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SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF LOCAL-TIME TYPE FUNCTIONALS OF MARKOV APPROXIMATIONS

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Abstract. A sufficient condition is obtained for the weak convergence of additive functionals defined on a sequence of Markov chains X_n approaching a Markov process X. The condition is expressed in terms of transient probabilities of the chains X_n. An application of the main result is given for the convergence on the Cantor set of local-time type functionals of random walks approaching an α-stable process with index $\alpha \le 1$.

1. Introduction

The aim of the paper is to find sufficient conditions for the convergence in distribution of functionals of stochastic processes approximating a homogeneous Markov process.

Suppose a sequence of stochastic processes X_n, $n \ge 1$, approximates in the Markov sense (see [1]) a homogeneous Markov process X. Let

$$t_{k,n} \stackrel{\text{def}}{=} \frac{k}{n}.$$

We consider functionals ϕ_n of the following form:

$$\phi_n^{s,t}(X_n) \stackrel{\text{def}}{=} \sum_{s \le t_{k,n} < t} g_n\left(X_n\left(t_{k,n}\right)\right), \qquad 0 \le s < t, \tag{1.1}$$

where g_n are nonnegative Borel functions. The functionals ϕ_n are stepwise functions with respect to each of their time argument. Consider the polygonal lines constructed from these functionals:

$$\psi_n^{s,t} = \phi_n^{t_{j-1,n},t_{k-1,n}} - (ns - j + 1)\phi_n^{t_{j-1,n},t_{j,n}} + (nt - k + 1)\phi_n^{t_{k-1,n},t_{k,n}},$$
$$s \in \left[t_{j-1,n}, t_{j,n}\right), \qquad t \in \left[t_{k-1,n}, t_{k,n}\right).$$

The lines ψ_n are viewed as random elements in the space $C(\mathbb{T}, \mathbb{R}^+)$, where

$$\mathbb{T} \stackrel{\text{def}}{=} \{(s,t) \mid 0 \le s \le t\}.$$

We consider the problem of the weak convergence of ψ_n in the space $C(\mathbb{T}, \mathbb{R}^+)$ to the element

$$\left\{\phi^{s,t}, (s,t) \in \mathbb{T}\right\},$$

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where ϕ is a Wiener W-functional of the limit process X. Theorem 1, the main result of the paper, contains sufficient conditions for such a convergence.

We show that this convergence follows from a condition that is a version of the local limit theorem for the processes X_n and from an estimate for the distributions of sequential exit times from a set with a curvilinear boundary. This estimate is of its own importance.

The proof of the main result is based on Theorem 1 of the paper [2], where we considered a general setting for the weak convergence of functionals (1.1). The main assumption of Theorem 1 in [2] is the convergence of the characteristics of the functionals ϕ_n to the characteristic of the functional ϕ. In this paper, we obtain sufficient conditions for the uniform convergence of the characteristics of ϕ_n. The method of the proof in [2] is a modification of Dynkin's approach (see [3, Chapter 6]) to the convergence of W-functionals.

As an application of the main result, we consider the problem of the weak convergence of local-time type functionals of random walks on the Cantor set. The functionals coincide, up to a normalizing factor, with the measure of the time spent by a random polygonal line constructed from a random walk in a neighborhood of the Cantor set. The random walk is assumed to approximate an α-stable process with $\alpha \le 1$.

2. Main result

Let $(\mathfrak{X}, \rho)$ be a locally compact metric space. Consider processes X and X_n, $n \ge 1$, defined on $\mathbb{R}^+$ and whose phase space is $\mathfrak{X}$. We assume that X is a homogeneous Markov process. Let the process X_n be Markovian for every $n \ge 1$ if the time parameter set is reduced to the set $\{t_{k,n}, k \ge 0\} \cap [0,T]$, where T is a fixed positive number.

The following definition from [1] is used in our main result.

Definition 1. A sequence $\{X_n\}$ approximates a process X in the Markov sense if, for all $\gamma > 0$ and $T > 0$, there exist a number $K(\gamma, T) \in \mathbb{N}$ and a sequence of two-component processes $\{\hat{Y}_n = (\hat{X}_n, \hat{X}^n)\}$ defined on a different probability space and such that

(i) $\hat{X}_n \overset{d}{=} X_n$, $\hat{X}^n \overset{d}{=} X$;

(ii) the process $\hat{Y}_n$ as well as the processes $\hat{X}_n$ and $\hat{X}^n$ possess the Markov property at the points $t_{iK(\gamma,T),n}$, $i \in \mathbb{N}$, with respect to the flow $\{\hat{\mathcal{F}}_t^n = \sigma(\hat{Y}_n(s), s \le t)\}$.

(iii)
$$\limsup_{n\to+\infty} \mathsf{P}\left(\sup_{i \le \frac{Tn}{K(\gamma,T)}} \rho\Big(\hat{X}_n\left(t_{iK(\gamma,T),n}\right), \hat{X}^n\left(t_{iK(\gamma,T),n}\right)\Big) > \gamma \right) < \gamma.$$

Assume that the transient probabilities of the process X_n have density for all n, that is,

$$\mathsf{P}\left(X_n\left(t_{i+k,n}\right) \in dy \mid X_n\left(t_{i,n}\right) = x\right) = p_{n,k}(x,y)\,dy, \qquad i \in \mathbb{Z}_+,\ k \in \mathbb{N},\ x, y \in \mathfrak{X}.$$

We also assume that the transient probability of the process X has density:

$$\mathsf{P}\left(X(t) \in dy \mid X(s) = x\right) = p_{t-s}(x,y)\,dy, \qquad 0 \le s < t \le T,\ x, y \in \mathfrak{X}.$$

Consider the functionals

$$\phi_n^{s,t}(X_n) \overset{\text{def}}{=} \sum_{k:\ s \le t_{k,n} < t} g_n\left(X_n\left(t_{k,n}\right)\right), \qquad 0 \le s < t,\ g_n \in \mathcal{B}(\mathfrak{X}, \mathbb{R}^+),\ n \ge 1. \tag{2.1}$$

We denote by $\mu_n(du) \overset{\text{def}}{=} n g_n(u)\,du$ the so-called symbols of the functionals ϕ_n. Following Dynkin's approach ([3, Chapter 6]), consider

$$f_n^{s,t}(x) \overset{\text{def}}{=} \mathsf{E}\left[\phi_n^{s,t}(X_n) \mid X_n(s) = x\right], \qquad s = t_{i,n},\ i \in \mathbb{Z}_+,\ 0 \le s < t < T,\ x \in \mathfrak{X}.$$

In what follows $f_n^{s,t}(x)$ is called the characteristic of ϕ_n.

We further assume that a W-functional (see [3]) $\phi = \phi(X)$ with characteristic f is given. It is known (see [3, Chapter 6]) that

$$\phi^{s,t} = L_2\text{-}\lim_{\varepsilon \to 0+} \int_s^t \frac{1}{\varepsilon} f^{0,\varepsilon}(X(r))\, dr.$$

This implies that

$$f^{s,t}(x) = \lim_{\varepsilon \to 0+} \int_s^t \int_{\mathfrak{X}} p_r(x,y) \frac{1}{\varepsilon} f^{0,\varepsilon}(y)\, dy\, dr.$$

We assume further that the measures $\varepsilon^{-1} f^{0,\varepsilon}\, du$ converge weakly as $\varepsilon \to 0+$ to a finite measure μ. We also assume that the characteristic f admits the representation

$$f^{s,t}(x) = \int_0^{t-s} \int_{\mathfrak{X}} p_r(x,y)\, \mu(dy)\, dr, \tag{2.2}$$

where

$$\int_0^T \left[\sup_{x \in \mathfrak{X}} \int_{\mathfrak{X}} p_r(x,y)\, \mu(dy) \right] dr < +\infty, \qquad T > 0. \tag{2.3}$$

Let $B(x,R) \equiv \{x \in \mathfrak{X} \mid \rho(x,y) < R\}$ and $\mathbb{I}_n \stackrel{\text{def}}{=} [0,T] \cap \frac{1}{n}\mathbb{Z}$.

Theorem 1. *Assume that there are constants $\beta, \gamma, \theta, A_0 > 0$ and $\varepsilon \in (0,1)$ such that*

(1) *the trajectories of the processes X_n are continuous and the sequence $\{X_n\}$ approximates the process X in the Markov sense;*
(2) *there exists a constant $C_{\beta,\delta,\varepsilon} > 0$ such that*
$$\sup_{x \in \mathbb{R}} \mathsf{P}_x \left(\rho(x, X_n(t)) > A t^{\delta} \right) \le C_{\beta,\delta,\varepsilon} A^{-\beta} t^{\varepsilon}$$
for all $n \ge 1$, $A > A_0$, and $t \in \mathbb{I}_n$;
(3) $\sup_{x \in \mathfrak{X}} g_n(x) \to 0$ *as $n \to \infty$;*
(4) *for a given $t_0 > 0$, the function $(t,x,y) \mapsto p_t(x,y)$ is uniformly continuous on $[t_0, +\infty) \times \mathfrak{X}^2$ and such that*
$$\sup_{x \notin B(y,R)} p_t(x,y) \to 0, \qquad R \to +\infty,$$
for all $y \in \mathfrak{X}$ and $t > 0$; moreover, let there exist a constant $C_\gamma > 0$ such that
$$\sup_{x,y \in \mathfrak{X}} p_t(x,y) \le C_\gamma t^{-\gamma}, \qquad t > 0;$$
(5) *there exists a sequence $\{\alpha_k\} \subset \mathbb{R}^+$ converging to zero and such that*
$$\sup_{x,y \in \mathfrak{X}} \left| p_{n,k}(x,y) - p_{\frac{k}{n}}(x,y) \right| \le \alpha_k \left(\frac{n}{k} \right)^{\gamma}, \qquad n,k \in \mathbb{N};$$
(6) *the measures μ_n are finite, weakly converge to a measure μ, and there are constants $C_\theta, c_\theta > 0$ such that*
$$\mu_n(B(x,R)) \le C_\theta R^{\theta}, \qquad x \in \mathfrak{X},\ n \in \mathbb{N},\ R > c_\theta n^{-\delta};$$
(7) *the constants γ, δ, and θ are such that*
$$\delta\theta + 1 > \gamma.$$

Then the random polygonal lines ψ_n constructed from the functionals ϕ_n converge in distribution to ϕ in the space $C(\mathbb{T}, \mathbb{R}^+)$.

Prior to the proof of the main result we consider an auxiliary result which is of its own interest.

3. Auxiliary results on the exit times

Assume that Z is a homogeneous Markov process taking values in a locally compact metric space $(\mathfrak{X},\rho)$ and defined on the set $\mathbb{I}_n$. As in the preceding section, let $t_{k,n} \overset{\text{def}}{=} k/n$.

For all A, $\delta>0$, and $n\in\mathbb{N}$, consider a sequence $\left\{\tau^{m,A,\delta}, m\ge 0\right\}$ of stopping times with respect to the filtration generated by the process Z:

$$\tau^{0,A,\delta} \overset{\text{def}}{=} 0,$$

$$\tau^{m+1,A,\delta} \overset{\text{def}}{=} \inf\left\{t_{i,n}\colon t_{i,n} > \tau^{m,A,\delta}, \rho\left(Z\left(t_{i,n}\right), Z\left(\tau^{m,A,\delta}\right)\right) > A\left(t_{i,n}-\tau^{m,A,\delta}\right)^{\delta}\right\}.$$

Lemma 1. *Assume that the following form of condition* (2) *in Theorem* 1 *holds for the process Z:*

$$\exists A_0, C, T, \beta, \delta > 0,\ \varepsilon\in(0,1)\colon$$
$$\forall A\ge A_0, \forall t\in\mathbb{I}_n \qquad \sup_{x\in\mathfrak{X}} \mathsf{P}\left(\rho\left(x, Z(t)\right) > At^{\delta} \mid Z(0)=x\right) \le CA^{-\beta}t^{\varepsilon}.$$

Then there exist constants $\widetilde{C}>0$, $\widetilde{C}_1>0$, and $A_1>0$ such that

$$\forall A\ge A_1, \forall t\in\mathbb{I}_n, \forall m\ge 1 \qquad \sup_{x\in\mathfrak{X}} \mathsf{P}\left(\tau^{m,A,\delta} < t \mid Z(0)=x\right) \le \frac{\widetilde{C}\left(\widetilde{C}_1 CA^{-\beta}\right)^m}{\Gamma^{\varepsilon}(m)} t^{m\varepsilon}.$$

We split the proof of Lemma 1 in two steps. The first step is to obtain a bound for the distribution of the first exit time $\tau^{1,A,\delta}$. The second step is to estimate the distribution of the convolutions of $\tau^{1,A,\delta}$.

Lemma 2. *If assumptions of Lemma* 1 *hold, then there exist constants $A_2>0$ and $\widetilde{C}>0$ such that*

$$\forall A > A_2, \forall t\in\mathbb{I}_n$$
$$\sup_{x\in\mathfrak{X}} \mathsf{P}\left(\sup_{u\colon u\in(0,t)\cap\frac{1}{n}\mathbb{Z}} \left(\rho\left(x, Z\left(u\right)\right) - Au^{\delta}\right) > 0 \mid Z(0)=x\right) \le \widetilde{C}t^{\varepsilon}A^{-\beta}.$$

Proof. Put $k=tn$ and $N=[\log_2 k]+1$. Then

$$\begin{aligned}
&\mathsf{P}\left(\sup_{l=1,\dots,k}\left(\rho\left(x, Z\left(t_{l,n}\right)\right) - A\left(t_{l,n}\right)^{\delta}\right) > 0 \;\middle|\; Z(0)=x\right)\\
&\quad\le \mathsf{P}\left(\sup_{i=0,\dots,N}\ \sup_{l=\left[\frac{nt}{2^{i+1}}\right],\dots,\left[\frac{nt}{2^{i}}\right]}\left(\rho\left(x, Z\left(t_{l,n}\right)\right) - A\left(t_{l,n}\right)^{\delta}\right) > 0 \;\middle|\; Z(0)=x\right)\\
&\quad\le \mathsf{P}\left(\sup_{i=0,\dots,N}\ \sup_{l=\left[\frac{nt}{2^{i+1}}\right],\dots,\left[\frac{nt}{2^{i}}\right]}\left(\rho\left(x, Z\left(t_{l,n}\right)\right) - A\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\delta}\right) > 0 \;\middle|\; Z(0)=x\right)\\
&\quad\le \sum_{i=0}^{N} P\left(\sup_{l=\left[\frac{nt}{2^{i+1}}\right],\dots,\left[\frac{nt}{2^{i}}\right]}\rho\left(x, Z\left(t_{l,n}\right)\right) - A\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\delta} > 0 \;\middle|\; Z(0)=x\right)\\
&\quad= \sum_{i=0}^{N} P\left(\sup_{l=\left[\frac{nt}{2^{i+1}}\right],\dots,\left[\frac{nt}{2^{i}}\right]}\rho\left(x, Z\left(t_{l,n}\right)\right) - A\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\delta} > 0 \;\middle|\; Z(0)=x\right).
\end{aligned}$$

Now we apply Skorokhod's inequality of Lemma 2 in [8, §9], where the proof is given for the case where the phase space is $\mathbb{R}$; the general case is treated similarly. Let ξ_j,

$j = 2, \dots, \left[\frac{nt}{2^i}\right] - \left[\frac{nt}{2^{i+1}}\right] + 1$, be equal to the values of the process at the points $\left[\frac{nt}{2^{i+1}}\right]$, $\left[\frac{nt}{2^{i+1}}\right] + 1, \dots, \left[\frac{nt}{2^i}\right]$, respectively, and let $\xi_1 \overset{\text{def}}{=} 0$. Since

$$\begin{aligned}
&\mathsf{P}\left(\rho\left(Z\left(t_{l,n}\right), Z\left(t_{\left[\frac{nt}{2^i}\right],n}\right)\right) \geq \frac{A}{2}\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\delta} \mid \mathcal{F}_{t_{l,n}}\right)\\
&\quad = \mathsf{P}\left(\rho\left(Z\left(t_{l,n}\right), Z\left(t_{\left[\frac{nt}{2^i}\right],n}\right)\right) \geq \frac{A}{2}\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\delta} \mid Z\left(t_{l,n}\right)\right)\\
&\quad \leq \mathsf{P}\left(\rho\left(Z\left(t_{l,n}\right), Z\left(t_{\left[\frac{nt}{2^i}\right],n}\right)\right) \geq \frac{A}{2}\left(t_{\left[\frac{nt}{2^{i+1}}\right],n} - t_{l,n}\right)^{\delta} \mid Z\left(t_{l,n}\right)\right)\\
&\quad \leq C 2^{\beta} A^{-\beta}\left(t_{\left[\frac{nt}{2^{i+1}}\right],n} - t_{l,n}\right)^{\varepsilon} \leq C 2^{\beta} A^{-\beta}\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\varepsilon},
\end{aligned}$$

we obtain the bound

$$\begin{aligned}
&\sum_{i=0}^{N} \mathsf{P}\left(\sup_{l=\left[\frac{nt}{2^{i+1}}\right],\dots,\left[\frac{nt}{2^i}\right]} \rho\left(x, Z\left(t_{l,n}\right)\right) > A\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\delta} \mid Z(0) = x\right)\\
&\qquad \leq C 2^{\beta} \sum_{i=0}^{N} A^{-\beta}\left(t_{\left[\frac{nt}{2^{i+1}}\right],n}\right)^{\varepsilon} \leq \widetilde{C} t^{\varepsilon} A^{-\beta}
\end{aligned} \tag{3.1}$$

for all $t \in \mathbb{I}_n$ and all sufficiently large A. This completes the proof of Lemma 2. □

Lemma 3. *Let a random variable τ assume values in $\frac{1}{n}\mathbb{Z}^+$. We denote by τ_m a random variable whose distribution is the m-th convolution of the distribution of τ. Let, for some $T > 0$ and $\varepsilon \in (0,1)$,*

$$\mathsf{P}(\tau \leq t_{k,n}) \leq C_1 t_{k,n}^{\varepsilon} \qquad \text{for all } k = 1, \dots, [nT].$$

Then there are $C_2 = C_2(\varepsilon) > 0$ and $C_3 = C_3(\varepsilon)$ such that

$$\mathsf{P}(\tau_m \leq t_{k,n}) \leq \frac{C_2 (C_3 C_1)^m}{\Gamma^{\varepsilon}(m)} t_{k,n}^{m\varepsilon} \quad \text{for all } k = 1, \dots, [nT].$$

Proof. Assume that

$$\exists C_m > 0\colon \qquad \mathsf{P}(\tau_m \leq t_{k,n}) \leq C_m t_{k,n}^{m\varepsilon}, \quad k = 1, \dots, [nT],$$

for some $m \geq 1$. To estimate the distribution function of the random variable τ_{m+1} we apply a discrete analogue of the integration by parts formula:

$$\begin{aligned}
\mathsf{P}\left(\tau_{m+1} \leq t_{k,n}\right) &= \sum_{i=0}^{k} \mathsf{P}\left(\tau_m \leq t_{k,n} - t_{i,n}\right) \mathsf{P}\left(\tau_1 = t_{i,n}\right)\\
&= \sum_{i=0}^{k} C_m \left(t_{k,n} - t_{i,n}\right)^{m\varepsilon} \left(\mathsf{P}\left(\tau_1 \leq t_{i,n}\right) - \mathsf{P}\left(\tau_1 \leq t_{i-1,n}\right)\right)\\
&= \sum_{i=0}^{k} C_m \left(t_{k-i,n}\right)^{m\varepsilon} \mathsf{P}\left(\tau_1 \leq t_{i,n}\right) - \sum_{i=-1}^{k-1} C_m \left(t_{k-i-1,n}\right)^{m\varepsilon} \mathsf{P}\left(\tau_1 \leq t_{i,n}\right)\\
&= \sum_{i=0}^{k-1} C_m C_1 \left[\left(t_{k-i,n}\right)^{m\varepsilon} - \left(t_{k,n} - t_{i+1,n}\right)^{m\varepsilon}\right] \left(t_{i,n}\right)^{\varepsilon}\\
&= \left(t_{k,n}\right)^{(m+1)\varepsilon} C_m C_1 \sum_{i=0}^{k-1} \left[\left(1 - \frac{i}{k}\right)^{m\varepsilon} - \left(1 - \frac{i+1}{k}\right)^{m\varepsilon}\right] \left(\frac{i}{k}\right)^{\varepsilon}.
\end{aligned}$$

In what follows we consider two cases, namely $m\varepsilon \geq 1$ and $m\varepsilon < 1$. First let $m\varepsilon < 1$. Then

$$\begin{aligned}\mathsf{P}\left(\tau_{m+1} \leq t_{k,n}\right) &= \left(t_{k,n}\right)^{(m+1)\varepsilon} C_m C_1 \sum_{i=0}^{k-1} \left[\left(1-\frac{i}{k}\right)^{m\varepsilon} - \left(1-\frac{i+1}{k}\right)^{m\varepsilon}\right]\left(\frac{i}{k}\right)^{\varepsilon} \\ &\leq \left(t_{k,n}\right)^{(m+1)\varepsilon} C_m C_1 \sum_{i=0}^{k-1} \left[\left(1-\frac{i}{k}\right)^{m\varepsilon} - \left(1-\frac{i+1}{k}\right)^{m\varepsilon}\right] \\ &= \left(t_{k,n}\right)^{(m+1)\varepsilon} C_m C_1.\end{aligned}$$

Therefore

$$\mathsf{P}\left(\tau_m \leq t_{k,n}\right) \leq C_1^m \left(t_{k,n}\right)^{m\varepsilon} \quad \text{for all } m \in \mathbb{Z} \cap \left[1, \frac{1}{\varepsilon}\right],\ k = 1, \dots, [nT].$$

Now we consider the case of $m\varepsilon \geq 1$. Then

$$\begin{aligned}&\left(t_{k,n}\right)^{(m+1)\varepsilon} C_1 C_m \sum_{i=0}^{k-1} \left[\left(1-\frac{i}{k}\right)^{m\varepsilon} - \left(1-\frac{i+1}{k}\right)^{m\varepsilon}\right]\left(\frac{i}{k}\right)^{\varepsilon} \\ &\quad\leq C_1 C_m \left(t_{k,n}\right)^{(m+1)\varepsilon} \int_0^1 u^{\varepsilon}\, d\left[-(1-u)^{m\varepsilon}\right] = C_1 C_m \left(t_{k,n}\right)^{(m+1)\varepsilon} m\varepsilon B(m\varepsilon, \varepsilon+1) \\ &\quad= C_1 C_m \left(t_{k,n}\right)^{(m+1)\varepsilon} \left(m\varepsilon \frac{\Gamma(m\varepsilon)\Gamma(\varepsilon+1)}{\Gamma(m\varepsilon+\varepsilon+1)}\right) \\ &\quad\leq C_1 C_m \left(t_{k,n}\right)^{(m+1)\varepsilon} \frac{\Gamma(m\varepsilon+1)}{\Gamma(m\varepsilon+1+\varepsilon)} \leq C_1 C_m \left(t_{k,n}\right)^{(m+1)\varepsilon} \frac{1}{m^{\varepsilon}}.\end{aligned}$$

We have used the following bound:

$$\begin{aligned}\frac{\Gamma(m\varepsilon+1)}{\Gamma(m\varepsilon+1+\varepsilon)} &= \frac{\Gamma(m\varepsilon-[m\varepsilon]+1)}{\Gamma(m\varepsilon+\varepsilon-[m\varepsilon]+1)} \cdot \prod_{i=0}^{[m\varepsilon]-1} \left(\frac{m\varepsilon-i}{m\varepsilon+\varepsilon-i}\right) \\ &\leq \exp\left[\sum_{i=0}^{[m\varepsilon]} \left(\frac{m\varepsilon-i}{m\varepsilon+\varepsilon-i} - 1\right)\right] \leq \exp\left[-\varepsilon \sum_{i=1}^{[m\varepsilon]-1} \frac{1}{i}\right] \\ &\leq \exp\left[-\varepsilon \ln(m\varepsilon)\right] = \frac{\varepsilon^{-\varepsilon}}{m^{\varepsilon}}.\end{aligned}$$

Combining this bound with the bound obtained in the first case, we conclude that there are constants $C_2 = \left(\Gamma(\varepsilon^{-1})\right)^{\varepsilon} > 0$ and $C_3 = \left(\varepsilon^{-1}\right)^{\varepsilon} > 0$ such that

$$\mathsf{P}(\tau_m \leq t_{k,n}) \leq \frac{C_2 (C_3 C_1)^m}{\Gamma^{\varepsilon}(m)} \left(t_{k,n}\right)^{m\varepsilon} \qquad \forall m \geq 1,\ \forall k = 1, \dots, [nT].$$

This result completes the proof of Lemma 3. □

Lemma 2 together with Lemma 3 proves Lemma 1.

4. The proof of Theorem 1

The following assertion is used to prove the convergence of functionals.

Proposition 1 ([2, Theorem 1]). *Assume that a sequence X_n approximates in the Markov sense a homogeneous Markov process X. Let the functionals $\{\phi_n \equiv \phi_n(X_n)\}$ be of the form* (1.1).

Assume that

(1) *the functions* $g_n(\cdot)$ *are bounded on* $\mathfrak{X}$ *and converge uniformly to zero, that is,*

$$\delta_n \stackrel{\text{def}}{=} \sup_{x\in\mathfrak{X}} g_n(x) \to 0, \qquad n\to\infty;$$

(2) *there exists a function* f *that is the characteristic (in the sense of* [3, Chapter 6]*) of a* W*-functional* $\phi=\phi(X)$ *of the limit Markov process* X *and such that*

$$\sup_{s=\frac{i}{n},\, t\in(s,T)} \left\| f_n^{s,t}(\cdot) - f^{s,t}(\cdot)\right\| \to 0, \qquad n\to\infty,$$

for all $T>0$*, where* $\|f(\cdot)\| \equiv \sup_{x\in\mathfrak{X}}|f(x)|$*;*

(3) *the limit function* f *is uniformly continuous with respect to the argument* x*, that is, for all* $T>0$*,*

$$\sup_{0\le s\le t<T} \left| f^{s,t}(x') - f^{s,t}(x'')\right| \to 0, \qquad |x'-x''|\to 0.$$

Then the convergence in distribution holds in the space $C(\mathbb{T},\mathbb{R}^+)$ *for the random polygonal lines* ψ_n *constructed from the functionals* ϕ_n*, that is,*

$$\psi_n(X_n) \Rightarrow \phi(X) \equiv \{\phi^{s,t}(X), (s,t)\in\mathbb{T}\}.$$

Note that the assumptions of Theorem 1 imply condition (1) of Proposition 2 below.

Condition (3) follows from inequality (2.2) and assumption (4) of Theorem 1. Indeed, let $\varepsilon>0$ be fixed. Consider r_0 such that

$$\int_0^{t_0}\left[\sup_{x\in\mathfrak{X}}\int_{\mathfrak{X}} p_r(x,y)\,\mu(dy)\right] dr < \frac{\varepsilon}{4}.$$

Choose $\delta>0$ such that

$$\mu(\mathfrak{X})T \sup_{\substack{x_1,x_2,y\in\mathfrak{X},\\ r\ge r_0,\\ \rho(x_1,x_2)<\delta}} |p_r(x_1,y)-p_r(x_2,y)| < \frac{\varepsilon}{2}.$$

Taking into account the inequality

$$\begin{aligned}|f^{s,t}(x_1)-f^{s,t}(x_2)| \le \int_0^{r_0}\int_{\mathfrak{X}} \left(p_r(x_1,y)+p_r(x_2,y)\right)\,\mu(dy)\,dr\\ + \int_{r_0}^{T}\int_{\mathfrak{X}} |p_r(x_1,y)-p_r(x_2,y)|\,\mu(dy)\,dr\end{aligned}$$

we get

$$\sup_{s,t\in[0,T]}\ \sup_{\substack{x_1,x_2\in\mathfrak{X},\\ \rho(x_1,x_2)<\delta}} |f^{s,t}(x_1)-f^{s,t}(x_2)| \le \varepsilon.$$

It remains to check condition (2). We show that

$$f_n^{s,t}(x) \rightrightarrows f^{s,t}(x), \qquad n\to\infty. \tag{4.1}$$

To prove (4.1) we proceed as follows. Consider the functionals $\hat\phi_{n,A}^{s,t}$ defined by

$$\hat\phi_{n,A}^{s,t} = \sum_{k=[sn]-1}^{[tn]} \phi_n^{s\vee t_{k,n},\,t\wedge t_{k+1,n}} \chi_{\{\rho(X_n(t_{k,n}),X_n(s))<A|t_{k,n}-s|^{\delta}\}}.$$

The expectations of $\hat{\phi}^{s,t}_{n,A}$ are estimated from above with the help of conditions (5) and (6) of Theorem 1; namely, we have

$$\begin{aligned}\mathsf{E}\left[\hat{\phi}^{s,t}_{n,A} \mid X_n(s)=x\right] &= \mathsf{E}_x \sum_{i=sn-1}^{tn} \phi_n^{t_{i,n},t_{i+1,n}} \chi_{\{\rho(X_n(t_{i,n}),X_n(s))<A|t_{i,n}-s|^\delta\}} \\ &\le g_n(x) + \frac{1}{n} \sum_{i=1}^{nt-ns-1} \int_{B\left(x,A\left(\frac{i}{n}\right)^\delta\right)} p_{n,i}(x,y)\,\mu_n(dy) \\ &\le \delta_n + C_\gamma + \frac{\alpha}{n} \sum_{i=1}^{nt-ns-1} \left(\frac{i}{n}\right)^{-\gamma} \mu_n\left(B\left(x, A\left(\frac{i}{n}\right)^\delta\right)\right) \\ &\le \delta_n + C_\gamma + \frac{\alpha}{n} \sum_{i=1}^{nt-ns-1} C_\theta A^\theta \left(\frac{i}{n}\right)^{-\gamma+\delta\theta} < B_1 + B_2(T)A^\theta\end{aligned}$$

for all $s,t\in\mathbb{I}_n$ such that $s<t$ and for some positive constants B_1, $B_2(T)$, $A>c_\theta$, and all $x\in\mathfrak{X}$, where $\alpha \overset{\text{def}}{=} \sup_k \alpha_k$.

Using the inequality

$$\begin{aligned}&\sup_{x\in\mathfrak{X}} \mathsf{E}\left[\phi_n^{s,t}\chi_{\{\forall i:\ ns\le i\le nt,\ \rho(X_n(t_{i,n}),X_n(s)|)<A|t_{i,n}-s|^\delta\}} \mid X_n(s)=x\right] \\ &\qquad \le \sup_{x\in\mathfrak{X}} \mathsf{E}\left[\hat{\phi}^{s,t}_{n,A} \mid X_n(s)=x\right] < \infty\end{aligned}$$

we get a bound for the expectation of the functionals $\phi^{s,t}_{n,A}$ on the set of ω for which the trajectory of the process X_n belongs to the domain

$$\left\{(t,u)\colon t\ge 0, |u|\le t^\delta\right\}.$$

We apply Lemma 1 to estimate the expectation of the functionals in the general case. Let

$$\tau_n^{0,A,\delta} \overset{\text{def}}{=} s, \qquad \tau^{0,A,\delta} \overset{\text{def}}{=} s,$$

$$\begin{aligned}\tau_n^{m+1,A,\delta} \overset{\text{def}}{=} \inf\Big\{&t_{u,n}\in[0,T]\colon t_{u,n}>\tau_n^{m,A,\delta}, \\ &\rho\left(X_n\left(t_{u,n}\right), X_n\left(\tau_n^{m,A,\delta}\right)\right) > A\left(t_{u,n}-\tau_n^{m,A,\delta}\right)^\delta\Big\},\end{aligned}$$

$$\tau^{m+1,A,\delta} \overset{\text{def}}{=} \inf\left\{z\in[0,T]: z>\tau^{m,A,\delta}, \rho\left(X\left(z\right), X\left(\tau^{m,A,\delta}\right)\right) > A\left(z-\tau^{m,A,\delta}\right)^\delta\right\}.$$

Since the constants A, $\widetilde{C}$, and $\widetilde{C}_1$ in Lemma 1 depend on β, δ, and ε and do not depend on the process Z, we obtain

$$\begin{aligned}&\exists A_1, C_1, C_2>0\colon\quad \forall m,n\ge 1, \forall s,t\in\mathbb{I}_n, \forall A>A_1 \\ &\qquad \sup_{x\in\mathfrak{X}} \mathsf{P}_x\left(\tau_n^{m,A,\delta}<t\right) \le \frac{C_1(C_2A^{-\beta})^m}{\Gamma^\varepsilon(m)}(t-s)^{m\varepsilon}.\end{aligned}$$

Now we prove the inequality

$$\begin{aligned}&\forall m\ge 1, \forall s,t\in(0,T), \forall A>A_1 \\ &\qquad \sup_{x\in\mathfrak{X}} \mathsf{P}_x\left(\tau^{m,A,\delta}<t\right) \le \frac{C_1(C_2A^{-\beta})^m}{\Gamma^\varepsilon(m)}(t-s)^{m\varepsilon}.\end{aligned} \tag{4.2}$$

First we check that

$$\begin{aligned}&\forall n\ge 1, \forall A>A_0, \forall t\in(0,T)\cap\mathbb{Q} \\ &\qquad \sup_{x\in\mathbb{R}} \mathsf{P}_x\left(\rho(x,X(t))>At^\delta\right) \le C_{\beta,\delta,\varepsilon}A^{-\beta}t^\varepsilon.\end{aligned} \tag{4.3}$$

Fix $\gamma > 0$ and recall that X_n approximates in the Markov sense the process X. Consider a sequence $(\widehat{X}^n, \widehat{X}_n)$ satisfying the conditions listed in Definition 1. For all $t \in (0,T)$, $x \in \mathfrak{X}$, and $n \geq 1$, we have

$$\begin{aligned}\mathsf{P}_x\left(\rho(x, X(t)) > At^\delta + \gamma\right) &= \mathsf{P}_x\big(\rho\big(x, \widehat{X}^n(t)\big) > At^\delta + \gamma\big)\\ &\leq \mathsf{P}_x\big(\rho\big(x, \widehat{X}_n(t)\big) > At^\delta\big) + \mathsf{P}_x\big(\rho\big(\widehat{X}^n(t), \widehat{X}_n(t)\big) > \gamma\big)\\ &< C_{\beta,\delta,\varepsilon}A^{-\beta}t^\varepsilon + \mathsf{P}_x\big(\rho\big(\widehat{X}^n(t), \widehat{X}_n(t)\big) > \gamma\big).\end{aligned}$$

Let $t = k_t/n_t$ for $k_t, n_t \in \mathbb{N}$, let $N_n \stackrel{\text{def}}{=} n \cdot n_t \cdot k(\gamma, T)$, and let $K_n \stackrel{\text{def}}{=} n \cdot k_t$, $n \geq 1$. Then

$$\begin{aligned}&\mathsf{P}_x\left(\rho\left(\widehat{X}^{N_n}\left(\frac{k_t}{n_t}\right), \widehat{X}_{N_n}\left(\frac{k_t}{n_t}\right) > \gamma\right)\right)\\ &\quad= \mathsf{P}_x\left(\rho\left(\widehat{X}^{N_n}\left(k(\gamma,T)\frac{K_n}{N_n}\right), \widehat{X}_{N_n}\left(k(\gamma,T)\frac{K_n}{N_n}\right) > \gamma\right)\right)\\ &\quad\leq \mathsf{P}_x\left(\sup_{i\in\mathbb{Z}^+:\ \frac{ik(\gamma,T)}{N_n}\in[0,T]}\rho\left(\widehat{X}^{N_n}\left(k(\gamma,T)\frac{i}{N_n}\right), \widehat{X}_{N_n}\left(k(\gamma,T)\frac{i}{N_n}\right) > \gamma\right)\right) \leq \gamma\end{aligned}$$

for sufficiently large n. Thus

$$\mathsf{P}_x\left(\rho(x, X(t)) > At^\delta + \gamma\right) < \gamma + C_{\beta,\delta,\varepsilon}A^{-\beta}t^\varepsilon.$$

Letting $\gamma \to 0$ we prove (4.3).

Consider the random variables

$$\begin{aligned}\widehat{\tau}_n^{0,A,\delta} &\stackrel{\text{def}}{=} s,\\ \widehat{\tau}_n^{m+1,A,\delta} &\stackrel{\text{def}}{=} \inf\Big\{t_{u,n} \in [0,T] \colon t_{u,n} > \widehat{\tau}_n^{m,A,\delta},\\ &\qquad \rho\left(X\left(t_{u,n}\right), X\left(\widehat{\tau}_n^{m,A,\delta}\right)\right) > A\left(t_{u,n} - \widehat{\tau}_n^{m,A,\delta}\right)^\delta\Big\}.\end{aligned}$$

Since

$$\mathsf{P}\left(\tau^{m,A,\delta} < t\right) = \lim_{n\to\infty}\mathsf{P}\left(\widehat{\tau}_n^{m,A,\delta} < t\right),$$

Lemma 1 proves (4.2).

For $s, t \in \frac{1}{n}\mathbb{N}$ such that $s < t$, we represent $\phi_n^{s,t}$ as follows:

$$\phi_n^{s,t} = \sum_{m=0}^{\infty}\phi_n^{\tau_m^{n,\delta}, t\wedge\tau_{m+1}^{n,\delta}}\chi_{\left\{\tau_n^{m,A,\delta} < t\right\}}.$$

Then

$$\begin{aligned}\mathsf{E}_x\,\phi_n^{\tau_n^{m,A,\delta},\tau_n^{m+1,A,\delta}\wedge t}\chi_{\left\{\tau_n^{m,A,\delta}<t\right\}} &= \mathsf{E}_x\left(\chi_{\left\{\tau_n^{m,A,\delta}<t\right\}}\,\mathsf{E}_x\left[\phi_n^{\tau_n^{m,A,\delta},\tau_n^{m+1,A,\delta}\wedge t}\;\Big|\;\mathcal{F}_{\tau_n^{m,A,\delta}}\right]\right)\\ &= \mathsf{E}_x\left(\chi_{\left\{\tau_n^{m,A,\delta}<t\right\}}\,\mathsf{E}_{X_n\left(\tau_n^{m,A,\delta}\right)}\left[\hat{\phi}_n^{s,\tau_n^{m+1,A,\delta}\wedge t-\tau_n^{m,A\delta}+s}\right]\right)\\ &\leq \mathsf{P}_x\left(\tau_n^{m,A,\delta} < t\right)\sup_{x\in\mathfrak{X}}\mathsf{E}_x\,\hat{\phi}_n^{s,t}.\end{aligned}$$

Now we apply Lemma 1 to the random variables $\tau_n^{m,1,\delta} - s$ for fixed n, δ, and

$$A = A_2 > \max(1, A_0, A_1).$$

This allows us to obtain an upper bound for the expectation of the increment of the functional in the whole interval (s,t):

$$\begin{aligned}\mathsf{E}\,\phi_n^{s,t} &\le \sup_{x\in\mathfrak{X}} \mathsf{E}_x\,\hat\phi_n^{s,t}\sum_{m=0}^{\infty}\mathsf{P}\left(\tau_n^{m,A_2,\delta}<t\right)\\ &\le \left(B_1+B_2(T)A_2^{\theta}\right)\sum_{m=0}^{\infty}\frac{C_1\left(C_2A_2^{-\beta}\right)^m}{\Gamma(m)^{\varepsilon}}(t-s)^{m\varepsilon}\le B_3(T).\end{aligned}$$

In what follows we need a bound for the conditional second moment of the functional $\phi_n^{s,t}$, namely for

$$\mathsf{E}\left[\left(\phi_n^{s,t}\right)^2 \mid X_n(s)=x\right].$$

We expand the second moment similarly to the case of the first moment and apply the strong Markov property:

$$\begin{aligned}\mathsf{E}_x\left(\phi_n^{s,t}\right)^2 &= \mathsf{E}_x\left(\sum_{i=0}^{\infty}\phi_n^{\tau_n^{m,A_2,\delta},t\wedge\tau_{m+1}^{n,\delta}}\chi_{\left\{\tau_n^{m,A_2,\delta}<t\right\}}\right)^2\\
&=2\,\mathsf{E}_x\sum_{i<j}^{\infty}\phi^{\tau_n^{i,A_2,\delta},\tau_n^{i+1,A_2,\delta}\wedge t}\phi^{\tau_n^{j,A_2,\delta},\tau_n^{j+1,A_2,\delta}\wedge t}\chi_{\left\{\tau_n^{j,A_2,\delta}<t\right\}}\\
&\quad+\sum_{i=0}^{\infty}\mathsf{E}_x\left(\phi^{\tau_n^{i,A_2,\delta},\tau_n^{i+1,A_2,\delta}\wedge t}\chi_{\left\{\tau_n^{i,A_2,\delta}<t\right\}}\right)^2\\
&\le 2\left(\sup_{x\in\mathfrak{X}}\mathsf{E}_x\,\hat\phi_n^{s,t}\right)^2\sum_{i=0}^{\infty}\mathsf{P}_x\left(\tau_n^{i,A_2,\delta}<t\right)+\sum_{i=0}^{\infty}\mathsf{E}_x\left(\phi^{\tau_n^{i,A_2,\delta},\tau_n^{i,1\delta}\wedge t}\chi_{\left\{\tau_n^{i,A_2,\delta}<t\right\}}\right)^2\\
&\le 2\left(\sup_{x\in\mathfrak{X}}\mathsf{E}_x\,\hat\phi_n^{s,t}\right)^2\sum_{i=0}^{\infty}\mathsf{P}_x\left(\tau_n^{i,A_2,\delta}<t\right)+\sup_{x\in\mathfrak{X}}\mathsf{E}_x\left[\left(\hat\phi_n^{s,t}\right)^2\right]\sum_{i=0}^{\infty}\mathsf{P}_x\left(\tau_n^{i,A_2,\delta}<t\right)\\
&\le 4\left(\sup_{x\in\mathfrak{X}}\mathsf{E}_x\,\hat\phi_n^{s,t}\right)^2\sum_{i=0}^{\infty}\mathsf{P}_x\left(\tau_n^{i,A_2,\delta}<t\right)\\
&\quad+\sup_{x\in\mathfrak{X}}\mathsf{E}_x\sum_{i=ns}^{nt}\left[\left(\hat\phi_n^{i/n,(i+1)/n}\right)^2\right]\sum_{i=0}^{\infty}\mathsf{P}_x\left(\tau_n^{i,A_2,\delta}<t\right)\\
&\le 4\left(\sup_{x\in\mathfrak{X}}\mathsf{E}_x\,\hat\phi_n^{s,t}\right)^2\sum_{i=0}^{\infty}\mathsf{P}_x\left(\tau_n^{i,A_2,\delta}<t\right)+\delta_n\left(\sup_{x\in\mathfrak{X}}\mathsf{E}_x\,\hat\phi_n^{s,t}\right)\sum_{i=0}^{\infty}\mathsf{P}_x\left(\tau_n^{i,A_2,\delta}<t\right)\\
&\le 4\left(B_1+B_2(T)A_2^{\theta}\right)B_3(T)+\delta_nB_3(T)\le B_4(T).\end{aligned}$$

A corresponding bound for the second moment of the functionals $\phi^{s,t}$ can be obtained similarly:

$$\sup_{x\in\mathfrak{X}}\mathsf{E}_x\left(\phi^{s,t}\right)^2\le B_4(T).$$

Now we prove the uniform convergence of the characteristics. Denote by $D_{n,A}^{s,t}$ the event $\left\{\tau_n^{1,A,\delta}\ge t\right\}$. Consider a nonincreasing Lipschitz function $\Psi\colon\mathbb{R}^+\to[0,1]$ such that $\Psi([0,1])=\{1\}$ and $\Psi([2,+\infty))=\{0\}$ and put

$$\Psi_r(x,y)=\Psi\left(r^{-1}\cdot|x-y|\right),\quad r>0,\ x,y\in\mathfrak{X},\qquad \Psi_0\equiv 1.$$

Given an arbitrary $r_0>0$, the function $\Psi\colon(r,x,y)\mapsto\Psi_r(x,y)$ is uniformly continuous on $[r_0,+\infty)\times\mathfrak{X}^2$.

Let $s \le t \le T$ and $A > A_0$ be fixed. We represent $\phi_n^{s,t}$ in the form $\phi_n^{s,t} = \eta_{n,A}^{s,t} + \zeta_{n,A}^{s,t}$, where

$$\eta_{n,A}^{s,t} = \frac{1}{n} \sum_{s \le \frac{k}{n} < t} n g_n \left(X_n \left(t_{k,n}\right)\right) \Psi_{A(t_{k,n}-s)^\delta} \left(X_n(s), X_n \left(t_{k,n}\right)\right).$$

If k is such that $s \le t_{k,n} < t$, then

$$\rho \left(X_n(s), X_n \left(t_{k,n}\right)\right) \le A \left(t_{k,n} - s\right)^\delta \Rightarrow \Psi_{A(t_{k,n}-s)^\delta} \left(X_n(s), X_n \left(t_{k,n}\right)\right) = 1$$

on the set $D_{n,A}^{s,t}$, whence $\{\phi_n^{s,t} = \eta_{n,A}^{s,t}\} \supset D_{n,A}^{s,t}$ and

$$\left\{\zeta_{n,A}^{s,t} \ne 0\right\} \subset \Omega \setminus D_{n,A}^{s,t}. \tag{4.4}$$

Using the bound for the second moment of the functionals ϕ_n and the inequality

$$0 \le \zeta_{n,A}^{s,t} \le \phi_n^{s,t},$$

we prove the estimate

$$\begin{aligned} \mathsf{E}\left[\zeta_{n,A}^{s,t} \mid X_n(s) = x\right] &\le \mathsf{E}\left[(\phi_n^{s,t})^2 \mid X_n(s) = x\right]^{1/2} \left[\mathsf{P}\big(\Omega \setminus D_{n,A}^{s,t} \mid X_n(s) = x\big)\right]^{1/2} \\ &\le B_5(T) A^{-\beta/2}. \end{aligned} \tag{4.5}$$

Similarly $\phi^{s,t} = \eta_A^{s,t} + \zeta_A^{s,t}$, where

$$\eta_A^{s,t} = \int_s^t \Psi_{A(r-s)^\delta}(X(s), X(r))\, d\phi^{s,r},$$

$$\mathsf{E}\left[\zeta_A^{s,t} \mid X(s) = x\right] \le B_6(T) A^{-\beta/2}. \tag{4.6}$$

Hence

$$\begin{aligned} &\left|\mathsf{E}\left[\eta_{n,A}^{s,t} \mid X_n(s) = x\right] - \mathsf{E}\left[\eta_A^{s,t} \mid X(s) = x\right]\right| \\ &\quad \le g_n(x) + \left|\frac{1}{n} \sum_{k=1}^{]n(t-s)[-1} \int_{\mathfrak{X}} p_{k,n}(x,y) \Psi_{A\left(\frac{k}{n}\right)^\delta}(x,y)\, \mu_n(dy)\right. \\ &\qquad\qquad \left. - \int_0^{t-s} \int_{\mathfrak{X}} p_r(x,y) \Psi_{Ar^\delta}(x,y)\, \mu(dy)\, dr\right| \\ &\quad \le \delta_n + \Delta_n^1(x,A,s,t) + \Delta_n^2(x,A,s,t) + \Delta_n^3(x,A,s,t), \end{aligned}$$

where $]z[\equiv \min\{N \in \mathbb{Z}, N \ge z\}$,

$$\Delta_n^1(x,A,s,t) = \left|\frac{1}{n} \sum_{k=1}^{]n(t-s)[-1} \int_{\mathfrak{X}} [p_{k,n}(x,y) - p_{\frac{k}{n}}(x,y)] \Psi_{A\left(\frac{k}{n}\right)^\delta}(x,y) \mu_n(dy)\right|,$$

$$\begin{aligned} \Delta_n^2(x,A,s,t) = &\left|\frac{1}{n} \sum_{k=1}^{]n(t-s)[-1} \int_{\mathfrak{X}} p_{\frac{k}{n}}(x,y) \Psi_{A\left(\frac{k}{n}\right)^\delta}(x,y)\, \mu_n(dy)\right. \\ &\qquad \left. - \int_0^{t-s} \int_{\mathfrak{X}} p_r(x,y) \Psi_{Ar^\delta}(x,y)\, \mu_n(dy)\, dr\right|, \end{aligned}$$

$$\Delta_n^3(x,A,s,t) = \left|\int_0^{t-s} \int_{\mathfrak{X}} p_r(x,y) \Psi_{Ar^\delta}(x,y) [\mu_n(dy) - \mu(dy)]\, dr\right|.$$

Put

$$\Delta_n^i(A) = \sup_{x \in \mathfrak{X}, s \le t \le T} \Delta_n^i(x,A,s,t), \qquad i = 1,2,3.$$

Since $\Psi_r(x,y) \in [0,1]$ and $\{\Psi_r(x,y) \neq 0\} \subset \{y \in B(x,2r)\}$, we have

$$\begin{aligned}\Delta_n^1(A) &\le \frac{1}{n}\sum_{k=1}^{]nT[-1} \alpha_k \left(\frac{n}{k}\right)^{\gamma} \mu_n\left(B\left(x, 2A\left(\frac{k}{n}\right)^{\delta}\right)\right)\\ &\le C_\theta (2A)^\theta \cdot \frac{1}{n}\sum_{k=1}^{]nT[-1} \alpha_k \left(\frac{k}{n}\right)^{\delta\theta-\gamma} \to 0, \qquad n \to +\infty,\end{aligned} \tag{4.7}$$

by the Toeplitz theorem.

The function

$$(r,x,y) \mapsto p_r(x,y)\Psi_r(x,y)$$

is uniformly continuous on $[r_0, +\infty) \times \mathfrak{X}^2$ if $r_0 > 0$. Thus an estimate, similar to (4.7), implies the uniform convergence (with respect to $x \in \mathfrak{X}$ and $(s,t) \in \mathbb{T}$) to zero of the sequence

$$\left|\frac{1}{n}\sum_{k=[r_0 n]+1}^{]n(t-s)[-1} \int_{\mathfrak{X}} p_{\frac{k}{n}}(x,y)\Psi_{A\left(\frac{k}{n}\right)^{\delta}}(x,y)\,\mu_n(dy) - \int_{r_0}^{t-s}\int_{\mathfrak{X}} p_r(x,y)\Psi_{Ar^\delta}(x,y)\,\mu_n(dy)\,dr\right|$$

(note that $\sup_n \mu_n(\mathfrak{X}) < +\infty$, since the measures μ_n weakly converge to μ). The same reasoning leads to the bound

$$\begin{aligned}\limsup_{n\to+\infty} \Delta_n^2(A) &\le \limsup_{n\to+\infty}\left[\frac{1}{n}\sum_{k=1}^{[r_0 n]} C_\gamma \left(\frac{n}{k}\right)^{\gamma} C_\theta \left(2A\left(\frac{k}{n}\right)^{\delta}\right)^{\theta}\right]\\ &\quad + \limsup_{n\to+\infty}\left[\int_0^{r_0} C_\gamma r^{-\gamma} C_\theta \left(2Ar^\delta\right)^\theta\, dr\right]\\ &= B_7(A)(r_0)^{\delta\theta-\gamma+1}\end{aligned}$$

for $A > c_\theta$ and arbitrary $r_0 > 0$ (the precise value of the constant $B_7(A)$ does not matter). This implies the convergence

$$\Delta_n^2(A) \to 0, \qquad n \to +\infty. \tag{4.8}$$

Finally, the weak convergence of μ_n to μ and the first part of condition (4) imply that

$$I_n(A,t) \equiv \sup_{x\in\mathfrak{X}}\left|\int_{\mathfrak{X}} p_t(x,y)\Psi_{Ar^\delta}(x,y)[\mu_n(dy) - \mu(dy)]\right| \to 0, \qquad n \to +\infty,$$

for all t. Since $I_n(A,t) \le C_\gamma t^{-\gamma} \cdot C_\theta(2At^\delta)^\theta$, the Lebesgue dominated convergence theorem yields

$$\Delta_n^3(A) \to 0, \qquad n \to +\infty. \tag{4.9}$$

Now we obtain from bounds (4.5)–(4.9) that

$$\limsup_{n\to+\infty} \sup_{x\in\mathfrak{X}, s\le t\le 1} \left|f_n^{s,t}(x) - f^{s,t}(x)\right| \le (B_5(T) + B_6(T))A^{-\beta/2}, \qquad A > \max(c_\theta, A_2).$$

Letting $A \to +\infty$ we get relation (4.1), and this completes the proof of Theorem 1.

5. Example

Assume that ξ_n, $n \ge 1$, is a sequence of independent identically distributed random variables with density such that the normalized sums

$$S_n \stackrel{\text{def}}{=} n^{-1/\alpha}\sum_{k=1}^{n} \xi_k$$

weakly converge to a stable random variable with the characteristic function

$$\varphi_0(\lambda) = \exp\bigl(ia\lambda - b|\lambda|^\alpha (1 + i\beta\omega(\lambda,\alpha))\bigr), \tag{5.1}$$

where $0 < \alpha \le 1$, $-1 \le \beta \le 1$, $a, b \in \mathbb{R}$, and

$$\omega(\lambda,\alpha) = \begin{cases} \frac{2}{\pi}\operatorname{sgn}\lambda \ln|\lambda|, & \alpha = 1, \\ \frac{2}{\pi}\operatorname{sgn}\lambda \tan\frac{\pi\alpha}{2}, & \alpha \ne 1. \end{cases}$$

We construct a sequence of processes $X_n(t)$ as follows. First we put

$$X_n(k/n) = \frac{1}{n^{1/\alpha}} \sum_{i=1}^{k} \xi_i, \qquad k = 0, \dots, n,$$

for $t \in [0,1]$. Then we use the linear interpolation to determine $X_n(t)$ for other $t \in [0,1]$. In Proposition 2 below we assume that the distribution of ξ_1 has bounded density on $\mathbb{R}$. Then Gnedenko's theorem ([7, Theorem 4.3.1]) implies that the densities of the random variables S_n converge uniformly to the density of a stable distribution with index α. Condition (1) of Theorem 1 is proved for this case in the paper [1], where the approximation in the Markov sense is applied to random polygonal lines for random variables belonging to the domain of attraction of a stable law.

Denote by K the Cantor set (that is, the set of points in the interval $(0,1)$ whose ternary numeral does not contain digit 1). Put

$$K_\varepsilon \overset{\text{def}}{=} \{x \in \mathbb{R} \mid \exists y \in K \colon |x - y| \le \varepsilon\}.$$

Let $\lambda_1(\cdot)$ be Lebesgue measure in the line, and let $h(n) \overset{\text{def}}{=} \left[\log_3 \frac{n}{2}\right]$. Consider the functionals $\phi_n^{s,t}$ of the form (2.1), where the function g_n is such that $ng_n(x) = a(n)\chi_{\{K_{r_n}\}}(x)$; that is, consider the functionals

$$\phi_n^{s,t}(Y) \overset{\text{def}}{=} \sum_{k : s \le k/n < t} a(n)\chi_{\{K_{r_n}\}}\left(Y\left(\frac{k}{n}\right)\right), \tag{5.2}$$

where $r_n \overset{\text{def}}{=} n^{-1}$ and

$$a(n) \overset{\text{def}}{=} \frac{1}{n}(\lambda_1\{K_{r_n}\})^{-1} = \frac{1}{n}\left(2^{h(n)}\left(\frac{1}{3^{h(n)}} + \frac{2}{n}\right)\right)^{-1}.$$

The meaning of these functionals for a fixed n is that they represent, up to a constant factor, the fraction of time spent by the random walk in the r_n-neighborhood of the Cantor set.

In the following assertion we prove that the random polygonal lines corresponding to functionals (5.2) converge to some W-functional of the limit process.

Proposition 2. *Assume that independent identically distributed random variables ξ_n have a bounded density and, for some $\alpha \in (1 - \ln 2/\ln 3, 1]$, the normalized sums*

$$S_n \overset{\text{def}}{=} n^{-1/\alpha} \sum_{k=1}^{n} \xi_k$$

weakly converge to a stable random variable whose characteristic function is given by equality (5.1). *Then the random polygonal lines ψ_n constructed from the functionals* (5.2) *converge in distribution in the space $C(\mathbb{T}, \mathbb{R}^+)$, that is,*

$$\psi_n(X_n) \Rightarrow \phi(X),$$

where $\phi(X)$ is a W-functional of the stable process X. It is natural to view the functional $\phi(X)$ as the local time of the process X in the Cantor set K.

We use the main result of this paper, Theorem 1, to prove Proposition 2. We already mentioned that condition (1) of Theorem 1 holds. The relation

$$p_t(x) = t^{1/\alpha} p_1\left(x t^{1/\alpha}\right)$$

(called the automodel property of a stable process) implies condition (4) of Theorem 1 for $\gamma = \alpha^{-1}$, since the density $p_1(\cdot)$ is bounded. The Gnedenko theorem and the second assumption of Proposition 2 imply condition (5), while condition (3) follows from the following estimate:

$$\frac{1}{n}\left(2^{h(n)}\left(\frac{1}{3^{h(n)}} + \frac{2}{n}\right)\right)^{-1} \le 2^{-h(n)}\left(\frac{n}{n/2} + 2\right)^{-1} \le \frac{1}{4} \cdot 2^{-h(n)} \to 0, \qquad n \to \infty.$$

Now we check condition (6). Recall that the measures mentioned in Theorem 1 are such that in the case under consideration we have

$$\mu_n(D) = a(n)\lambda_1\left\{K_{r_n} \cap D\right\} \quad \text{for all } D \in \mathcal{B}(\mathbb{R}),$$

where λ_1 is Lebesgue measure on the line. Let R be a number such that

$$\frac{1}{3^{h(n)}} + \frac{1}{n} < R < 1,$$

and let $H(R) \stackrel{\text{def}}{=} -[\log_3 R]$. Then

$$\frac{1}{3^{H(R)+1}} < R \le \frac{1}{3^{H(R)}}.$$

Thus

$$\begin{aligned}\mu_n((0,R)) &= n \cdot a(n)\lambda_1\left\{K_{r_n} \cap (0,R)\right\} \le n \cdot a(n)\lambda_1\left\{K_{r_n} \cap \left(0, 3^{-H(R)}\right)\right\} \\ &\le n \cdot a(n) 2^{h(n)-H(R)} \cdot \frac{\lambda_1\left\{K_{r_n}\right\}}{2^{h(n)}} = 2^{-H(R)} = R^{-(\ln 2/\ln R) H(R)} \\ &\le R^{-(\ln 2/\ln R)(-\ln R/\ln 3+1)} \le R^{\ln 2/\ln 3}.\end{aligned}$$

If $R < 1$ and $0 < R < 3^{-h(n)} + n^{-1}$, then

$$\mu_n((0,R)) = Rn \cdot a(n) \le R \le R^{\ln 2/\ln 3}.$$

Moreover one can choose $\theta = \ln 2/\ln 3$.

In a similar way we prove the weak convergence of μ_n to the measure μ constructed from the Cantor function, which is known to be a distribution function (see [6, Chapter 5, §8]). Let

$$\mu_n(x) \stackrel{\text{def}}{=} \mu_n([0,x]), \qquad \mu(x) \stackrel{\text{def}}{=} \mu([0,x]), \qquad x \in [0,1].$$

Since the distribution function $\mu(\cdot)$ is continuous, we need to check the pointwise convergence of $\mu_n(\cdot)$. We prove that for all $x \in (\frac{1}{3}, \frac{2}{3})$,

$$\mu_n(x) \to \mu(x).$$

If n_0 is chosen so that $x \notin K_{r_n}$ for $n \ge n_0$, then

$$\mu_n(x) = n a(n) 2^{h(n)-1} \frac{\lambda_1\left\{K_{r_n}\right\}}{2^{h(n)}} = \frac{1}{2} = \mu(x)$$

for $n \ge n_0$. Similarly we prove that

$$\mu_n(x) \to \mu(x), \qquad n \to \infty,$$

for all $x \in [0,1] \setminus K$.

Since the functions $\mu_n(\cdot)$ are monotone, $\mu(\cdot)$ also is monotone and continuous. The set $[0,1] \setminus K$ is everywhere dense in $[0,1]$, whence the pointwise convergence follows for all points of the interval $[0,1]$. Thus condition (6) holds.

If ξ is an arbitrary random variable and $C > 0$ is an arbitrary constant, then

$$\mathsf{P}\left[|\xi| \geq C\right] \leq 7C \int_0^{1/C} |1 - \varphi(t)| \, dt,$$

where $\varphi\colon t \to \mathsf{E}\left[e^{it\xi}\right]$ is the characteristic function of the random variable ξ (see [9, Lemma 3, §3, Chapter 3]). Denote by $\varphi_1(\cdot)$ the characteristic function of the random variable ξ_1. According to Theorem 1.4 in [4],

$$\begin{aligned} &\exists \varepsilon \in (0,1), a, b > 0\colon \quad \forall \lambda, |\lambda| < \varepsilon \\ &\qquad \ln \varphi_1(\lambda) = ai\lambda - b\,|\lambda|^\alpha \left(1 + i\beta\omega(\lambda,\alpha)\right)(1 + r(\lambda)), \qquad r(\lambda) \to 0,\ \lambda \to 0. \end{aligned}$$

Thus we may assume that

$$|b| \cdot |1 + i\beta\omega(\lambda,\alpha)| \cdot |\lambda|^\alpha \, |1 + r(\lambda)| < \frac{1}{2} \quad \text{and} \quad |a\lambda| < \frac{1}{2} \tag{5.3}$$

for λ such that $|\lambda| < \varepsilon$.

Put

$$r_1(\cdot) \overset{\text{def}}{=} \operatorname{Re}(r(\cdot)), \qquad r_2(\cdot) \overset{\text{def}}{=} \operatorname{Im}(r(\cdot)).$$

If $c_0 > 1/\varepsilon$ is fixed, then

$$\begin{aligned} \mathsf{P}\left\{\frac{S_n}{n^{1/\alpha}} > c\right\} &\leq 7c \int_0^{1/c} \left|1 - \varphi_1^n\left(\lambda n^{-1/\alpha}\right)\right| d\lambda \\ &\leq 7c \int_0^{1/c} \left|1 - e^{\left(a \cdot i\lambda \cdot n^{1-1/\alpha} - b(1+i\beta\omega(\lambda,\alpha))|\lambda|^\alpha\left(1 + r\left(\lambda \cdot n^{-1/\alpha}\right)\right)\right)}\right| d\lambda \end{aligned} \tag{5.4}$$

for all $c > c_0$.

Let

$$h \overset{\text{def}}{=} \max\left(\tan \pi\frac{\alpha}{2}, \frac{2}{\pi}\right).$$

If $x, y \in \mathbb{R}$ are two arbitrary real numbers such that $|x| + |y| < 1$, then

$$\left|e^{x+iy} - 1\right| \leq \left|e^{x+iy} - e^x\right| + |1 - e^x| \leq e\left|e^{iy} - 1\right| + e\,|x| \leq 2e(|x| + |y|).$$

This inequality allows us to estimate the last term in (5.4). First we consider the case of $\alpha < 1$:

$$\begin{aligned} &14ec \int_0^{c^{-1}} \left|a \cdot i\lambda \cdot n^{1-1/\alpha} - b(1 + i\beta\omega(\lambda,\alpha))\,|\lambda|^\alpha \left(1 + r\left(\lambda \cdot n^{-1/\alpha}\right)\right)\right| d\lambda \\ &\quad \leq 14ec \int_0^{c^{-1}} \left(|a| \cdot |\lambda| + 2\,|b|\,h\,|\lambda|^\alpha\right) d\lambda \\ &\quad \leq 28ec \int_0^{c^{-1}} (|a| + |bh|)\lambda^\alpha \, d\lambda \\ &\quad \leq 28ec^{-\alpha}(|a| + h\,|b|). \end{aligned}$$

Now let $\alpha = 1$ and $c > e$. Then

$$\begin{aligned}14ec\int_0^{c^{-1}} &\left|a\cdot i\lambda - b(1+i\beta\omega(\lambda,1))\,|\lambda|\left(1+r\left(\lambda\cdot n^{-1}\right)\right)\right|\,d\lambda\\&\le 14ec\int_0^{c^{-1}}\left(|a|\cdot|\lambda| + 2\,|b|\,h(1+|\ln\lambda|)\,|\lambda|\right)\,d\lambda\\&\le 28ec\int_0^{c^{-1}}(|a|+|bh|\,(1+|\ln\lambda|))\lambda\,d\lambda\\&\le 28ec^{-1}(|a|+h\,|b|)+28e|b|hc\int_0^{c^{-1}}(-\lambda\ln\lambda)\,d\lambda\\&\le 28ec^{-1}(|a|+h\,|b|)+28e|b|hc\left(\frac{1}{2}c^{-2}\ln c+\frac{1}{4}c^{-2}\right)\\&\le 56ec^{-1/2}(|a|+h|b|).\end{aligned}$$

By assumption, $\alpha \in (1-\ln 2/\ln 3, 1]$. Thus there exists δ such that

$$(1/\alpha - 1)\,\frac{\ln 3}{\ln 2} < \delta < 1/\alpha.$$

If the parameters γ, δ, and θ are chosen as indicated above, then

$$\delta\theta + 1 - \gamma = \delta\frac{\ln 2}{\ln 3} + 1 - \frac{1}{\alpha} > 0;$$

that is, condition (7) of Theorem 1 holds. Condition (2) holds if

$$\mathsf{P}\left(\frac{S_k}{n^{1/\alpha}} > A\left(\frac{k}{n}\right)^{\delta}\right) \le CA^{-\beta}\left(\frac{k}{n}\right)^{\varepsilon}$$

for some $\beta > 0$ and $\varepsilon \in (0,1)$. The last condition holds, indeed, since, as we have proved above,

$$\mathsf{P}\left(\frac{S_k}{n^{1/\alpha}} > A\left(\frac{k}{n}\right)^{\delta}\right) = \mathsf{P}\left(\frac{S_k}{k^{1/\alpha}} > A\left(\frac{k}{n}\right)^{\delta-1/\alpha}\right) \le \left[28e(|a|+h|b|)\right]A^{-\alpha}\left(\frac{k}{n}\right)^{(1-\alpha\delta)}$$

for sufficiently large A and $\alpha \in (1-\ln 2/\ln 3, 1)$, and

$$\mathsf{P}\left(\frac{S_k}{n} > A\left(\frac{k}{n}\right)^{\delta}\right) \le \left[56e(|a|+h|b|)\right]A^{-1/2}\left(\frac{k}{n}\right)^{(1-\delta)}$$

for $\alpha = 1$.

We have checked all the assumptions of Theorem 1, and this proves Proposition 2.

Remark 1. The paper [2] contains two results on the sufficient conditions for the convergence of characteristics. The feature of the example considered in Section 5 is that neither of the results of [2] can be applied. The crucial assumption of Theorem 2 in [2] is that the trajectories are Hölderian. This assumption does not hold for the above-mentioned example, since the trajectories of the process X are discontinuous almost surely. Theorem 3 of [2] also cannot be applied, since $\gamma = \alpha^{-1} \ge 1$. Therefore Proposition 2 shows that Theorem 1 of this paper is a nontrivial extension of the preceding results of [2].

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