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ANALYTICAL PROBLEMS OF THE ASYMPTOTIC BEHAVIOR OF MARKOV FUNCTIONALS. I

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Abstract. Some results are given for the asymptotic behavior of Markov functionals of a homogeneous ergodic Markov process.

1. Introduction

Let $X(t)$ be a homogeneous Markov ergodic process [1] with either discrete or continuous time, and let $X(t)$ take values in a phase space $(E, \mathcal{B})$. For the sake of simplicity we assume that E is a complete separable metric space, that $\mathcal{B}$ is the σ-algebra of Borel sets, and that $X(t)$ is a stochastically continuous process in the case of continuous time. The transient probability and invariant probability distribution of the process $X(t)$ are denoted by $P(t, x, A)$, $t \geq 0$, $x \in E$, $A \in \mathcal{B}$, and $\pi(A)$, $A \in \mathcal{B}$, respectively.

Consider a family $\xi_\varepsilon(t)$ of Markov functionals that depends on a small parameter $\varepsilon > 0$ and is asymptotically degenerate. The latter means that

$$\xi_\varepsilon(0) = \xi(0), \qquad \varepsilon > 0, \tag{1}$$

$$\lim_{\varepsilon \to 0} \mathbb{P}_{x,i}\{\xi_\varepsilon(t) \neq i\} = 0 \tag{2}$$

for all $x \in E$, $i \in I$, and $t \geq 0$. Here $\mathbb{P}_{x,i}$ is the regular conditional probability given $X(0) = x$, $\xi(0) = i$.

Conditions (1)–(2) are equivalent to the condition that the joint distribution of the vector $\{X(t), \xi_\varepsilon(t)\}$ converges as $\varepsilon \to 0$ to the distribution of the vector $\{X(t), \xi(0)\}$.

Below we consider Markov functionals with a countable space of states $I = \{1, 2, \dots\}$. In the case of continuous time, we assume that the trajectories of $\{X(t), \xi(t)\}$ are right continuous and have limits on the left with probability one.

The process $X(t)$ is assumed to be aperiodic in the case of continuous time; that is,

$$\lim_{t \to \infty} \int_E \varphi(y)\, P_t(x, dy) = \int_E \pi(dy)\varphi(y) \tag{3}$$

for all $x \in E$ and all continuous bounded functions $\varphi(y)$. If the time is discrete, we additionally require that

$$\lim_{k \to \infty} P_k(x, A) = \pi(A) \tag{4}$$

for all $x \in E$ uniformly in $A \in \mathcal{B}$.

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One can prove that there exists a Markov moment τ and a set $D \in \mathcal{B}$ such that $\pi(D) > 0$ and

$$\mathbb{P}_x\{X(\tau) \in A\} = \pi_D(A), \tag{5}$$

$$\sup_{x\in D}[1 - \mathbb{P}_{x,i}\{\xi_\varepsilon(\tau) = i\}] \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{6}$$

$$\sup_{x\in D} \mathbb{P}(\tau, \tau > T) \underset{T\to\infty}{\longrightarrow} 0, \tag{7}$$

$$\tau \geq 1 \qquad \mathbb{P}_x\text{-almost surely} \tag{8}$$

for all $i \in I$, where $\pi_D(A) = \pi(AD)/\pi(D)$.

Put $m = \mathbb{P}_{\pi_D,i}\tau$. Here and in what follows,

$$\mathbb{P}_{\pi_D,i}(\cdot) = \int_D \pi_D(dx)\mathbb{P}_{x,i}(\cdot).$$

2. Main results

The first result of this section deals with the case of continuous time.

Theorem 1. *Let conditions* (1), (2), *and* (3) *hold. Assume also that*

$$\sup_{\varepsilon>0} \mathbb{P}_{x,i}\{\xi_\varepsilon(t) = i, \xi_\varepsilon(s) \neq i\} \underset{s\to t}{\longrightarrow} 0 \tag{9}$$

for all $t \geq 0$, $x \in E$, and $i \in I$. If there exists a matrix $C = \|c_{ij}\|_{i,j=1}^{\infty}$ such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon m}[\mathbb{P}_{\pi_D,i}\{\xi_\varepsilon(\tau) = i\} - 1] = c_{ii}, \tag{10}$$

$$\lim_{\varepsilon\to 0} \sum_{j\neq i} \left[\frac{1}{\varepsilon m}\mathbb{P}_{\pi_D,i}\{\xi_\varepsilon(\tau) = j\} - c_{ij}\right] = 0, \tag{11}$$

$$\sum_j c_{ij} = 0, \qquad \sup_i |c_{ii}| < \infty, \tag{12}$$

then

$$\mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = j] - p_{ij}(u) \int_E \pi(dy)\varphi(y) \underset{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow} 0 \tag{13}$$

for all $x \in E$, $i \in I$, and all continuous bounded functions $\varphi(y)$ where $p_{ij}(u)$ is the entry (i, j) of the matrix e^{uC}, $C = \|c_{ij}\|_{i,j=1}^{\infty}$.

Remark 1. Assume that

$$-\infty < c < c_{ii} \quad \text{for all } i \in I \tag{14}$$

instead of (12). Put $-c_{ii} = \lambda_i$ and $c_{ij} = \lambda_i\pi_{ij}$ for $i \neq j$. If $\lambda_i > 0$, then we let $\pi_{ii} = 0$; otherwise, that is, if $\lambda_i = 0$, we let $\pi_{ij} = 0$ for $i \neq j$ and $\pi_{ii} = 1$ for all i. Then

$$p_{ij}(u) = \sum_{n=0}^{\infty} p_{ij}^{(n)}(u)$$

under assumption (13) where

$$p_{ij}^0(u) = \delta^{ij}e^{-\lambda_i u},$$

$$p_{ij}^{(n+1)}(u) = \sum_{k\neq i} \int_0^u \lambda_i e^{-\lambda_i(u-s)}\pi_{ik}p_{kj}^{(n)}(s)\, ds, \qquad n = 0, 1, 2, \ldots .$$

Proof. Fix a bounded continuous function $\varphi(x) \geq 0$ and a state $l \in \{1, 2, \dots\}$. Let

$$f^i_\varepsilon(x,t) = \mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = l].$$

By the law of total probability,

$$\begin{aligned} f^i_\varepsilon(x,t) &= \mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = l, t < \tau] \\ &\quad + \int_E \int_0^t \sum_j \mathbb{P}_{x,i}\big\{X(\tau) \in dy, \tau \in du, \xi_\varepsilon(\tau) = j\big\} f^j_\varepsilon(y, t-u). \end{aligned} \tag{15}$$

For $i \neq j$, put

$$\begin{aligned} Q^i_\varepsilon(x, dy \times du) &= \mathbb{P}_{x,i}\{X(\tau) \in dy, \tau \in du, \xi_\varepsilon(\tau) = i\}, \\ Q^{ij}_\varepsilon(x, dy \times du) &= \mathbb{P}_{x,i}\{X(\tau) \in dy, \tau \in du, \xi_\varepsilon(\tau) = j\}, \\ g^i_\varepsilon(x,t) &= \mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = l, t < \tau]. \end{aligned}$$

Now equation (15) can be rewritten as follows

$$f^i_\varepsilon(x,t) = g^i_\varepsilon(x,t) + Q^i_\varepsilon * f^i_\varepsilon(x,t) + \sum_{j \neq i} Q^{ij}_\varepsilon * f^j_\varepsilon(x,t). \tag{16}$$

This implies that

$$f^i_\varepsilon(x,t) = H^i_\varepsilon * g^i_\varepsilon(x,t) + \sum_{j \neq i} R^{ij}_\varepsilon * f^j_\varepsilon(x,t), \tag{17}$$

where $H^i_\varepsilon(x, dy \times dt)$ is the potential of the kernel $Q^i_\varepsilon(x, dy \times dt)$, that is,

$$\begin{aligned} H^i_\varepsilon(x, dy \times dt) &= \sum_{r=0}^{\infty} (Q^i_\varepsilon)^{*r}(x, dy \times dt), \\ R^{ij}_\varepsilon(x, dy \times dt) &= H^i_\varepsilon * Q^{ij}_\varepsilon(x, dy \times dt), \qquad i \neq j. \end{aligned}$$

First we consider the case of

$$\sup_{x \in E} \sum_{j \neq i} R^{ij}_\varepsilon(x, E \times [0, \infty)) \leq r < 1 \tag{18}$$

for all sufficiently small $\varepsilon > 0$ and all $i \in I$.

Put

$$h^i_\varepsilon(x,t) = H^i_\varepsilon * f^i_\varepsilon(x,t)$$

and define the sequence of functions $h^i_{\varepsilon,k}(x,t)$ by

$$\begin{aligned} h^i_{\varepsilon,0}(x,t) &= h^i_\varepsilon(x,t), \\ h^i_{\varepsilon,k+1}(x,t) &= \sum_{j \neq i} R^{ii}_\varepsilon * h^j_{\varepsilon,k}(x,t). \end{aligned} \tag{19}$$

Starting with

$$f^i_{\varepsilon,0}(x,t) = f^i_\varepsilon(x,t)$$

we similarly define the sequence of functions $f^i_{\varepsilon,k}(x,t)$. It is clear that

$$f^i_\varepsilon(x,t) = \sum_{k=0}^{N-1} h^i_{\varepsilon,k}(x,t) + f^i_{\varepsilon,N}(x,t) \tag{20}$$

for all natural numbers N. Fix an arbitrary number $\delta > 0$ and choose a natural number $N = N(\delta)$ so large that

$$\sup_i \sup_{x \in E} \sup_{t \geq 0} f^i_{\varepsilon,N}(x,t) < \delta \tag{21}$$

for all sufficiently small $\varepsilon > 0$. Such a number N exists in view of condition (18). In what follows we need the following auxiliary result.

Fix $\alpha \geq 0$, $i \neq j$, and put

$$\widehat{Q}^i_{\varepsilon,\alpha}(x,A) = \int_0^\infty e^{-\alpha s/t} Q^i_\varepsilon(x, A \times ds),$$
$$\widehat{Q}^{ij}_{\varepsilon,\alpha}(x,A) = \int_0^\infty e^{-\alpha s/t} Q^{ij}_\varepsilon(x, A \times ds),$$
$$\widehat{H}^i_{\varepsilon,\alpha}(x,A) = \int_0^\infty e^{-\alpha s/t} H^i_\varepsilon(x, A \times ds),$$
$$\widehat{R}^{ij}_\varepsilon(x,A) = \int_0^\infty e^{-\alpha s/t} R^{ij}_\varepsilon(x, A \times ds).$$

It is clear that

$$\widehat{Q}^i_\varepsilon(x,A) \leq \pi_D(A), \tag{22}$$

$$\lim_{\varepsilon\to 0} \widehat{Q}^i_{\varepsilon,\alpha}(x,A) = \pi_D(A) \tag{23}$$

uniformly in $x \in D$, $A \in \mathcal{B}$.

Denote by $\lambda^i_{\varepsilon,\alpha}$ the Perronian root of the kernel $\widehat{Q}^i_{\varepsilon,\alpha}$, and by $e^i_{\varepsilon,\alpha}(x)$ and $\rho^i_{\varepsilon,\alpha}(A)$ the eigenfunction and eigenmeasure, respectively, that correspond to the Perronian root $\lambda^i_{\varepsilon,\alpha}$, that is,

$$\int_D \widehat{Q}^i_{\varepsilon,\alpha}(x,dy) e^i_{\varepsilon,\alpha}(y) = \lambda^i_{\varepsilon,\alpha} e^i_{\varepsilon,\alpha}(x), \qquad x \in D,$$
$$\int_D \rho^i_{\varepsilon,\alpha}(dx) \widehat{Q}^i_{\varepsilon,\alpha}(x,A) = \lambda^i_{\varepsilon,\alpha} \rho^i_{\varepsilon,\alpha}(A), \qquad A \in \mathcal{B}.$$

For the sake of brevity we put

$$e^i_\varepsilon(x) = e^i_{\varepsilon,0}(x), \qquad \rho^i_\varepsilon(A) = \rho^i_{\varepsilon,0}(A)$$

and assume that

$$\rho^i_\varepsilon(D) = 1, \qquad \int_D \rho^i_\varepsilon(dx) e^i_\varepsilon(x) = 1, \tag{24}$$

$$\begin{aligned} &\int_D \rho^i_{\varepsilon,\alpha}(dx) e^i_\varepsilon(x) = 1, \qquad \alpha > 0, \\ &\int_D \rho^i_{\varepsilon,\alpha}(dx) e^i_{\varepsilon,\alpha}(x) = 1, \qquad \alpha > 0. \end{aligned} \tag{25}$$

Equality (23) and the theorem on the perturbation of separated parts of the spectrum [3, Chapter 4, §4, Theorem 3.16] imply that

$$\lambda^i_{\varepsilon,\alpha} \to 1,$$
$$e^i_{\varepsilon,\alpha}(x) \to 1$$

uniformly in $x \in D$, and

$$\rho^i_{\varepsilon,\alpha}(A) \underset{\varepsilon\to 0}{\longrightarrow} \pi_D(A)$$

uniformly in $A \in \mathcal{B}$ for all $\alpha \geq 0$.

Finally we put

$$\rho^{ij}_\varepsilon(A) = \frac{1}{1-\lambda^i_{\varepsilon,0}} \rho^i_\varepsilon Q^{ij}_\varepsilon(A) = \frac{1}{1-\lambda^i_{\varepsilon,0}} \int_D \rho^i_\varepsilon(dx) Q^{ij}_\varepsilon(x,A)$$

for $i \neq j$, where

$$Q^{ij}_\varepsilon(x,A) = \widehat{Q}^{ij}_{\varepsilon,0}(x,A) = \mathbb{P}_{x,i}\{X(\tau) \in A, \xi_\varepsilon(\tau) = j\}. \qquad \square$$

Lemma 1. *If all the assumptions of Theorem* 1 *hold, then*

$$\sum_{j\neq i}\left|\widehat{R}^{ii}_{\varepsilon,\alpha}(x,A)-\frac{c_{ii}}{c_{ii}u-\alpha}\rho^{ij}_{\varepsilon}(A)\right| \underset{\substack{t\to\infty\\ \varepsilon\to 0\\ \varepsilon t\to u}}{\longrightarrow} 0$$

for all $u\geq 0$ *and* $\alpha\geq 0$ *uniformly in* $x\in D$, $A\in\mathcal{B}$.

Proof. The kernels $\widehat{Q}^{i}_{\varepsilon,\alpha}$, $\widehat{Q}^{ij}_{\varepsilon,\alpha}$, $\widehat{H}^{i}_{\varepsilon,\alpha}$, and $\widehat{R}^{i}_{\varepsilon,\alpha}$ are such that

$$\widehat{H}^{i}_{\varepsilon,\alpha}=\left(1-Q^{i}_{\varepsilon,\alpha}\right)^{-1},$$
$$\widehat{R}^{ij}_{\varepsilon,\alpha}=\widehat{H}^{i}_{\varepsilon,\alpha}\widehat{Q}^{ij}_{\varepsilon,\alpha}.$$

It is also clear that

$$\widehat{Q}^{i}_{\varepsilon,\alpha}(x,A)=\mathbb{P}_{x,i}\left[e^{-\alpha\tau/t},X(\tau)\in A,\xi_\varepsilon(\tau)=i\right],$$
$$\widehat{Q}^{ij}_{\varepsilon,\alpha}(x,A)=\mathbb{P}_{x,i}\left[e^{-\alpha\tau/t},X(\tau)\in A,\xi_\varepsilon(\tau)=j\right].$$

According to the theorem on the isolated eigenvalue,

$$\widehat{H}^{i}_{\varepsilon,\alpha}(x,A)-\frac{1}{1-\lambda^{i}_{\varepsilon,\alpha}}e^{i}_{\varepsilon,\alpha}(x)\rho^{i}_{\varepsilon,\alpha}(A)\underset{\varepsilon\to 0}{\longrightarrow} I(x,A)-\pi_D(A)$$

uniformly in $x\in D$ and $A\in\mathcal{B}$ (see [1, Chapter 3, §3, Theorem 1]). This together with (23) implies that

$$\sum_{j\neq i}\left[\widehat{R}^{ij}_{\varepsilon,\alpha}(x,A)-\frac{m\varepsilon}{1-\lambda^{i}_{\varepsilon,\alpha}}e^{i}_{\varepsilon,\alpha}(x)\rho^{ij}_{\varepsilon,\alpha}(A)\right]\underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{26}$$

uniformly in $x\in D$ and $A\in\mathcal{B}$, where

$$\rho^{ij}_{\varepsilon,\alpha}(A)=\frac{1}{m\varepsilon}\rho^{i}_{\varepsilon,\alpha}Q^{ij}_{\varepsilon,\alpha}(A).$$

Now relations (22) and (23) imply that

$$1-\lambda^{i}_{\varepsilon,\alpha}\sim 1-\mathbb{P}_{\pi_D,i}\left[e^{-\alpha\tau/t},\xi_\varepsilon(\tau)=i\right] \tag{27}$$

as $\varepsilon\to 0$. Applying the latter relation for $\alpha=0$, we obtain from (10) that

$$\frac{1-\lambda^{i}_{\varepsilon,0}}{\varepsilon}\underset{\varepsilon\to 0}{\longrightarrow} -mc_{ii}. \tag{28}$$

The right hand side of (27) for $\alpha>0$ is equal to

$$1-\mathbb{P}_{\pi_D,i}\{\xi_\varepsilon(\tau)=i\}+\mathbb{P}_{\pi_D,i}\left[\left(1-e^{-\alpha\tau/t}\right),\xi_\varepsilon(\tau)=i\right]. \tag{29}$$

The second term in (29) divided by α/t approaches m as $t\to\infty$. Thus the whole expression (29) equals

$$1-\mathbb{P}_{\pi_D,i}\{\xi_\varepsilon(\tau)=i\}+m\frac{\alpha}{t}+o\left(\frac{1}{t}\right). \tag{30}$$

Using equality (10) we evaluate the limit:

$$\lim_{\substack{t\to\infty\\ \varepsilon\to 0\\ \varepsilon t\to u}} t\left[1-\mathbb{P}_{\pi_D,i}\{\xi_\varepsilon(\tau)=i\}+m\frac{\alpha}{t}+o\left(\frac{1}{t}\right)\right]=m(-c_{ii}u+\alpha),$$

whence

$$t\left(1-\lambda^{i}_{\varepsilon,\alpha}\right)\underset{\substack{t\to\infty\\ \varepsilon\to 0\\ \varepsilon t\to u}}{\longrightarrow} m(-c_{ii}u+\alpha). \tag{31}$$

The next step is to prove that

$$t\left[\rho^i_{\varepsilon,\alpha}(A)-\rho^i_\varepsilon(A)\right]\underset{\substack{\varepsilon\to 0\\ t\to\infty}}{\longrightarrow}\alpha m\pi_D(A)-\alpha\mathbb{P}_{\pi_D}[\tau,X(\tau)\in A] \tag{32}$$

uniformly in $A\in\mathcal{B}$.

By the definition of $\rho^i_{\varepsilon,\alpha}$ and ρ^i_ε we have

$$\left(\rho^i_{\varepsilon,\alpha}-\rho^i_\varepsilon\right)\left(Q^i_\varepsilon-\lambda^i_\varepsilon\right)=\rho^i_{\varepsilon,\alpha}\left(Q^i_\varepsilon-\widehat{Q}^i_{\varepsilon,\alpha}\right)+\left(\lambda^i_{\varepsilon,\alpha}-\lambda^i_\varepsilon\right)\rho^i_{\varepsilon,\alpha} \tag{33}$$

where $Q^i_\varepsilon=\widehat{Q}^i_{\varepsilon,0}$ and $\lambda^i_\varepsilon=\lambda^i_{\varepsilon,0}$.

According to (31) and (28), we have

$$t\left[\lambda^i_{\varepsilon,\alpha}-\lambda^i_\varepsilon\right]\underset{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow}-m\alpha \tag{34}$$

for $u>0$. Furthermore, put

$$\overline{Q}^i_\varepsilon(x,A)=\mathbb{P}_{x,i}[\tau,X(\tau)\in A,\xi_\varepsilon(\tau)=i].$$

Since

$$Q^i_\varepsilon(x,A)-\widehat{Q}^i_{\varepsilon,\alpha}(x,A)=\mathbb{P}_{x,i}\left[\left(1-e^{-\alpha\tau/t}\right),X(\tau)\in A,\xi_\varepsilon(\tau)=i\right],$$

we get

$$\begin{aligned}0&\le\alpha\overline{Q}^i_\varepsilon(x,A)-t\left[Q^i_\varepsilon(x,A)-\widehat{Q}^i_{\varepsilon,\alpha}(x,A)\right]\\&\le\mathbb{P}_{x,i}\left[\left(\alpha\tau-\frac{1-e^{-\alpha\tau/t}}{\alpha/t}\right),X(\tau)\in A,\xi_\varepsilon(\tau)=i\right]\\&\le\mathbb{P}_{x,i}\left(\alpha\tau-\frac{1-e^{-\alpha\tau/t}}{\alpha/t}\right).\end{aligned}$$

In view of (7), the last expression approaches zero uniformly in $x\in D$ as $t\to\infty$. Moreover, (6) and (7) imply

$$\overline{Q}^i_\varepsilon(x,A)\underset{\varepsilon\to 0}{\longrightarrow}\overline{Q}(x,A)=\mathbb{P}_x[\tau,X(\tau)\in A]$$

uniformly in $x\in D$ and $A\in\mathcal{B}$. Thus,

$$t\left[Q^i_\varepsilon(x,A)-\widehat{Q}^i_{\varepsilon,\alpha}(x,A)\right]\underset{\substack{t\to\infty\\ \varepsilon\to 0}}{\longrightarrow}\alpha\overline{Q}(x,A) \tag{35}$$

uniformly in $x\in D$ and $A\in\mathcal{B}$.

Furthermore, let V_ε be the generalized inverse operator of $Q^i_\varepsilon-\lambda^i_\varepsilon$; that is,

$$V_\varepsilon=\frac{1}{2\pi i}\oint\left(z-Q^i_\varepsilon\right)^{-1}\frac{dz}{z-\lambda^i_\varepsilon}, \tag{36}$$

where $C=\{z\in\mathbb{C}\colon|z|=\frac{1}{3}\}$ is the circle of radius $\frac{1}{3}$ in the complex plane centered at the origin. Since $\lambda^i_\varepsilon>\frac{1}{3}$ for sufficiently small ε, the operator V_ε is well defined. Therefore

$$\left(Q^i_\varepsilon-\lambda^i_\varepsilon\right)V_\varepsilon(x,A)=I(x,A)-e_\varepsilon(x)\rho_\varepsilon(A). \tag{37}$$

It is also clear that

$$V_\varepsilon(x,A)\underset{\varepsilon\to 0}{\longrightarrow}\pi_D(A)-I(x,A)$$

uniformly in $x\in D$ and $A\in\mathcal{B}$.

Using (24) and (25) we obtain from (33) and (37) that

$$t\left[\rho^i_{\varepsilon,\alpha}(A)-\rho^i_\varepsilon(A)\right]=t\rho^i_{\varepsilon,\alpha}\left(Q^i_\varepsilon-Q^i_{\varepsilon,\alpha}\right)V_\varepsilon(A)+t\left(\lambda^i_{\varepsilon,\alpha}-\lambda^i_\varepsilon\right)\rho^i_{\varepsilon,\alpha}V_\varepsilon(A). \tag{38}$$

Combining (35), (34), and (38) and collecting the like terms we prove (32).

We see from (26), (31), and (32) that the lemma follows from

$$\sup_{x\in D}\sup_{A\in\mathcal{B}}\sum_{j\neq i} t\left|Q_\varepsilon^{ij}(x,A)-\widehat{Q}_{\varepsilon,\alpha}^{ij}(x,A)\right| \underset{\substack{\varepsilon\to 0\\ t\to\infty}}{\longrightarrow} 0 \tag{39}$$

and

$$\sup_{\varepsilon>0}\sum_{j\neq i}\rho_\varepsilon^{ij}(D)<\infty. \tag{40}$$

We start with the proof of (39). The sum in (39) is nonnegative and does not exceed

$$\sum_{j\neq i} t\mathbb{P}_{x,i}\left[\left(1-e^{-\alpha\tau/t}\right),\xi_\varepsilon(\tau)=j\right].$$

In turn, the last expression does not exceed

$$\alpha\mathbb{P}_x[\tau,\tau>T]+\alpha T[1-\mathbb{P}_{x,i}\{\xi_\varepsilon(\tau)=i\}]$$

for any $T>0$. The first term in the last expression approaches zero uniformly in $x\in D$ as $T\to\infty$ in view of (14), while the second term, for every fixed $T>0$, approaches zero uniformly in $x\in D$ as $\varepsilon\to 0$. Relation (39) is proved.

To prove (40) note that

$$\begin{aligned}\sum_{j\neq i}\rho_\varepsilon^{ij}(D)&\le\frac{1}{1-\lambda_\varepsilon^i}\int_D\rho_\varepsilon^i(dx)[1-\mathbb{P}_{x,i}\{\xi_\varepsilon(\tau)=i\}]\\&=\frac{1}{1-\lambda_\varepsilon^i}\left[\rho_\varepsilon^i(D)-\rho_\varepsilon^iQ_\varepsilon^i(D)\right]=1.\end{aligned}$$

In fact, we obtained a stronger result, namely

$$\limsup_{\varepsilon\to 0}\sum_{j\neq i}\rho_\varepsilon^{ij}(D)\le 1. \tag{41}$$

The lemma is proved. □

Lemma 2. *Suppose the assumptions of Lemma 1 hold and a bounded function $\Psi(x,t)$ is measurable with respect to the variables $x\in D$ and $t\ge 0$. We also assume that*

$$\sup_{x\in D}|\Psi(x,t+s)-\Psi(x,t)|\underset{s\to 0}{\longrightarrow}0 \tag{42}$$

for all $t\ge 0$. Then

$$\sum_{j\neq i}\left|\int_D\int_0^t R_\varepsilon^{ij}(x,dy\times t\,ds)\Psi(y,s)-c_{ij}u\int_0^t e^{c_{ii}us}\Psi(y,s)\,ds\right|\underset{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow}0 \tag{43}$$

uniformly in $x\in D$ and $0\le t\le T$, where $T>0$, $u>0$, and $i\in I$ are arbitrary.

Proof. Note that relation (42) holds uniformly in t belonging to an arbitrary finite interval in view of compactness, that is,

$$\sup_{t\in[0,T]}\sup_{x\in D}|\Psi(x,t+s)-\Psi(x,t)|\underset{s\to 0}{\longrightarrow}0 \tag{44}$$

for all $T>0$. Lemma 1 and the continuity theorem for the Laplace transform imply

$$\sum_{j\neq i}\left|\int_D\int_0^t R_\varepsilon^{ij}(x,dy\times[ts_1,ts_2))\Psi(y)-\int_D\rho_\varepsilon^{ij}(dy)\Psi(y)\int_{s_1}^{s_2}(-c_{ii})ue^{c_{ii}us}\,ds\right|\underset{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow}0 \tag{45}$$

for all $0\le s_1\le s_2$ and all bounded $\mathcal{B}$-measurable functions $\Psi(y)$.

For a fixed $\delta > 0$, put

$$\Psi_\delta^+(x,t) = \sup_{k\delta \le s \le k\delta+\delta} \Psi(x,s), \qquad k\delta \le t \le k\delta + \delta,$$
$$\Psi_\delta^-(x,t) = \inf_{k\delta \le s \le k\delta+\delta} \Psi(x,s), \qquad k\delta \le t \le k\delta + \delta.$$

Since the function $\Psi(x,t)$ is continuous in t, the functions $\Psi_\delta^\pm(x,t)$ are measurable with respect to all their arguments. Relation (44) implies that for all $T > 0$,

$$\sup_{t\in[0,T]} \sup_{x\in D} \left|\Psi_\delta^\pm(x,t) - \Psi(x,t)\right| \underset{\delta\to 0}{\longrightarrow} 0. \tag{46}$$

Now relation (45) implies that

$$\sum_{j\ne i} \left| \int_D \int_0^t R_\varepsilon^{ij}(x, dy \times t\,ds)\Psi_\delta^\pm(y,s) - \int_D \rho_\varepsilon^{ij}(dy) \int_0^t (-c_{ii}) u e^{c_{ii} u s} \Psi_\delta^\pm(y,s)\,ds \right| \underset{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow} 0$$

for all $\delta > 0$. The limit

$$\sum_{j\ne i} \left| \rho_\varepsilon^{ij}(D) - \frac{c_{ij}}{(-c_{ii})} \right| \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{47}$$

exists by (10) and (11). Hence

$$\sum_{j\ne i} \left| \int_D \int_0^t R_\varepsilon^{ij}(x, dy + t\,ds)\Psi_\delta^\pm(y,s) - c_{ij} u \int_0^t e^{c_{ii} u s} \Psi_\delta^\pm(y,s)\,ds \right| \underset{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow} 0.$$

This together with (46) implies (43). The lemma is proved. □

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