indices \( s_1 \) and \( s_1' \) in the sums run in the intervals

\[
0 \leq s_1 \leq 2^r k \quad \text{and} \quad 0 \leq s_1' \leq 2^r k,
\]

respectively. The upper bounds for \( s_1 \) and \( s_1' \) in the latter restrictions follow from (27), (16), and the inclusions \( i \in A(\gamma_1, \ldots, \gamma_q) \) and \( j \in B(\gamma_1, \ldots, \gamma_q) \).

Considering the inequalities \( s_1 \leq 2^r k \) and \( s_1' \leq 2^r k \) we rewrite relation (32) in the following form:

\[
\sigma_0 \leq 2^{2^r - 1} 2^{(r-1)n} 2 r^{2^r + 1} \phi(n) \left( \sum_{s_1=0}^{2^r k} C_{n-\rho(n)}^{s_1} \right) \left( \sum_{s_1'=0}^{2^r k} C_{\rho(n)}^{s_1'} \right).
\]

Since

\[
2^{r+1} \varphi(n) \leq \rho(n)
\]

for \( 1 \leq r \leq 7 \lambda^* \), inequality (33) implies that

\[
\sigma_0 \leq 2^{2^r - 1} 2^{(r-1)n} 2 r^{2^r + 1} \varphi(n) (2^r \varphi(n))^2 C_{\rho(n)}^{2^r k} C_{\max(\rho(n), n-\rho(n))}^{2^r k}.
\]
whence
\begin{equation}
\sigma_0 \leq 2^{2r-1}2^{(r-1)n}2^{r}\varphi(n) \left( \frac{e\rho(n)}{\varepsilon\varphi(n)} \right) 2^{r+1}\varphi(n)
\end{equation}
by Stirling's formula.

According to equality (25) the fraction
\begin{equation}
S^{(0)}(n,r;Q) \over 2^{rN_{2n}}
\end{equation}
is equal to
\begin{equation}
1 - \frac{\sigma_0}{2^r} + |O(u)|.
\end{equation}
This implies that relation (24) is equivalent to
\begin{equation}
\left( \frac{\sigma_0}{2^r} + |O(u)| \right) \left( e^{2\lambda^*} \sqrt{\lambda^*} \right) \rightarrow 0, \quad n \rightarrow \infty.
\end{equation}
It is easy to check that
\begin{equation}
u \left( e^{2\lambda^*} \sqrt{\lambda^*} \right) \rightarrow 0, \quad n \rightarrow \infty,
\end{equation}
by condition (6).

Now
\begin{equation}
\frac{\sigma_0}{2^r} \left( e^{2\lambda^*} \sqrt{\lambda^*} \right) \rightarrow 0, \quad n \rightarrow \infty,
\end{equation}
in view of condition (3) and relations (34) and (20).

Using (36) and (37) we obtain (35).

In its turn (35) and the equality \( E(Y)_r = 2^r m \) imply (24).

Having obtained (21) and (24) we complete the proof of (8) if
\begin{equation}
1 \over 2^{rN + 2n m} \left( \sum_{\Delta = 1}^{2^r - 1} S^{(\Delta)}(n,r;Q) \right) \left( e^{2\lambda^*} \sqrt{\lambda^*} \right) \rightarrow 0, \quad n \rightarrow \infty,
\end{equation}
for \( 1 \leq r \leq 7 \lambda^* \) and \( \Delta \geq 1 \).

Denote by \( M_1 (\tilde{M}_1) \) the collection of all \( i \in I \) (or \( j \in J \)) that do not belong to \( I_{\omega_{\alpha}} \)
(or to \( J_{\omega_{\alpha}} \), \( \alpha = 1, \ldots, \Delta \)). Put \( M_2 = I \setminus M_1 \) and \( \tilde{M}_2 = J \setminus \tilde{M}_1 \).

Let \( z \) be the minimal integer number such that
\begin{equation}
\Delta \leq 2^z - 1, \quad 1 \leq z \leq r.
\end{equation}
Then the number of elements of the set \( M_1 (\tilde{M}_1) \) can be estimated as follows:
\begin{equation}|M_1| \leq 2^{r-z} - 1, \quad |\tilde{M}_1| \leq 2^{r-z} - 1 \end{equation}
by Proposition 2.1 in [3].

If (22) holds, then (41) implies
\begin{equation}
|Q| \leq 2^{zN}Q_0
\end{equation}
for the left hand side of (38), where
\begin{equation}
Q_0 = 1 + 2^{-z} (2^r - \Delta - 1) \sum_{i=1}^N \exp \left\{ -2 \sum_{t \in T_i} \delta_{i t}(n)C_{i \varphi(n)}^t \right\} + O(u^2).
\end{equation}
We show that \(45\) follows from the following condition \((R_0)\):

\[
\rho(n) - k + 1 \leq s' \leq \rho(n),
\]

where \(s'\) is the index of summation introduced for equality \(43\).

Let \(S^\Delta_{((R_0):\gamma_1,\ldots,\gamma_\Delta)}(n, r; 1)\) be defined similarly to \(S^0_{((R_0):\gamma_1,\ldots,\gamma_\Delta)}(n, r; 1)\) but with an extra restriction \((R_0)\). If \((R_0)\) holds, then

\[
\sum_{\Delta=1}^{2^r-1} S^{\Delta}(n, r; Q) = \sum_{r=1}^{2^r-1} \sum_{\Delta=2^r-1}^{2^r-1} S^{\Delta}_{((R_0):\gamma_1,\ldots,\gamma_\Delta)}(n, r; Q).
\]

It follows from \(40\) that every term on the right hand side of \(42\) admits the following upper bound:

\[
S^{\Delta}_{((R_0):\gamma_1,\ldots,\gamma_\Delta)}(n, r; Q) \leq 2^{2N} S^{\Delta}_{((R_0):\gamma_1,\ldots,\gamma_\Delta)}(n, r; 1)Q_0.
\]

Using \(40\) we deduce that

\[
S^\Delta_{((R_0):\gamma_1,\ldots,\gamma_\Delta)}(n, r; 1)
\leq \sum_{s=0}^{n-\rho(n)} C_{n-\rho(n)}^s (2^{r-s} - 1)^s \sum_{s_2=0}^{s} C_{s_2}^r (2^r - 1)^r (2^{r-s} - 1)^r (2^r - 1)^r \rho(n)
\times \sum_{s'=\rho(n)-k+1}^{\rho(n)} C_{\rho(n)}^{s'} \sum_{s'_2=0}^{s'} C_{s'_2}^{s'_2} (2^r - 1)^{s'_2}.
\]

Relations \(39\), \(41\), and \(11\)–\(44\) imply that

\[
\sum_{\Delta=1}^{2^r-1} S^{\Delta}(n, r; Q)
\leq 2^{2^r+r\mu-m} \left(1 - \frac{1}{2^r-1}\right)^{\rho(n)} (2^r - 1)^{2(2^r-1)\varepsilon\varphi(n)} \times \left(\sum_{s_2=0}^{(2^r-1)k} C_{n-\rho(n)}^{s_2} \sum_{s'_2=0}^{(2^r-1)k} C_{s'_2}^{s'_2}\right) \sum_{s'=0}^{k} C_{\rho(n)}^{s'} Q_0.
\]

Note that

\[
\sum_{s_2=0}^{(2^r-1)k} C_{n-\rho(n)}^{s_2} \leq \left(\frac{n-\rho(n)}{2^r-1}\varphi(n)\right)^{(2^r-1)\varepsilon\varphi(n)} \sqrt{(2^r-1)\varepsilon\varphi(n)},
\]

\[
\sum_{s'_2=0}^{(2^r-1)k} C_{s'_2}^{s'_2} \leq \left(\frac{\rho(n)e}{(2^r-1)\varphi(n)}\right)^{(2^r-1)\varepsilon\varphi(n)} \sqrt{(2^r-1)\varepsilon\varphi(n)},
\]

\[
\sum_{s'=0}^{k} C_{\rho(n)}^{s'} \leq \left(\frac{\rho(n)e}{\varepsilon\varphi(n)}\right)^{\varphi(n)} \sqrt{\varepsilon\varphi(n)}.
\]
Thus estimate (45) can be rewritten as follows:

\[
\sum_{\Delta=1}^{2r-1} S(\Delta)(n, r; Q) \leq 2^{2r+rn-n} \left(1 - \frac{1}{2r-1}\right) \frac{\rho(n)}{\rho(n) \max(\rho(n), n - \rho(n)) e^2} \left(2^r - 1\right) \varepsilon(n)
\]

(48)

\[
\times \left(\frac{\rho(n)}{\varepsilon(n)}\right) \left(2^r - 1\right) \left(\varepsilon(n)\right)^{3/2} Q_0.
\]

If condition \((R_0)\) holds for \(1 \leq r \leq 7\lambda^*,\) then we use (2), (3), (20), and (48) to prove that

\[
\frac{1}{2rN + rm} \sum_{\Delta=1}^{2r-1} S(\Delta)(n, r; Q) \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}}
\]

\[
\leq \exp\left\{\frac{\rho(n)}{2^{2\lambda^* - 1}} \left(1 + \psi_1(2\rho(n)) - \frac{2^{14\lambda^* - 1}}{\rho(n)} \ln \left(\frac{\max(\rho(n), n - \rho(n)) e^2}{2^2 \varepsilon(n)}\right)\right)
\]

\[
- \frac{2^{7\lambda^* - 1}}{\rho(n)} \ln \left(\frac{\rho(n) e}{\varepsilon(n)}\right) - \frac{3 \cdot 2^{7\lambda^* - 1}}{\rho(n)} \ln \left(\varepsilon(n)\right) - \frac{2^{7\lambda^* - 1}}{\rho(n)} \ln \left(1 + \frac{2^r - \Delta - 1}{2^2 (2^r - 1) u + O(u^2)}\right)\right\} \to 0
\]

as \(n \to \infty,\) since \(u \to 0\) as \(n \to \infty,\) where

\[\psi_1(2\rho(n)) = -\frac{2^{14\lambda^*} \ln 2}{2\rho(n)} + \frac{2^{7\lambda^*} m \ln 2}{2\rho(n)} - \frac{2^{7\lambda^*} \lambda^*}{\rho(n)} + \frac{2^{7\lambda^*} \ln \lambda^*}{4\rho(n)}\]

This means that (53) holds under the condition \((R_0)\).

Now we show that (58) holds under the condition \((R_1)\):

\[s' \leq \rho(n) - k\]

and there exists \(i \in M_2\) and (or) there exists \(j \in \widetilde{M}_2\) such that \(i \in (0; k]\) and (or) \(j \in (0; k]\).

Condition \((R_1)\) implies that

\[
\sum_{\Delta=1}^{2r-1} S(\Delta)(n, r; Q) = \sum_{z=1}^{r} \sum_{\Delta=2^{z-1} \leq \gamma_1 < \cdots < \gamma_\Delta \leq 2^{r-1}} S(\Delta)(n, r; Q),
\]

(51)

where \(S(\Delta)(n, r; Q)\) is defined similarly to \(S^{(0)}(\gamma_1, \ldots, \gamma_\Delta)(n, r; Q)\) but with an additional restriction \((R_1)\).

Using inequality (17) and \((R_1)\) we find that

\[
\Gamma_{t,r}^{\omega_0} \geq C_t^{r-1} \left(s^{(\alpha)} + \tilde{s}^{(\alpha)}\right)
\]

(52)

for all \(t \in \{2, \ldots, k\}\) and some \(\alpha = 1, \ldots, \Delta,\) where \(s^{(\alpha)} = \sum_{i \in I_{\omega_0}} i\) and \(\tilde{s}^{(\alpha)} = \sum_{j \in J_{\omega_0}} j.\)
For \( i = 1, \ldots, N \), we obtain the estimate
\[
\left| \prod_{t=1}^N (1 - 2\rho_t)^{\tau_t} \right| \leq \exp \left\{ -2 \sum_{t \in T_i} \delta_{it}(n) C_k^{\ell - 1} \left( s^{(\alpha)} + \tilde{s}^{(\alpha)} \right) \right\}
\]
by (62) and (4). The latter estimate together with (23) implies that
\[
Q \leq 2^{z_N} Q_1
\]
for
\[
\Delta = 2^z - 1, \quad s^{(\alpha)} + \tilde{s}^{(\alpha)} \geq 1,
\]
and some \( \alpha \in \{ 1, 2, \ldots, \Delta \} \), where
\[
Q_1 = \exp \left\{ -\frac{N}{2^z} + \frac{1}{2^z} \sum_{i=1}^N \exp \left\{ -2 \sum_{t \in T_i} \delta_{it}(n) C_k^{\ell - 1} \right\} + \frac{2^r - \Delta - 1}{2^z(2^r - 1)} u \right\}.
\]
We deduce from (51) and (53) that
\[
\left\{ S(\Delta)_{(R_1); \gamma_1, \ldots, \gamma_{\Delta}}(n, r; Q) \leq 2^{z_N} S(\Delta)_{(R_1); \gamma_1, \ldots, \gamma_{\Delta}}(n, r; 1) Q_1. \right.
\]
We use (40) to estimate the factor \( S(\Delta)_{(R_1); \gamma_1, \ldots, \gamma_{\Delta}}(n, r; 1) \), namely
\[
S(\Delta)_{(R_1); \gamma_1, \ldots, \gamma_{\Delta}}(n, r; 1) \leq (2^{r - z})^n (2^r - 1) 2^{(2^r - 1) z \varphi(n)} \sum_{s_2 = 0}^{(2^r - 1)k} C^{s_2}_{\epsilon} \sum_{s_2 = 0}^{(2^r - 1)k} C^{s_2}_{\rho(n)}.
\]
Combining relations (51), (53), (55), and (56) we obtain
\[
\sum_{\Delta = 1}^{2^{r - 1}} S(\Delta)(n, r; Q)
\]
\[
\leq 2^{r + r_n - m} \left( \frac{\rho(n) \max(\rho(n), n - \rho(n)) e^2}{\varepsilon^2 \varphi^2(n)} \right)^{(2^r - 1) z \varphi(n)} (2^r - 1) \varepsilon \varphi(n)
\]
\[
\times \exp \left\{ -2^{-z} N \left( 1 - \frac{1}{N} \sum_{i=1}^N \exp \left\{ -2 \sum_{t \in T_i} \delta_{it}(n) C_k^{\ell - 1} \right\} - \frac{2^r - \Delta - 1}{N(2^r - 1)} u \right) \right\}.
\]
Further we use conditions (2), (3), and \( (7) \) and relations (20) and (57) to check relation (55) under the restrictions \( (R_1) \) and (74) for \( 1 \leq r \leq 7 \lambda^* \):
\[
\sum_{\Delta = 1}^{2^{r - 1}} S(\Delta)(n, r; Q) \leq \frac{1}{2^N} \sum_{\Delta = 1}^{2^{r - 1}} S(\Delta)(n, r; Q) \frac{e^{2 \lambda^*}}{\sqrt{\lambda^*}}
\]
\[
\leq \exp \left\{ -\frac{N}{2^{7 \lambda^*}} \left( 1 + \psi(N) - \frac{1}{N} B(k, 1) - \frac{2^r - \Delta - 1}{N(2^r - 1)} u \right) \right\} \to 0, \quad n \to \infty,
\]
where
\[
\psi(N) = \psi_1(N) - \frac{7 \lambda^*}{N} \ln \left( \frac{\max(\rho(n), n - \rho(n)) \rho(n) e^2}{\varepsilon^2 \varphi^2(n)} \right)
\]
\[
- \frac{7 \lambda^* \ln(\varepsilon \varphi(n))}{N} - \frac{2^{7 \lambda^*} 2 \lambda^*}{N} \ln(\varepsilon \varphi(n)).
\]
Now let \( \Delta < 2^z - 1 \). Then the term \( Q \) on the right hand side of (51) is estimated as follows:
\[
Q \leq 2^{z_N} \left( 1 - \frac{1}{2^z} \right)^N \exp \left\{ \frac{2^r - \Delta - 1}{(2^z - 1)(2^r - 1)} u \right\}.
\]
If \((R_1)\) holds and \(\Delta < 2^z - 1\), then \((38)\) also holds, namely

\[
\frac{1}{2rN + rm} \sum_{\Delta=1}^{2^r-1} S^{(\Delta)}(n, r; Q) \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}}
\leq \exp \left\{ \frac{-N}{2\lambda^*} \left( 1 + \psi(N) \frac{2\lambda^*}{N} \right) \right\} \to 0, \quad n \to \infty,
\]

by conditions \((2)\) and \((3)\), and relations \((20), (46), (47), (51), (56),\) and \((59)\).

Then we refer to \([3, \text{Remark 2.2}]\) to show that there exists \(\alpha\)
and restriction \((50)\) holds.

If \(z = r\) (or \(r \in \{1, 2\}\)), then there exists \(\alpha \in \{1, 2, \ldots, \Delta\}\) such that \(\xi_\alpha \leq 2\). If \(z = r - 1\), then we refer to \([3, \text{Remark 2.2}]\) to show that there exists \(\alpha \in \{1, 2, \ldots, \Delta\}\) such that \(\xi_\alpha \leq 2\). Further, if \(\xi_\alpha \leq 2\) for some \(\alpha \in \{1, 2, \ldots, \Delta\}\), then \(s^{(\alpha)} + \tilde{s}^{(\alpha)} \geq 1\) by relations \((14)\) and \((15)\). Thus inequality \(s^{(\alpha)} + \tilde{s}^{(\alpha)} \geq 1\) holds for \(\alpha, r,\) and \(z\) specified above.

Now we show that the inequality

\[s^{(\alpha)} + \tilde{s}^{(\alpha)} \geq 1\]

follows for some \(\alpha \in \{1, 2, \ldots, \Delta\}\) if either \(r \in \{1, 2\}\) or \(z \in \{r, r - 1\}\) in inequality \((39)\).

If \(r = 1\) (or \(r \in \{1, 2\}\)), then we refer to \([3, \text{Remark 2.2}]\) to show that there exists \(\alpha \in \{1, 2, \ldots, \Delta\}\) such that \(\xi_\alpha \leq 2\). Further, if \(\xi_\alpha \leq 2\) for some \(\alpha \in \{1, 2, \ldots, \Delta\}\), then \(s^{(\alpha)} + \tilde{s}^{(\alpha)} \geq 1\) by relations \((14)\) and \((15)\). Thus inequality \(s^{(\alpha)} + \tilde{s}^{(\alpha)} \geq 1\) holds for \(\alpha, r,\) and \(z\) specified above.

We remain to show that \((38)\) holds under the condition \((R_2)\):

\[s^{(\alpha)} + \tilde{s}^{(\alpha)} = 0,\]

\[\xi_\alpha \geq 3, \quad \alpha = 1, \ldots, \Delta, \quad \Delta = 2^z - 1, \quad 1 \leq z \leq r - 2, \quad 3 \leq r < \infty,\]

and restriction \((50)\) holds.

Indeed, if condition \((R_2)\) holds, then

\[
\sum_{\Delta=1}^{2^r-1} S^{(\Delta)}(n, r; Q) = \sum_{z=1}^{r} \sum_{\Delta=2^z-1}^{2^r-1} S^{(\Delta)}_{(\{R_2\}; \gamma_1, \ldots, \gamma_\Delta)}(n, r; Q),
\]

where \(S^{(\Delta)}_{(\{R_2\}; \gamma_1, \ldots, \gamma_\Delta)}(n, r; 1)\) is defined similarly to \(S^{(0)}_{(\gamma_1, \ldots, \gamma_\Delta)}(n, r; 1)\) with an extra restriction \((R_2)\).

An estimate for \(Q\) on the right hand side of \((62)\) can be obtained by using \((2\iota)\) and \((R_2)\), namely

\[Q \leq 2zN Q_2,\]

where

\[Q_2 = \exp \left\{ \frac{2^r - \Delta - 1}{2^r (2^r - 1)} \right\}.
\]

According to \((62)\) and \((63)\) we have

\[S^{(\Delta)}_{(\{R_2\}; \gamma_1, \ldots, \gamma_\Delta)}(n, r; Q) \leq 2zN S^{(\Delta)}_{(\{R_2\}; \gamma_1, \ldots, \gamma_\Delta)}(n, r; 1) Q_2.
\]

If \(|\tilde{M}_1| < 2^r - z - 1\), then we find that

\[S^{(\Delta)}_{(\{R_2\}; \gamma_1, \ldots, \gamma_\Delta)}(n, r; 1) \leq \left(2^{r-z} \right)^{n-\rho(n)} \left(2^{r-z} - 1 \right)^{\rho(n)}.
\]
If \((R_2)\) holds and \(|\tilde{M}_1| < 2^{r-z} - 1, 1 \leq r \leq 7\lambda^*, \) then \((38)\) follows from conditions \((2)\) and \((\Delta)\) and relations \((20)\) and \((62)-(65)\), since

\[
\frac{1}{2^{n+1}\rho(n)} \sum_{\Delta=1}^{2^r n} S(\Delta) \left( \frac{\ln 2}{\rho(n)} \right) \geq 2^{2\lambda^*} \text{exp} \left\{ -\frac{1}{2^{r-z}} \left( 1 - \frac{2^{r-z} 2^{\lambda^*} \ln 2}{\rho(n)} + \frac{2^{r-z} \ln 2}{\rho(n)} - \frac{2^{r-z} 2^{\lambda^*}}{\rho(n)} \right) + \frac{2^{r-z} \ln \lambda^*}{2 \rho(n)} \right\} \leq 0, \quad n \to \infty.
\]

Now we check \((38)\) under the conditions \((R_2)\) and \((67)\)

\[
|\tilde{M}_1| = 2^{r-z} - 1.
\]

If \((61)\) and \((67)\) are satisfied, then the set \(\tilde{M}_1\) contains at least three elements

\[
j_{m_\nu} \in \tilde{M}_1, \quad \nu = 1, 2, 3,
\]

such that

\[
|\omega_\alpha \cap m_\nu| = 2, \quad \nu = 1, 2, 3, \quad |\omega_\alpha \cap (a \cup b)| = 3
\]

for some \(\alpha \in \{1, 2, \ldots, \Delta\}\) and all \(a, b \in \{m_\nu : \nu = 1, 2, 3\}\), \(a \neq b\) (see \([3\), Proposition 2.2]). For this \(\alpha\),

\[
\Gamma_{\nu, r}^{\alpha} \geq \gamma_t^{\{a \cup b\}}, \quad t \in \{2, \ldots, k\},
\]

by \((68)\) and by the representation for \(\Gamma_{r, t}^{\{u_1, \ldots, u_\nu\}}\), \(1 \leq u_1 < \cdots < u_\nu \leq r, \nu = 1, \ldots, r, t = 1, \ldots, n\), obtained in \([3\) equality \((2.11)\)].

The right hand side of \((69)\) is estimated according to \([3\) estimate \((2.16)\)], namely

\[
\gamma_t^{\{a \cup b\}} \geq t^{-1} j_t \left( j^* - 2^{-1} (j_\nu - 1) \right) C_t^{r-2} \left( \frac{1}{j^*/2} + (3j_\nu/4) - (5/4) \right)
\]

provided \(j_\nu \geq t\), where \(j_\nu = \min \{j_a, j_b\}\) and \(j^* = \max \{j_a, j_b\}\). If \(j_\nu \geq [\sqrt{\varepsilon}\varphi(n)]\), then the inequality \(j_\nu \geq t, t \in \{2, \ldots, k\}\), holds for \(0 < \varepsilon < 1\). Thus \((69)\) and \((70)\) imply that

\[
\Gamma_{\nu, r}^{\alpha} \geq \varepsilon \left( 2t \right)^{-1} \varphi_t^2 (n) C_t^{r-2} \left( \frac{1}{\sqrt{\varepsilon}\varphi(n)} \right) \left( \frac{1}{(5\sqrt{\varepsilon}\varphi(n))} - (5/4) \right)
\]

for sufficiently large \(n\) and for \(j_\nu\) and \(t\) as specified above. The latter relation contradicts \((22)\).

This means that if \((61)\) and \((67)\) hold, then at least one element \(j_\nu\) of the set \(\tilde{M}_1\), \(j_\nu \in \tilde{M}_1\), satisfies the inequality \(j_\nu < [\sqrt{\varepsilon}\varphi(n)]\). This result and the restrictions \((R_2)\) and \(|\tilde{M}_1| = 2^{r-z} - 1\) imply

\[
(71) \quad S(\Delta)_{(R_2); (\gamma_1, \ldots, \gamma_\Delta)} (n, r; 1) \leq 2^{2r-z} \left( \frac{1}{2^{r-z}} \right)^{\rho(n)} \frac{(\sqrt{\varepsilon}\varphi(n) \ln \rho(n))}{(\sqrt{\varepsilon}\varphi(n))!}.
\]

Now we use \((62)-(65)\) and \((\Delta)\) to get

\[
\sum_{\Delta=1}^{2^r n} S(\Delta) (n, r; Q) \leq 2^{2r + m - n} \left( \frac{1}{2^{r-z}} \right)^{\rho(n)} \frac{(\sqrt{\varepsilon}\varphi(n))^{\rho(n)}}{(\sqrt{\varepsilon}\varphi(n))!} \exp \left\{ \sqrt{\varepsilon}\varphi(n) \ln \rho(n) + \frac{2^r - \Delta - 1}{2^z (2^r - 1)} u \right\}.
\]
The latter inequality and condition (3) allow us to check that
\[
\frac{1}{2^{\nu+1}} \sum_{\Delta=1}^{2^{\nu}-1} S^{(\Delta)}(n, r; Q) \frac{n^{2\lambda^*}}{\sqrt{\lambda^*}} \leq \exp \left\{ -\rho(n) \frac{2^{-\nu} (1 + o(1))}{2^{-\nu}} \right\} \to 0, \quad n \to \infty,
\]
under the restrictions (R2) and \(|M_1| = 2^{\nu-2} - 1\), where \(o(1) \to 0\) as \(n \to \infty\).

Thus if (R2) holds, then (66) and (72) imply (38).

Considering the restrictions (R0), (R1), and (R2) and taking into account relations (49), (58), (60), (66), and (72) we prove that (38) holds for all possible values of the indices \(s, s'\) and \(i \in I, j \in J\) involved in the sums on the right hand side of (124) and such that inequality (22) is satisfied for \(\Delta \geq 1\).

Relations (21), (24), and (38) prove condition (8) of Proposition 2.1, where \(Y\) is a random variable with the Poisson distribution with parameter \(2m\).

Therefore all the assumptions of Proposition 2.1 are checked; hence
\[
\max_{1 \leq t \leq 2 \lambda^*} \left| P \left\{ \nu_n \geq t \right\} - P \left\{ Y \geq t \right\} \right| \to 0 \quad \text{as} \quad n \to \infty.
\]

Relation (73) implies that
\[
\max_{-\sqrt{\lambda^*} \leq w \leq \sqrt{\lambda^*}} \left| P \left\{ \frac{\nu_n - \lambda^*}{\sqrt{\lambda^*}} \geq w \right\} - P \left\{ Y - \frac{2m}{\sqrt{\lambda^*}} \geq w \right\} \right| \to 0, \quad n \to \infty,
\]
where \(w = (t - \lambda^*)/\sqrt{\lambda^*}\).

One can easily check with the help of (3), (5), and (20) that the limit distributions of the random variables \(\frac{\nu_n - \lambda^*}{\sqrt{\lambda^*}}\) and \(\frac{2m - \lambda}{\sqrt{\lambda}}\) coincide as \(n \to \infty\) (this is also true for the random variables \(\frac{Y - \lambda^*}{\sqrt{\lambda^*}}\) and \(\frac{\nu_n - \lambda^*}{\sqrt{\lambda^*}}\)). Since the random variable \(\frac{Y - \lambda^*}{\sqrt{\lambda^*}}\) has the standard normal distribution as \(n \to \infty\), the limit distribution of the random variable \(\frac{\nu_n - \lambda^*}{\sqrt{\lambda^*}}\) also is normal with parameters \((0, 1)\) as \(n \to \infty\) (this is due to relation (74)).

The theorem is proved.

**Bibliography**


Department of Probability Theory and Mathematical Statistics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

E-mail address: vimasol@ukr.net

Department of Probability Theory and Mathematical Statistics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

E-mail address: sv_yaras@rambler.ru

Received 22/MAR/2006

Translated by S. KVASKO