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ON SOME PROPERTIES OF ASYMPTOTIC QUASI-INVERSE FUNCTIONS

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Abstract. A characterization of normalizing functions connected with the limiting behavior of ratios of asymptotic quasi-inverse functions is discussed. For nondecreasing functions, conditions are obtained that are necessary and sufficient for their asymptotic quasi-inverse functions to belong to the class of (so-called) O-regularly varying functions or to some of its subclasses.

1. Introduction

This paper is a continuation of Buldygin et al. [10, 11].

Various problems of mathematical analysis and its applications in the theory of probability are connected with the question of finding conditions under which the following implication holds true:

$$\lim_{t\to\infty}\frac{g(t)}{f(t)} = a \Longrightarrow \lim_{t\to\infty}\frac{g^{(-1)}(t)}{f^{(-1)}(t)} = b, \tag{1}$$

where $f(t)\to\infty$ and $g(t)\to\infty$ as $t\to\infty$, $f^{(-1)}$ and $g^{(-1)}$ are certain "inverse" functions, and $a, b\in[0,\infty]$. This problem was studied, for instance, by Djurčić and Torgašev [16] in the case where $f^{(-1)}$ and $g^{(-1)}$ are inverse or generalized inverse functions, or by Buldygin et al. [8, 11] in the case where $f^{(-1)}$ and $g^{(-1)}$ are inverse or quasi-inverse or asymptotic quasi-inverse functions. A motivation for an application of related characterizations in probability theory is, for example, the correspondence between the strong law of large numbers for random walks and the renewal theorem for counting processes (see, e.g., Gut et al. [20]). Applications to the asymptotic behavior of renewal processes and of solutions of stochastic differential equations were considered in Klesov et al. [25] and Buldygin et al. [8], [11]–[13].

A general result (see Buldygin et al. [11]) shows that, under certain conditions,

$$\lim_{t\to\infty}\frac{g(t)}{f(t)} = a\in(0,\infty) \Longrightarrow \lim_{t\to\infty}\frac{g^{\sim}(t)}{f^{\sim}(t/a)} = 1, \tag{2}$$

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where $f^{\tilde{}}$ and $g^{\tilde{}}$ are *asymptotic quasi-inverse functions* (see the definition below in Section 2). The right-hand side of (2) differs from the right-hand side of (1). If g is a *regularly varying function* with index $\rho > 0$, then the right-hand side of (2) has the following form:

$$\lim_{t\to\infty} \frac{g^{\tilde{}}(t)}{f^{\tilde{}}(t)} = \left(\frac{1}{a}\right)^{1/\rho}, \tag{3}$$

which is similar to the right-hand side in (1).

We prove below (see Remark 3.1) that, under some natural conditions, the relation (1), with $a \neq 1$, $b \neq 1$, and $f^{(-1)}$ and $g^{(-1)}$ asymptotic quasi-inverse functions, is only possible for normalizing functions f from the so-called class of *functions with nondegenerate groups of regular points* (see Buldygin et al. [9]).

In this paper we are also concerned with the following question. Let f be a nondecreasing function and let its asymptotic quasi-inverse function $f^{\tilde{}}$ belong to some of the classes of functions studied in the *Karamata theory* of regular variation or some of its extensions. What are the conditions to be imposed on f to assure certain characteristic properties? The simplest case is when f is continuous, strictly increasing, and belongs to the class of regularly varying functions with positive index. Then it is well known that its inverse function f^{-1} exists and belongs to the same class. In other words, this class of functions is invariant with respect to the transformations $f \mapsto f^{-1}$ and $f^{-1} \mapsto f$. It turns out that there are natural extensions of this class that are invariant with respect to the transformations $f \mapsto f^{\tilde{}}$ and $f^{\tilde{}} \mapsto f$. The descriptions of these classes are obtained here as a byproduct of our main results.

The paper is organized as follows. In Section 2, we recall the definitions and some properties of various classes of regularly varying functions and their extensions. Also in this section, asymptotic quasi-inverse functions are discussed. In Section 3, we study relations between the limits of the ratio of two functions and the limits of the ratio of their quasi-inverse functions. Some known results on the connections of the asymptotic properties of f and $f^{\tilde{}}$, respectively, are discussed in Section 4. Then, our new results about these connections of f and $f^{\tilde{}}$ are presented; the proofs are given in Section 5. In Section 6, we briefly introduce *asymptotic right quasi-inverses* and *asymptotic left quasi-inverses* and discuss their properties.

2. Classes of regularly varying functions and their extensions

Let $\mathbf{R}$ be the set of real numbers, and $\mathbf{R}_+$ the set of positive reals. Also let $\mathbb{F} = \mathbb{F}(\mathbf{R}_+)$ be the space of real-valued functions $f = (f(t), t \geq 0)$, and

$$\mathbb{F}_+ = \bigcup_{A>0} \{f \in \mathbb{F} : f(t) > 0, t \in [A, \infty)\}.$$

Thus $f \in \mathbb{F}_+$ if and only if f is eventually positive.

Let $\mathbb{F}^{(\infty)}$ be the space of functions $f \in \mathbb{F}_+$ such that

(i) $\sup_{0\leq t\leq T} f(t) < \infty$ for all $T > 0$;
(ii) $\limsup_{t\to\infty} f(t) = \infty$.

Furthermore, let $\mathbb{F}^{\infty}$, respectively $\mathbb{F}^{\infty}_{\mathrm{ndec}}$, be the spaces of functions $f \in \mathbb{F}^{(\infty)}$ such that $f(t) \to \infty$ as $t \to \infty$, respectively f is nondecreasing for large t. We also use the subspace $\mathbb{C}^{\infty}$ of continuous functions in $\mathbb{F}^{\infty}$.

Throughout the paper, "measurability" means "Lebesgue measurability".

For a given $f \in \mathbb{F}_+$, we introduce the *upper* and *lower limit functions*

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} \quad \text{and} \quad f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}, \qquad c > 0,$$

which take values in $[0, \infty]$.

2.1. **RV functions.** In a stimulating paper of 1930, Karamata [22] introduced the notion of *regular variation* and proved some fundamental theorems for *regularly varying functions* (see also Karamata [23]). These results (together with later extensions and generalizations) turned out to be fruitful in various fields of mathematics (cf. Seneta [31] and Bingham et al. [7] for excellent surveys on this topic and for the history of the theory and its applications).

Recall that a measurable function $f \in \mathbb{F}_+$ is called *regularly varying* (RV) if

$$f_*(c) = f^*(c) = \varkappa(c) \in (0, \infty)$$

for all $c > 0$. For any RV function f, $\varkappa(c) = c^\rho$, $c > 0$, for some real number ρ, which is called the *index* of the function f. The case $\rho = 0$ corresponds to a (so-called) *slowly varying* (SV) function.

Denote by $\mathcal{RV}$ the class of all RV functions, by $\mathcal{RV}_\rho$ the class of all RV functions with index ρ, and by $\mathcal{RV}_+$ the class of all RV functions with positive index. In particular, $\mathcal{RV}_0$ denotes the class of SV functions.

The boundary case $\rho = \infty$ corresponds to functions f for which

$$\lim_{t\to\infty} \frac{f(ct)}{f(t)} = \infty \quad \text{for all } c > 1.$$

The functions $f \in \mathcal{RV}_\infty$ are called *rapidly varying* functions.

2.2. **ORV functions.** After the work of Karamata, various generalizations of the notion of regular variation appeared in the literature. One of them is a generalization due to Avakumović [2], which was further investigated by Karamata [24], Feller [18], and Aljančić and Arandelović [1]. The functions studied by these and several other authors are known in the literature as *O-regularly varying* (ORV) functions. Bari and Stechkin [3], for example, independently studied ORV functions and their applications in the theory of best function approximation.

A measurable function $f \in \mathbb{F}_+$ is called *O-regularly varying* if

$$f^*(c) < \infty \quad \text{for all } c > 0. \tag{4}$$

Recall that monotone ORV functions are known in the literature as *dominatedly varying*, and a monotone function $f \in \mathbb{F}_+$ is an ORV function if and only if (4) holds for some $c > 1$ (see Feller [18]).

2.3. **OSV functions.** Drasin and Seneta [17] introduced the so-called *O-slowly varying* (OSV) functions. An ORV function f is called OSV if

$$\sup_{c>0} f^*(c) < \infty.$$

Let $\mathcal{OSV}$ denote the class of OSV functions. Any slowly varying function f belongs to $\mathcal{OSV}$, since $f^*(c) = 1$, $c > 0$. OSV functions play an important role in integral representation theorems for functions with a nondegenerate group of regular points (see Buldygin et al. [9]).

2.4. **PRV functions.** For any RV function f, we have $f^*(c) \to 1$ as $c \to 1$. This condition is characteristic for a wider class of measurable functions $f \in \mathbb{F}_+$, the so-called *pseudo-regularly varying functions* (PRV). Note that every PRV function is an ORV function, but a quickly varying function is not PRV.

PRV functions and their various applications have been studied by Korenblyum [26], Matuszewska [27], Matuszewska and Orlicz [28], Stadtmüller and Trautner [32, 33], Berman [4, 5], Yakymiv [35], Cline [14], Djurčić [15], Djurčić and Torgašev [16], Klesov et al. [25], and Buldygin et al. [8], [10]–[13]. Note that the PRV functions are called *regularly oscillating* in Berman [4], *weakly oscillating* in Yakymiv [35], *intermediate regularly varying* in Cline [14], and CRV in Djurčić [15]. We use the term PRV introduced in Buldygin et al. [8].

One of the well-known properties of PRV functions is that they and only they preserve the equivalence of sequences (cf., e.g., Buldygin et al. [8]). Recall that a function f *preserves the equivalence of sequences* if

$$\lim_{n\to\infty} \frac{f(u_n)}{f(v_n)} = 1$$

for all sequences of positive numbers $\{u_n\}$ and $\{v_n\}$ such that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$$

and $\lim_{n\to\infty} u_n/v_n = 1$.

Lemma 2.1 ([8])**.** *A measurable function $f \in \mathbb{F}_+$ preserves the equivalence of sequences if and only if it is PRV.*

In what follows we will denote by $\mathcal{ORV}$ $(\mathcal{OSV}, \mathcal{PRV})$ the set of all ORV (OSV, PRV) functions. Note that

$$\mathcal{RV}_0 \subset \mathcal{RV} \subset \mathcal{PRV} \subset \mathcal{ORV}. \tag{5}$$

2.5. **SQI functions.** Let $\mathcal{SQI}$ denote the subclass of measurable functions $f \in \mathbb{F}_+$ satisfying the following condition:

$$f_*(c) > 1 \quad \text{for all } c > 1.$$

We call it the class of *sufficiently quickly increasing* (SQI) functions. These functions have also been used by Yakymiv [34], Djurčić and Torgašev [16], and Buldygin et al. [8]–[13]. Note that a slowly varying function is not SQI.

2.6. **PI functions.** Consider the following condition:

$$f_*(c_0) > 1 \quad \text{for some } c_0 > 1. \tag{6}$$

Note that condition (6) has been used by Bingham and Goldie [6], de Haan and Stadtmüller [21], Buldygin et al. [8, 11], and Rogozin [30]. The functions with property (6) are called *positively increasing* (PI).

Let $\mathcal{PI}$ be the subclass of measurable functions in $\mathbb{F}_+$ that satisfy condition (6). Then the following inclusions hold:

$$\mathcal{RV}_\infty \subset \mathcal{RV}_+ \cup \mathcal{RV}_\infty \subset \mathcal{SQI} \subset \mathcal{PI}. \tag{7}$$

2.7. **Functions with nondegenerate groups of regular points.** Consider $f \in \mathbb{F}_+$. A number $\lambda > 0$ is called a *regular point* of the function f, denoted $\lambda \in \mathbb{G}_r(f)$, if $f_*(\lambda) = f^*(\lambda) \in (0, \infty)$; that is, the limit

$$\kappa_f(\lambda) = \lim_{t\to\infty} \frac{f(\lambda t)}{f(t)}$$

exists, and is positive and finite. The set $\mathbb{G}_r(f)$ of regular points of f is a multiplicative subgroup of $\mathbf{R}_+$ with $1 \in \mathbb{G}_r(f)$. If $\mathbb{G}_r(f) = \{1\}$, then $\mathbb{G}_r(f)$ is called *degenerate*; otherwise, it is *nondegenerate*.

An ORV function f is called *a function with nondegenerate group of regular points* if $\mathbb{G}_r(f)$ is nondegenerate (cf. Buldygin et al. [9]). It turns out that functions with nondegenerate groups of regular points serve as normalizing sequences in problems connected with partial attraction to certain limit laws (cf., e.g., Grinevich and Khokhlov [19]).

2.8. **Asymptotic quasi-inverse functions.** The notion of an asymptotic quasi-inverse function is a natural generalization of the notion of an inverse function. Let $f \in \mathbb{F}^{(\infty)}$. If f is continuous and strictly increasing, then its *inverse function* f^{-1} exists and is determined by the following two properties:

(i) $f(f^{-1}(t)) = t$, $t \in [f(t_0), \infty)$, and
(ii) $f^{-1}(f(t)) = t$, $t \in [t_0, \infty)$.

However, if f is either discontinuous or is not strictly increasing, then its inverse function f^{-1} does not exist. Therefore we consider *asymptotic quasi-inverse functions* $f^{\sim}$ defined, for a given function f, by the properties $f(f^{\sim}(t)) \sim t$ and $f^{\sim}(t) \to \infty$ as $t \to \infty$ (cf. Buldygin et al. [10]). (Here $f(t) \sim g(t)$ as $t \to \infty$ means that $f(t)/g(t) \to 1$ as $t \to \infty$.) That is, we only keep condition (i) in the definition of inverse functions in an asymptotic sense. If, in addition, $f^{\sim}(f(t)) \sim t$ as $t \to \infty$, then $f^{\sim}$ is called an *asymptotic inverse function* of f (cf. Bingham et al. [7]). In Section 6, we also introduce *asymptotic right quasi-inverse functions*, satisfying condition (i) above, and *asymptotic left quasi-inverse functions*, for which only condition (ii) holds true.

Note that an asymptotic quasi-inverse function is not unique, even when the original function is continuous and strictly increasing. On the other hand, not all functions f have an asymptotic quasi-inverse function. However, for continuous functions f, and in various other cases, both their *generalized inverse functions* $f^{\leftarrow}(s) = \inf\{t \in [t_0, \infty) \colon f(t) > s\}$ and $\psi(s) = \sup\{t \ge 0 \colon f(t) \le s\}$, and some other versions of those, are asymptotic quasi-inverse functions (see Buldygin et al. [10]).

The book by Resnick [29] is a standard reference for the properties of generalized inverse functions, in particular in relation to regular variation.

For a given f, let $\mathbb{AQI}(f)$ be the set of all its asymptotic quasi-inverse functions.

3. On limits of the ratio of functions and their quasi-inverse functions

Let $\mathbb{LP}(\varphi)$ be the set of limit points of a function φ, and let $\mathbb{LP}(\{z_n\})$ be the set of limit points of a sequence $\{z_n\}$. Put

$$l_*(c; f, \{z_n\}) = \liminf_{n\to\infty} \frac{z_n}{f(cf^{\sim}(z_n))}, \qquad l^*(c; f, \{z_n\}) = \limsup_{n\to\infty} \frac{z_n}{f(cf^{\sim}(z_n))}, \qquad c > 0.$$

Lemma 3.1. *Let f, $g \in \mathbb{F}^{(\infty)}$, $f^{\sim} \in \mathbb{AQI}(f)$, and $g^{\sim} \in \mathbb{AQI}(g)$. Assume that f is PRV. If*

$$b \in \mathbb{LP}(g^{\sim}/f^{\sim}) \cap \mathbf{R}_+,$$

then, for any sequence of positive numbers $\{s_n\}$ such that $s_n \to \infty$ and

$$\lim_{n\to\infty} \frac{g^{\sim}(s_n)}{f^{\sim}(s_n)} = b, \tag{8}$$

one has

$$l_*(b; f, \{s_n\}) \in \mathbb{LP}(g/f), \qquad l^*(b; f, \{s_n\}) \in \mathbb{LP}(g/f).$$

Proof of Lemma 3.1. Since, by Lemma 2.1, the function f preserves the equivalence of sequences, we have, in view of (8), the following implications:

$$\begin{aligned} \lim_{n\to\infty} \frac{g^{\sim}(s_n)}{bf^{\sim}(s_n)} = 1 \Longrightarrow \lim_{n\to\infty} \frac{f(g^{\sim}(s_n))}{f(bf^{\sim}(s_n))} = 1 \Longrightarrow \lim_{n\to\infty} \frac{f(bf^{\sim}(s_n))}{f(g^{\sim}(s_n))} = 1 \\ \Longrightarrow \lim_{n\to\infty} \left(\frac{s_n}{f(g^{\sim}(s_n))}\right)\left(\frac{f(bf^{\sim}(s_n))}{s_n}\right) = 1 \\ \Longrightarrow \lim_{n\to\infty} \left(\frac{g(t_n)}{f(t_n)}\right)\left(\frac{f(bf^{\sim}(s_n))}{s_n}\right) = 1, \end{aligned}$$

where $t_n = g^{\sim}(s_n) \to \infty$. Thus

$$\liminf_{n\to\infty} \frac{g(t_n)}{f(t_n)} = 1 \Big/ \limsup_{n\to\infty} \frac{f(bf^{\sim}(s_n))}{s_n} = l_*(b; f, \{s_n\})$$

and

$$\limsup_{n\to\infty} \frac{g(t_n)}{f(t_n)} = 1 \Big/ \liminf_{n\to\infty} \frac{f(bf^{\sim}(s_n))}{s_n} = l^*(b; f, \{s_n\}).$$

This proves Lemma 3.1, since

$$\liminf_{n\to\infty} \frac{g(t_n)}{f(t_n)} \in \mathbb{LP}(g/f), \qquad \limsup_{n\to\infty} \frac{g(t_n)}{f(t_n)} \in \mathbb{LP}(g/f). \qquad \square$$

Corollary 3.1. *Let* $f, g \in \mathbb{F}^{(\infty)}$, $f \in \mathcal{PRV}$, $f^{\sim} \in \mathbb{AQI}(f)$, *and let* $g^{\sim} \in \mathbb{AQI}(g)$. *Then*

$$1 \in \mathbb{LP}(g^{\sim}/f^{\sim}) \Longrightarrow 1 \in \mathbb{LP}(g/f).$$

Corollary 3.2. *Let* $f, g \in \mathbb{F}^{(\infty)}$, $f \in \mathcal{PRV}$, $f^{\sim} \in \mathbb{AQI}(f)$, $g^{\sim} \in \mathbb{AQI}(g)$, *and let* $f^{\sim} \in \mathbb{C}^{\infty}$. *If the limits*

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = a \in \mathbb{R}_+ \quad \text{and} \quad \lim_{s\to\infty} \frac{g^{\sim}(s)}{f^{\sim}(s)} = b \in \mathbb{R}_+ \tag{9}$$

exist, then also

$$\lim_{t\to\infty} \frac{f(t/b)}{f(t)} \in \mathbb{R}_+ \tag{10}$$

and

$$a = \lim_{t\to\infty} \frac{f(t/b)}{f(t)}. \tag{11}$$

Proof of Corollary 3.2. Note that, by Lemma 3.1 and (9),

$$l_*(b; f, \{s_n\}) = l^*(b; f, \{s_n\}) = \lim_{t\to\infty} \frac{g(t)}{f(t)}$$

for any sequence of positive numbers $\{s_n\}$ such that $s_n \to \infty$. This means that the limit

$$\lim_{t\to\infty} \frac{t}{f(bf^{\sim}(t))}$$

exists and

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = \lim_{t\to\infty} \frac{t}{f(bf^{\sim}(t))} = \lim_{t\to\infty} \frac{f(f^{\sim}(t))}{f(bf^{\sim}(t))}.$$

But

$$\lim_{t\to\infty} \frac{f(f^{\sim}(t))}{f(bf^{\sim}(t))} = \lim_{t\to\infty} \frac{f(t)}{f(bt)} = \lim_{t\to\infty} \frac{f(t/b)}{f(t)} \tag{12}$$

if $f^{\sim} \in \mathbb{C}^{\infty}$. Thus (10) and (11) are proved. □

The following theorem is similar to Corollary 3.2, but includes other conditions on the functions f and $f^{\sim}$.

Theorem 3.1. *Let $f \in \mathbb{F}^{\infty}_{\text{ndec}}$, $g \in \mathbb{F}^{(\infty)}$, $f^{\sim} \in \mathbb{AQI}(f)$, and $g^{\sim} \in \mathbb{AQI}(g)$. Assume that f is PRV and $f^{\sim}$ is nondecreasing. If* (9) *holds, then* (10) *and* (11) *follow.*

For the proof of Theorem 3.1 we need the following result.

Lemma 3.2. *Let $f \in \mathbb{F}^{\infty}_{\text{ndec}}$ and $f^{\sim} \in \mathbb{AQI}(f)$. Assume that $f^{\sim}$ is nondecreasing. Then, for some $c_0 > 0$, the limit*

$$l_1(c_0) = \lim_{t\to\infty} \frac{f(c_0 f^{\sim}(t))}{t} \in [0,\infty]$$

exists if and only if the limit

$$l_2(c_0) = \lim_{t\to\infty} \frac{f(c_0 t)}{f(t)} \in [0,\infty]$$

exists. In the latter case, $l_1(c_0) = l_2(c_0)$.

Proof of Lemma 3.2. First assume that the limit $l_2(c_0)$ exists. Then

$$\lim_{t\to\infty} \frac{f(c_0 t)}{f(t)} = \lim_{t\to\infty} \frac{f(c_0 f^{\sim}(t))}{f(f^{\sim}(t))} = \lim_{t\to\infty} \frac{f(c_0 f^{\sim}(t))}{t},$$

that is, the limit $l_1(c_0)$ exists and $l_1(c_0) = l_2(c_0)$.

Now assume that $l_1(c_0)$ exists. Consider a sequence of positive numbers $\{s_n\}$ tending to ∞. Since $f^{\sim}$ is nondecreasing, for any $n \geq 1$ there exist positive numbers t_n and τ_n such that $t_n \to \infty$, $t_n/\tau_n \to 1$, and $f^{\sim}(t_n) \leq s_n \leq f^{\sim}(\tau_n)$. Then

$$f^{\sim}(t_n) \leq s_n \leq f^{\sim}(\tau_n).$$

Hence

$$\begin{aligned} l_1(c_0) &= \liminf_{n\to\infty} \frac{f(c_0 f^{\sim}(t_n))}{t_n} = \liminf_{n\to\infty} \frac{f(c_0 f^{\sim}(t_n))}{\tau_n} = \liminf_{n\to\infty} \frac{f(c_0 f^{\sim}(t_n))}{f(f^{\sim}(\tau_n))} \\ &\leq \liminf_{n\to\infty} \frac{f(c_0 s_n)}{f(s_n)} \leq \limsup_{n\to\infty} \frac{f(c_0 s_n)}{f(s_n)} \leq \limsup_{n\to\infty} \frac{f(c_0 f^{\sim}(\tau_n))}{f(f^{\sim}(t_n))} \\ &= \limsup_{n\to\infty} \frac{f(c_0 f^{\sim}(\tau_n))}{t_n} = \limsup_{n\to\infty} \frac{f(c_0 f^{\sim}(\tau_n))}{\tau_n} = l_1(c_0), \end{aligned}$$

since f is nondecreasing. This means that, for any sequence of positive numbers $\{s_n\}$ tending to ∞,

$$\lim_{n\to\infty} \frac{f(c_0 s_n)}{f(s_n)} = l_1(c_0).$$

Hence $l_2(c_0)$ exists and $l_1(c_0) = l_2(c_0)$.

This completes the proof of Lemma 3.2. □

Proof of Theorem 3.1. It is the same as the proof of Corollary 3.2, just making use of Lemma 3.2 in (12). □

Remark 3.1. Assume the conditions of Corollary 3.2 or Theorem 3.1. Then (10) and (11) hold true. If (9) holds with $b = 1$, then (10) is trivial and, by (9) and (11), one has

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = \lim_{t\to\infty} \frac{g^{\sim}(t)}{f^{\sim}(t)} = 1.$$

But, if $b > 0$ and $b \neq 1$, then (10) is nontrivial, and in this case f is a function with a nondegenerate group of regular points (see Section 2). Note also that, by (11), $b \neq 1$ if $a \neq 1$.

Assume that (9) holds with $b \neq 1$. Then $1/b \in \mathbb{G}_r(f)$ (see Section 2) and

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = a = \lim_{t\to\infty} \frac{f(t/b)}{f(t)} = \left(\frac{1}{b}\right)^{\rho},$$

where ρ denotes the index of f, i.e.,

$$\rho = \lim_{\lambda\to\infty} \frac{\log f^*(\lambda)}{\log \lambda} = \lim_{t\to\infty} \frac{\log f(t)}{\log t}$$

(see Theorem 5.1, Corollary 5.3, and Corollary 7.2 in Buldygin et al. [9]). If $\rho > 0$, then

$$\lim_{t\to\infty} \frac{g^{\sim}(t)}{f^{\sim}(t)} = b = \left(\frac{1}{a}\right)^{1/\rho}$$

(cf. equality (3)).

4. Connections between the asymptotic properties of f and $f^{\sim}$

In the sequel, we assume that $f \in \mathbb{F}^{(\infty)}$, $f^{\sim}$ is an asymptotic quasi-inverse function of f, and both f and $f^{\sim}$ are measurable functions.

4.1. **Some earlier results.** It is well known that, for any $\rho > 0$,

$$f \in \mathcal{RV}_{1/\rho} \Longleftrightarrow f^{\sim} \in \mathcal{RV}_{\rho},$$

and if $f \in \mathcal{RV}_+$, then $f^{\sim}$ is an asymptotic inverse function of f (see, e.g., Bingham et al. [7], Theorem 1.5.12, and Buldygin et al. [11], Theorem 8.4). One can rewrite this assertion in a different form:

$$f \in \mathcal{RV}_+ \Longleftrightarrow f^{\sim} \in \mathcal{RV}_+. \tag{13}$$

In other words, asymptotic quasi-inverse functions are in the same class as their originals.

If f is nondecreasing, then

$$f \in \mathcal{SQI} \Longrightarrow f^{\sim} \in \mathcal{PRV},$$

and if, in addition, $f^{\sim}$ is an asymptotic inverse function of f, then

$$f \in \mathcal{SQI} \Longleftrightarrow f^{\sim} \in \mathcal{PRV}$$

(cf. Buldygin et al. [10], Proposition 7.1 and Theorem 7.1; for continuous functions f, see also Buldygin et al. [8], Proposition 6.1). On the other hand, if either f or $f^{\sim}$ is nondecreasing, then

$$f \in \mathcal{PRV} \Longrightarrow f^{\sim} \in \mathcal{SQI},$$

and if $f^{\sim}$ is an asymptotic inverse function of f and f is nondecreasing, then

$$f \in \mathcal{PRV} \Longleftrightarrow f^{\sim} \in \mathcal{SQI}$$

(cf. Buldygin et al. [11], Proposition 8.1, and Buldygin et al. [10], Corollary 7.1; for increasing and continuous functions f, see also Buldygin et al. [8]).

The following statement is similar to (13):

$$f \in \mathcal{SQI} \cap \mathcal{PRV} \Longleftrightarrow f^{\sim} \in \mathcal{SQI} \cap \mathcal{PRV},$$

and, if $f \in \mathcal{SQI} \cap \mathcal{PRV}$, then $f^{\sim}$ is an asymptotic inverse function of f (cf. Buldygin et al. [11], Theorem 8.4; for increasing and continuous functions f, see also Djurčić and Torgašev [16], and Buldygin et al. [8]).

The corresponding characterizations are open for the classes $\mathcal{ORV}$, $\mathcal{RV}_0$, $\mathcal{RV}_\infty$, and some other related classes such as $\mathcal{OSV}$.

4.2. **Some new results.** The following theorems hold true.

Theorem 4.1. *Let f be nondecreasing. Then*

$$f \in \mathcal{PI} \Longrightarrow f^{\sim} \in \mathcal{ORV}.$$

Theorem 4.2. *Let both functions f and $f^{\sim}$ be nondecreasing. Then*

(i) $f \in \mathcal{PI} \Longleftrightarrow f^{\sim} \in \mathcal{ORV}$;
(ii) $f \in \mathcal{ORV} \Longleftrightarrow f^{\sim} \in \mathcal{PI}$.

Theorem 4.3. *Let both functions f and $f^{\sim}$ be nondecreasing. Then*

(iii) $f \in \mathcal{RV}_\infty \Longleftrightarrow f^{\sim} \in \mathcal{RV}_0$;
(iv) $f \in \mathcal{RV}_0 \Longleftrightarrow f^{\sim} \in \mathcal{RV}_\infty$.

The last two theorems provide characterizations for the classes $\mathcal{RV}_0$, $\mathcal{RV}_\infty$, $\mathcal{ORV}$, and $\mathcal{PI}$. The next result follows from Theorem 4.2.

Theorem 4.4. *Let both functions f and $f^{\sim}$ be nondecreasing. Then*

$$f \in \mathcal{ORV} \cap \mathcal{PI} \Longleftrightarrow f^{\sim} \in \mathcal{ORV} \cap \mathcal{PI}.$$

The intersection of the corresponding classes in (5) and (7) gives

$$\mathcal{RV}_+ \subset \mathcal{PRV} \cap \mathcal{SQI} \subset \mathcal{ORV} \cap \mathcal{PI},$$

where all the classes $\mathcal{RV}_+$, $\mathcal{PRV} \cap \mathcal{SQI}$, and $\mathcal{ORV} \cap \mathcal{PI}$ are invariant with respect to both transformations $f \mapsto f^{\sim}$ and $f^{\sim} \mapsto f$.

Next we turn to the class of OSV functions. Consider the following condition:

$$\lim_{t\to\infty} \frac{f(c_0 f^{\sim}(t))}{t} = \infty \quad \text{for some } c_0 > 1. \tag{14}$$

It is obvious that (14) is equivalent to

$$\lim_{t\to\infty} \frac{f(c_0 t)}{f(t)} = \infty \quad \text{for some } c_0 > 1 \tag{15}$$

if $f^{\sim}$ is continuous. It is less obvious that (14) and (15) are equivalent if both functions f and $f^{\sim}$ are nondecreasing; see Lemma 3.2.

Let $\mathcal{QV}$ and $\mathcal{QV}_1$ be the subclasses of measurable functions of $\mathbb{F}^{(\infty)}$ satisfying conditions (14) and (15), respectively.

Theorem 4.5. *Let f be nondecreasing. Then $f \in \mathcal{QV} \Longleftrightarrow f^{\sim} \in \mathcal{OSV}$.*

Theorem 4.6. *Let both functions f and $f^{\sim}$ be nondecreasing. Then*

(v) $f \in \mathcal{QV}_1 \Longleftrightarrow f^{\sim} \in \mathcal{OSV}$;
(vi) $f \in \mathcal{OSV} \Longleftrightarrow f^{\sim} \in \mathcal{QV}_1$.

5. Proof of Theorems 4.1–4.6

Proof of Theorem 4.1. First assume that $f \in \mathcal{PI}$, but $f^{\sim} \notin \mathcal{ORV}$. Since $f^{\sim} \notin \mathcal{ORV}$, there exists a number $a > 0$ such that

$$(f^{\sim})^*(a) = \limsup_{t\to\infty} \frac{f^{\sim}(at)}{f^{\sim}(t)} = \infty. \tag{16}$$

Hence, for any $\varepsilon > 0$, there is an $a = a(\varepsilon) \in (0, 1+\varepsilon)$ such that (16) holds, since $(f^{\sim})^*(\sqrt{u}) \geq \sqrt{(f^{\sim})^*(u)}$ for every $u > 1$. This means that, for any $\varepsilon > 0$, there exist a number $a = a(\varepsilon) \in (0, 1+\varepsilon)$ and a sequence of positive numbers $\{s_n\}$, increasing to ∞, such that, for any $A > 0$,

$$f^{\leftarrow}(a(\varepsilon)s_n) \geq Af^{\leftarrow}(s_n) \tag{17}$$

for sufficiently large n.

Moreover, since $f \in \mathcal{PI}$, there exists a number $\delta > 0$ such that, for any sequence of positive numbers $\{t_n\}$ tending to ∞,

$$f(c_0 t_n) > (1+\delta) f(t_n) \tag{18}$$

for sufficiently large n, where c_0 is the constant from condition (6).

Now, by relations (17) and (18), for $\varepsilon = \delta$ and $A = c_0$, we have

$$\begin{aligned} 1+\delta > a(\delta) &= \lim_{n\to\infty} \frac{f(f^{\leftarrow}(a(\delta)s_n))}{s_n} \geq \limsup_{n\to\infty} \frac{f(c_0 f^{\leftarrow}(s_n))}{s_n} \\ &\geq (1+\delta) \lim_{n\to\infty} \frac{f(f^{\leftarrow}(s_n))}{s_n} = 1+\delta. \end{aligned}$$

This contradiction proves Theorem 4.1. □

The proof of Theorem 4.2 makes use of the following auxiliary result.

Lemma 5.1. *Let both functions f and $f^{\leftarrow}$ be nondecreasing. Then*

$$f^{\leftarrow} \in \mathcal{ORV} \Longrightarrow f \in \mathcal{PI}.$$

Proof of Lemma 5.1. Let $f^{\leftarrow} \in \mathcal{ORV}$, but $f \notin \mathcal{PI}$. Since $f^{\leftarrow} \in \mathcal{ORV}$, for any fixed $a > 1$, there exists a number $A > 1$ such that

$$f^{\leftarrow}(as) \leq Af^{\leftarrow}(s) \tag{19}$$

for sufficiently large s. Moreover, there is a sequence of positive numbers $\{s_n\}$, strictly increasing to ∞, such that

$$\lim_{n\to\infty} \frac{f(As_n)}{f(s_n)} = 1, \tag{20}$$

since $f \notin \mathcal{PI}$ and f is nondecreasing.

Put $t_n = \sup\{t \geq 0 \colon f^{\leftarrow}(t) \leq s_n\}$ and $\tau_n = \inf\{t \geq 0 \colon f^{\leftarrow}(t) > s_n\}$, $n \geq 1$ (here $\sup \varnothing := 0$). Then $t_n \to \infty$ and $t_n = \tau_n$, $n \geq 1$, since $f^{\leftarrow}$ is nondecreasing and unbounded.

Hence, for $u_n = t_n - 1$ and $v_n = t_n + 1$, $n \geq 1$, we have

$$f^{\leftarrow}(u_n) \leq s_n \leq f^{\leftarrow}(v_n), \qquad n \geq 1, \tag{21}$$

and

$$u_n \sim f(s_n) \sim v_n, \qquad n \to \infty, \tag{22}$$

since $f^{\leftarrow}$ and f are nondecreasing.

Thus, by (19)–(22),

$$\begin{aligned} 1 < a &= \lim_{n\to\infty} \frac{f(f^{\leftarrow}(au_n))}{u_n} \leq \limsup_{n\to\infty} \frac{f(Af^{\leftarrow}(u_n))}{u_n} \leq \limsup_{n\to\infty} \frac{f(As_n)}{u_n} \\ &= \limsup_{n\to\infty} \frac{f(As_n)}{f(s_n)} \frac{f(s_n)}{u_n} = \lim_{n\to\infty} \frac{f(As_n)}{f(s_n)} = 1. \end{aligned}$$

This contradiction proves Lemma 5.1. □

Proof of Theorem 4.2. Statement (i) follows from a combination of Theorem 4.1 and Lemma 5.1.

Next we prove statement (ii). First we assume that $f \notin \mathcal{ORV}$, but $f^{\leftarrow} \in \mathcal{PI}$. Since $f^{\leftarrow} \in \mathcal{PI}$, there exist numbers $c_0 > 1$ and $b > 1$ such that

$$f^{\leftarrow}(c_0 t) \geq bf^{\leftarrow}(t)$$

for sufficiently large t. Since f is nondecreasing, $f^*(a) = \infty$ for all $a > 1$, as in the proof of Theorem 4.1. Thus, given constants $a > 1$ and $A > 1$, there is an increasing sequence $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$ and

$$f(as_n) \geq Af(s_n), \qquad n \geq 1.$$

Now, we define sequences $\{u_n\}$ and $\{v_n\}$ as in the proof of Lemma 5.1 and choose $a = b$ and $A = 2c_0$. Then

$$\begin{aligned} c_0 = \lim_{n\to\infty} \frac{f\left(f^{\sim}(c_0 v_n)\right)}{v_n} &\geq \limsup_{n\to\infty} \frac{f\left(bf^{\sim}(v_n)\right)}{v_n} \geq \limsup_{n\to\infty} \frac{f\left(bs_n\right)}{v_n} \\ &\geq 2c_0 \limsup_{n\to\infty} \frac{f(s_n)}{v_n} = 2c_0. \end{aligned}$$

This contradiction proves that $f \in \mathcal{ORV}$.

Assume now that $f \in \mathcal{ORV}$, but $f^{\sim} \notin \mathcal{PI}$. Since $f \in \mathcal{ORV}$, for any $a > 1$, there exists an $A > 1$ such that

$$f(at) \leq Af(t)$$

for sufficiently large t. At the same time, since $f^{\sim} \notin \mathcal{PI}$, for any $c > 1$ and any number $b > 1$, there exists a sequence of positive numbers $\{s_n\}$, strictly increasing to ∞, such that

$$f^{\sim}(cs_n) \leq bf^{\sim}(s_n), \qquad n \geq 1.$$

For a fixed $a > 1$, choose $b = a$ and $c = 2A$. Then

$$2A \leq \lim_{n\to\infty} \frac{f(f^{\sim}(cs_n))}{s_n} \leq \limsup_{n\to\infty} \frac{f(af^{\sim}(s_n))}{s_n} \leq A \limsup_{n\to\infty} \frac{f(f^{\sim}(s_n))}{s_n} = A.$$

This contradiction proves the implication $f^{\sim} \in \mathcal{ORV} \Longrightarrow f \in \mathcal{PI}$, which completes the proof of (ii). □

Proof of Theorem 4.3. We first prove statement (iii). Assume that $f \in \mathcal{RV}_\infty$, but $f^{\sim} \notin \mathcal{RV}_0$. Since $f^{\sim} \notin \mathcal{RV}_0$, there exist numbers $c_0 > 1$, $a > 1$, and a sequence of positive numbers $\{t_n\}$, strictly increasing to ∞, such that

$$f^{\sim}(c_0 t_n) \geq af^{\sim}(t_n), \qquad n \geq 1.$$

Thus, since $f \in \mathcal{RV}_\infty$,

$$\infty = \lim_{n\to\infty} \frac{f\left(af^{\sim}(t_n)\right)}{f\left(f^{\sim}(t_n)\right)} \leq \lim_{n\to\infty} \frac{f\left(f^{\sim}(c_0 t_n)\right)}{f\left(f^{\sim}(t_n)\right)} = c_0.$$

This contradiction proves the implication $f \in \mathcal{RV}_\infty \Longrightarrow f^{\sim} \in \mathcal{RV}_0$.

Now assume that $f^{\sim} \in \mathcal{RV}_0$, but $f \notin \mathcal{RV}_\infty$. Since $f^{\sim} \in \mathcal{RV}_0$, for all $c > 1$ and all $a > 1$,

$$f^{\sim}(ct) \leq af^{\sim}(t)$$

for sufficiently large t.

Moreover, as $f \notin \mathcal{RV}_\infty$, there exist numbers $\lambda > 1$ and $A > 0$, and a sequence $\{s_n\}$, tending to ∞, such that

$$f(\lambda s_n) \leq Af(s_n), \qquad n \geq 1.$$

Choose $a = \lambda$, $c > A$, and sequences $\{u_n\}$ and $\{v_n\}$ as in the proof of Lemma 5.1. Then

$$\begin{aligned} c = \lim_{n\to\infty} \frac{f\left(f^{\sim}(cu_n)\right)}{u_n} &\leq \limsup_{n\to\infty} \frac{f\left(af^{\sim}(u_n)\right)}{u_n} \leq \limsup_{n\to\infty} \frac{f\left(as_n\right)}{u_n} \\ &\leq A \limsup_{n\to\infty} \frac{f\left(s_n\right)}{u_n} = A. \end{aligned}$$

This contradiction completes the proof of (iii).

Next we prove statement (iv). First assume that $f \in \mathcal{RV}_0$, but $f^{\sim} \notin \mathcal{RV}_\infty$. Since $f^{\sim} \notin \mathcal{RV}_\infty$, there are numbers $c > 1$, $a > 0$, and a sequence of positive numbers $\{t_n\}$, strictly increasing to ∞, such that

$$f^{\sim}(ct_n) \le af^{\sim}(t_n), \qquad n \ge 1,$$

whence $f\left(f^{\sim}(ct_n)\right) \le f\left(af^{\sim}(t_n)\right)$ for all $n \ge 1$. Moreover, since $f \in \mathcal{RV}_0$,

$$1 < c = \lim_{n\to\infty} \frac{f\left(f^{\sim}(ct_n)\right)}{t_n} \le \lim_{n\to\infty} \frac{f\left(af^{\sim}(t_n)\right)}{t_n} = \lim_{n\to\infty} \frac{f\left(f^{\sim}(t_n)\right)}{t_n} = 1.$$

This contradiction proves the implication $f \in \mathcal{RV}_0 \Longrightarrow f^{\sim} \in \mathcal{RV}_\infty$.

Now assume that $f^{\sim} \in \mathcal{RV}_\infty$, but $f \notin \mathcal{RV}_0$. Since $f \notin \mathcal{RV}_0$, there exist $c_0 > 1$, $a > 1$, and a sequence of positive numbers $\{s_n\}$, strictly increasing to ∞, such that

$$f(c_0 s_n) \ge af(s_n), \qquad n \ge 1. \tag{23}$$

Let $c > c_0$. Then, since $f^{\sim} \in \mathcal{RV}_\infty$, for every $\lambda > 1$

$$f^{\sim}(\lambda t) \ge cf^{\sim}(t). \tag{24}$$

Choose two sequences of positive numbers $\{u_n\}$ and $\{v_n\}$, increasing to ∞, such that relations (21) and (22) hold (confer the proof of Lemma 5.1 for details). On account of (24), for $t = v_n$, and using (21) and (23), we get

$$f\left(f^{\sim}(\lambda v_n)\right) \ge f\left(cf^{\sim}(v_n)\right) \ge f(c_0 s_n) \ge af(s_n)$$

for sufficiently large n. Thus, by (22),

$$1 = \limsup_{n\to\infty} \frac{f(s_n)}{v_n} = \limsup_{n\to\infty} \frac{\lambda f(s_n)}{f\left(f^{\sim}(\lambda v_n)\right)} \le \frac{\lambda}{a}$$

for any $\lambda > 1$. Letting $\lambda \downarrow 1$ we get a contradiction. This proves the implication $f^{\sim} \in \mathcal{RV}_\infty \Longrightarrow f \in \mathcal{RV}_0$ and completes the proof of (iv). □

Proof of Theorem 4.5. Let $f^{\sim} \in \mathcal{OSV}$. Then there exists a number $A > 0$ such that $f^{\sim}(ct) \le Af^{\sim}(t)$ for all $c > 0$ and sufficiently large t, whence $f(f^{\sim}(ct)) \le f(Af^{\sim}(t))$ and

$$c = \lim_{t\to\infty} \frac{f(f^{\sim}(ct))}{t} \le \liminf_{t\to\infty} \frac{f(Af^{\sim}(t))}{t}.$$

Since c is arbitrary, (14) holds with $c_0 = A$. Thus $f \in \mathcal{QV}$ and the implication

$$f^{\sim} \in \mathcal{OSV} \Longrightarrow f \in \mathcal{QV}$$

is proved.

Now assume that $f \in \mathcal{QV}$, but $f^{\sim} \notin \mathcal{OSV}$. Then, for any $A > 0$, there exist a number $c > 0$ and a sequence of positive numbers $\{t_n\}$ tending to ∞, such that

$$f^{\sim}(ct_n) \ge Af^{\sim}(t_n), \qquad n \ge 1.$$

In particular, for $A = c_0$, there exist $c > 0$ and $\{t_n\}$ such that $f^{\sim}(ct_n) \ge c_0 f^{\sim}(t_n)$, $n \ge 1$, whence $f(f^{\sim}(ct_n)) \ge f(c_0 f^{\sim}(t_n))$, $n \ge 1$, and

$$c = \lim_{n\to\infty} \frac{f(f^{\sim}(ct_n))}{t_n} \ge \liminf_{n\to\infty} \frac{f(c_0 f^{\sim}(t_n))}{t_n} \ge \liminf_{t\to\infty} \frac{f(c_0 f^{\sim}(t))}{t},$$

which contradicts the assumption that $f \in \mathcal{QV}$. Thus $f^{\sim} \in \mathcal{OSV}$ and the implication $f \in \mathcal{QV} \Longrightarrow f^{\sim} \in \mathcal{OSV}$ is proved. □

Proof of Theorem 4.6. Statement (v) follows from a combination of Theorem 4.5 and Lemma 3.2.

Let $f^{\sim} \in \mathcal{QV}_1$. Then there exists a number c_0 such that

$$\lim_{t\to\infty} \frac{f^{\sim}(c_0 t)}{f^{\sim}(t)} = \infty.$$

Thus, for every $A > 0$,

$$f^{\sim}(c_0 t) \geq A f^{\sim}(t)$$

for all sufficiently large t.

Since f is nondecreasing,

$$c_0 = \lim_{t\to\infty} \frac{f(f^{\sim}(c_0 t))}{t} \geq \limsup_{t\to\infty} \frac{f(Af^{\sim}(t))}{t} \tag{25}$$

for all $A > 0$. Next we show that

$$\sup_{A>0} f^*(A) \leq c_0.$$

Otherwise, there is an $A_0 > 0$ such that $f^*(A_0) > c_0$. Pick B with $c_0 < B < f^*(A_0)$ and choose a sequence of positive numbers $\{s_n\}$, strictly increasing to ∞, such that

$$f(A_0 s_n) \geq B f(s_n), \qquad n \geq 1.$$

Upon defining $\{u_n\}$ and $\{v_n\}$ as in the proof of Lemma 5.1, we have

$$f(A_0 f^{\sim}(v_n)) \geq f(A_0 s_n) \geq B f(s_n) \geq B f(f^{\sim}(u_n)),$$

and, in view of (22) and (25),

$$c_0 \geq \limsup_{n\to\infty} \frac{f(A_0 f^{\sim}(v_n))}{v_n} \geq B \limsup_{n\to\infty} \frac{f(f^{\sim}(v_n))}{v_n} = B.$$

This contradiction proves that $f^{\sim} \in \mathcal{QV}_1 \implies f \in \mathcal{OSV}$.

To prove the converse, let $f \in \mathcal{OSV}$ and $f^{\sim} \notin \mathcal{QV}_1$. Then there exists a number $c_0 \geq 1$ such that $f^*(A) \leq c_0$ for every $A > 0$. Now, $f^{\sim} \notin \mathcal{QV}_1$ implies that, for any given $c > 1$,

$$\liminf_{t\to\infty} \frac{f^{\sim}(ct)}{f^{\sim}(t)} < \infty.$$

Then we find $B > 0$ such that

$$f^{\sim}(ct) \leq B f^{\sim}(t)$$

for infinitely many large t. On choosing $c > c_0$, we get

$$c = \lim_{t\to\infty} \frac{f(f^{\sim}(ct))}{t} \leq \liminf_{t\to\infty} \frac{f(Bf^{\sim}(t))}{f(f^{\sim}(t))} \leq \limsup_{s\to\infty} \frac{f(Bs)}{f(s)} \leq c_0.$$

This contradiction proves that $f \in \mathcal{OSV} \implies f^{\sim} \in \mathcal{QV}_1$.

Thus, statement (vi) is proved. □

6. Asymptotic right and left quasi-inverse functions

Let $f \in \mathbb{F}^{(\infty)}$. As discussed above, an asymptotic quasi-inverse function $f^{\sim}$ for a given function f is characterized by the property

$$(f \circ f^{\sim})(t) \sim t \quad \text{as } t \to \infty, \tag{26}$$

where the symbol $\circ$ stands for the superposition of functions. For obvious reasons, any function $f_r^{\sim}$ satisfying (26), with $f_r^{\sim}$ instead of $f^{\sim}$ and such that $f_r^{\sim}(t) \to \infty$ as $t \to \infty$, can be called an *asymptotic right quasi-inverse function* (of f).

Another asymptotic property, inherited from the definition of an inverse function, is that

$$(f_{\ell}^{\sim} \circ f)(t) \sim t \quad \text{as } t \to \infty. \tag{27}$$

Any $f_{\ell}^{\sim}$ satisfying (27) and such that

$$f_{\ell}^{\sim}(t) \to \infty \quad \text{as } t \to \infty$$

may be called an *asymptotic left quasi-inverse function* (of f). The generalized inverse function $f^{\leftarrow}$, for example, is one version of both, an asymptotic right quasi-inverse function of f if f is continuous, and an asymptotic left quasi-inverse function of f if f is strictly increasing.

Generally, if g is an asymptotic right quasi-inverse function of f, then f is an asymptotic left quasi-inverse function of g, and *vice versa*. In view of this symmetry, Theorems 4.2–4.5 can be reformulated as follows.

Theorem 6.1. *Let both functions f and $f_{\ell}^{\sim}$ be nondecreasing. Then*

(a) $f \in \mathcal{PI} \iff f_{\ell}^{\sim} \in \mathcal{ORV}$;
(b) $f \in \mathcal{ORV} \iff f_{\ell}^{\sim} \in \mathcal{PI}$;
(c) $f \in \mathcal{ORV} \cap \mathcal{PI} \iff f_{\ell}^{\sim} \in \mathcal{ORV} \cap \mathcal{PI}$.

Theorem 6.2. *Let both functions f and $f_{\ell}^{\sim}$ be nondecreasing. Then*

(a) $f \in \mathcal{RV}_{\infty} \iff f_{\ell}^{\sim} \in \mathcal{RV}_0$;
(b) $f \in \mathcal{RV}_0 \iff f_{\ell}^{\sim} \in \mathcal{RV}_{\infty}$.

Theorem 6.3. *Let both functions f and $f_{\ell}^{\sim}$ be nondecreasing. Then*

(a) $f \in \mathcal{OSV} \iff f_{\ell}^{\sim} \in \mathcal{QV}_1$;
(b) $f \in \mathcal{QV}_1 \iff f_{\ell}^{\sim} \in \mathcal{OSV}$.

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