

REPRESENTATIONS AND PROPERTIES OF WEIGHT FUNCTIONS IN TAUBERIAN THEOREMS

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ABSTRACT. We continue the studies of weight functions in Tauberian theorems for random fields. We obtain the rate of convergence of function series in the representation of a weight function and prove a recurrence relation for weight functions in spaces of various dimensions.

1. INTRODUCTION

Abelian and Tauberian theorems are often used when studying asymptotic properties of random fields. Most of those theorems establish a relationship between the asymptotic behavior of the spectral function at infinity and the covariance function at zero.

The variance of the integrals of random fields is studied in the paper [1]. The explicit form of the weight function $g_{n,r,a}(|t|)$ corresponding to the spectral function

$$\Phi_a(\lambda) = \Phi(a + \lambda) - \Phi(a)$$

is obtained in [1]. A new method for the evaluation of weight functions with the help of recurrence relations is proposed in the paper [2].

The current paper continues the studies of weight functions in Tauberian theorems for random fields. We consider the rate of convergence of function series in the representation of the function $g_{n,r,a}(|t|)$. A recurrence relation for weight functions is found for spaces of various dimensions. We present some numerical examples by using *Mathematica 5.0*. We also use *Maple 9.5* to check all numerical results and compare them with the tables in [9] whenever possible.

2. SOME DEFINITIONS AND NOTATION

Let \mathbf{R}^n be the Euclidean space of dimension $n \geq 2$ and let $\xi(t)$, $t \in \mathbf{R}^n$, be a real-valued measurable mean-square continuous random field, homogeneous and isotropic in the wide sense (see [4]). Assume that its mean value is zero and denote the covariance function by

$$\mathbb{B}_n(r) = \mathbb{B}_n(\|t\|) = \mathbf{E} \xi(0)\xi(t), \quad t \in \mathbf{R}^n.$$

It is known that there exists a bounded nondecreasing function $\Phi(x)$, $x \geq 0$, called the spectral function of the field $\xi(t)$, such that

$$\mathbb{B}_n(r) = 2^{(n-2)/2} \Gamma\left(\frac{n}{2}\right) \int_0^\infty \frac{J_{(n-2)/2}(rx)}{(rx)^{(n-2)/2}} d\Phi(x),$$

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where $J_\nu(z)$ is the Bessel function of the first kind and of order $\nu > -\frac{1}{2}$ (see, for example, [3, 4]).

Some asymptotic properties of the variance of integrals of random fields

$$b_n(r) = \text{Var} \left[\int_{v(r)} \xi(t) dt \right] = (2\pi)^n r^{2n} \int_0^\infty \frac{J_{n/2}^2(rx)}{(rx)^n} d\Phi(x),$$

where $v(r) = \{t \in \mathbf{R}^n : |t| \leq r\}$ is a ball in \mathbf{R}^n , are studied in the papers [3, 4] in terms of both covariance and spectral functions.

The asymptotic behavior of the function $\Phi_a(\lambda) := \Phi(a + \lambda) - \Phi(a)$ as $\lambda \rightarrow +0$ is considered in the paper [1]. The Tauberian theorem in [1] is stated in terms of the functionals

$$\tilde{b}^a(r) := (2\pi)^n \int_0^\infty \frac{J_{n/2}^2(rx)}{(rx)^n} d\Phi_a(x).$$

It is also shown in [1] that there exists a real-valued function $g_{n,r,a}(|t|)$ such that

$$\tilde{b}^a(r) = D \left[\int_{\mathbf{R}^n} g_{n,r,a}(|t|) \xi(t) dt \right].$$

Moreover

$$(1) \quad g_{n,r,a}(|t|) = \frac{1}{|t|^{n/2-1}} \int_0^\infty \underbrace{(\lambda + a)^{n/2} J_{n/2-1}(|t|(\lambda + a)) \frac{J_{n/2}(r\lambda)}{(r\lambda)^{n/2}}}_{G(\lambda)} d\lambda, \quad |t| \neq r.$$

In what follows we use the same symbol C for different constants whose exact values are not important and may vary in the course of a proof.

3. DISCUSSION OF THE PROBLEM

The rate of convergence of series in the representation of a weight function $f_{n,r,a}(|t|)$ in Tauberian theorems is obtained in the paper [5]. Some numerical results for the approximation of weight functions by partial sums of the representation are also given in [5]. We are interested in obtaining similar and new results of this type for the function $g_{r,a}(|t|)$.

Using the asymptotic behavior of the Bessel function (see [7])

$$(2) \quad J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right), \quad z \rightarrow \infty,$$

we derive the asymptotic behavior of $G(\lambda)$ given in (1), namely

$$\begin{aligned} G(\lambda) &\sim \frac{C}{\lambda} \cos\left(|t|(\lambda + a) - \frac{\pi n}{4} + \frac{\pi}{4}\right) \cos\left(r\lambda - \frac{n\pi}{4} - \frac{\pi}{4}\right) \\ &\sim \frac{C}{\lambda} \left(\cos\left((r + |t|)\lambda + |t|a - \frac{\pi n}{2}\right) - \sin\left((|t| - r)\lambda + |t|a\right) \right), \quad \lambda \rightarrow \infty. \end{aligned}$$

Thus the integral in representation (1) converges conditionally.

The classical approach uses the Poisson formula ([7, §3.3])

$$(3) \quad J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^1 (1 - x^2)^{\nu-1/2} \cos(zx) dx$$

and is based on the following representation of the functional $g_{r,a}(|t|)$:

$$\begin{aligned} g_{n,r,a}(|t|) &= \frac{1}{2^{n/2-1} \Gamma(\frac{n-1}{2}) \sqrt{\pi} r^{n/2}} \\ &\times \int_0^\infty \frac{(\lambda + a)^{n-1}}{\lambda^{n/2}} J_{n/2}(r\lambda) \int_{-1}^1 (1 - x^2)^{\frac{n-3}{2}} \cos(|t|(\lambda + a)x) dx d\lambda \end{aligned}$$

and on the change of variables formula

$$g_{n,r,a}(|t|) = \frac{1}{2^{n/2-1}\Gamma(\frac{n-1}{2})\sqrt{\pi r^{n/2}}} \times \left(\int_{-1}^1 (1-x^2)^{\frac{n-3}{2}} \cos(|t|ax) \int_0^\infty \frac{(\lambda+a)^{n-1}}{\lambda^{n/2}} J_{n/2}(r\lambda) \cos(|t|\lambda x) d\lambda dx - \int_{-1}^1 (1-x^2)^{\frac{n-3}{2}} \sin(|t|ax) \int_0^\infty \frac{(\lambda+a)^{n-1}}{\lambda^{n/2}} J_{n/2}(r\lambda) \sin(|t|\lambda x) d\lambda dx \right).$$

The rest of the proof is devoted to evaluating the above integrals.

Unfortunately, one cannot use such a change of variables in the above integral (the same is true in the case of the functional $f_{r,a}(|t|)$ considered in [5]), since the integrals with respect to λ are divergent for $n \in \mathbb{N}$ (this follows from the asymptotic formula (2)).

Another approach is used in the paper [1]. This approach is based on a representation of the function $g_{r,a}(|t|)$ in the form of the sum of the following function series:

$$(4) \quad g_{n,r,a}(|t|) = \frac{a^{n/2}\Gamma(\frac{n-2}{2})}{2^{n/2}|t|^{n/2}} \times \sum_{m=0}^\infty (-1)^m \binom{\frac{n}{2} + m - 1}{m} \binom{n+m-3}{m} J_{\frac{n-2}{2}+m}(|t|a) \times \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \times \begin{cases} {}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}, \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right), & |t| > r, \\ \frac{|t|^{m+1} {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1}\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{n-m-k+1}{2}\right)}, & |t| < r, \end{cases}$$

where the corresponding term vanishes in the cases of $|t| < r$ and

$$(m+k-n-1)/2 \in \mathbf{N} \cup \{0\},$$

since $\Gamma((n-m-k+1)/2) = \infty$.

We will use the following notation:

$$(5) \quad d_m(n, r, a, t) := (-1)^m \binom{\frac{n}{2} + m - 1}{m} \binom{n+m-3}{m} J_{\frac{n-2}{2}+m}(|t|a) \times \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \times \frac{|t|^{m+1} {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1}\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{n-m-k+1}{2}\right)},$$

$$(6) \quad s_m(n, r, a, |t|) := (-1)^m \binom{\frac{n}{2} + m - 1}{m} \binom{n+m-3}{m} J_{\frac{n-2}{2}+m}(|t|a) \times \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \times \frac{{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}, \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right)}{|t|^k \Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m+n-k-1}{2}\right)}.$$

4. ASYMPTOTIC PROPERTIES OF A HYPERGEOMETRIC FUNCTION

We need some properties of the hypergeometric function

$${}_2F_1(a, b, c; z)$$

to evaluate the sum in (4) (see [8]). First, consider the asymptotic behavior of

$${}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right)$$

as $m \rightarrow \infty$. We use the Stirling formula,

$$\Gamma(k+1) \sim \sqrt{2\pi}k^{k+1/2}e^{-k}e^{\theta_k/k} \quad \text{and} \quad k! \sim \sqrt{2\pi}k^{k+1/2}e^{-k}e^{\theta_k/k},$$

where $\theta_k \in (0; \frac{1}{12})$.

The Watson result (see [8, §2.3.2]) is often helpful when deriving the asymptotic results. Following Watson's method with

$$a = \frac{k+1}{2}, \quad c = \frac{n}{2}+1, \quad b = \frac{k-n+3}{2}, \quad \lambda = \frac{m}{2}, \quad \frac{2}{1-z} = \left(\frac{|t|}{r}\right)^2,$$

we obtain

$$\begin{aligned} & {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right) \\ & \sim \frac{\sqrt{2\pi}\Gamma(n/2+m)2^{k+2-n/2}}{\Gamma\left(\frac{k-n+1+m}{2}\right)\Gamma\left(n+\frac{m-k-1}{2}\right)m^{1/2}} \left(\frac{r}{|t|}\right)^{m+k+1} O(C^m) \\ & \sim O\left(\frac{m^{n/2+m-1/2}C^m}{m^{1/2}m^{(m+k-n)/2}m^{(m-k+2n-2)/2}}\right) = O(C^m) \end{aligned}$$

as $m \rightarrow \infty$. This formula holds in the domain $z \in \mathbf{C} \setminus (-\infty, 1)$ and, unfortunately, cannot be used directly in our case. Nevertheless the asymptotic behavior of order $O(C^m)$ can be proved.

Lemma 4.1. For $|t| < r$,

$${}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right) = O(C^m), \quad m \rightarrow \infty.$$

Proof. Using the definition of the hypergeometric function (see [5, 3]),

$$\begin{aligned} & {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}, \frac{n}{2}+m; \left(\frac{|t|}{r}\right)^2\right) = \sum_{l=0}^{\infty} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{m+k+1-n}{2}\right)_l}{\left(\frac{n}{2}+m\right)_l l!} z^l \\ & = \sum_{l=0}^{\infty} \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)\Gamma\left(\frac{n}{2}+m\right)\Gamma\left(\frac{m+k+1-n}{2}+l\right)z^l}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{n}{2}+m+l\right)\Gamma\left(\frac{m+k+1-n}{2}\right)\Gamma(l+1)} \\ & = 1 + \frac{\Gamma\left(\frac{n}{2}+m\right)}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{m+k+1-n}{2}\right)} \sum_{l=1}^{\infty} \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)\Gamma\left(\frac{m+k+1-n}{2}+l\right)z^l}{\Gamma\left(\frac{n}{2}+m+l\right)\Gamma(l+1)}. \end{aligned}$$

According to Stirling's formula, we get

$$(7) \quad \frac{\Gamma\left(\frac{n}{2}+m\right)}{\Gamma\left(\frac{m+k+1}{2}\right)\Gamma\left(\frac{m+k+1-n}{2}\right)} \sim \frac{2^{m+k-n/2}m^{n-k-1/2}}{\sqrt{2\pi}}$$

as $m \rightarrow \infty$. Using Stirling's formula again, we rearrange and estimate the series:

$$\begin{aligned}
 & \sum_{l=1}^{\infty} \frac{\Gamma\left(\frac{m+k+1}{2} + l\right) \Gamma\left(\frac{m+k+1-n}{2} + l\right) z^l}{\Gamma\left(\frac{n}{2} + m + l\right) \Gamma(l+1)} \\
 (8) \quad & \leq C \sum_{l=1}^{\infty} \frac{\left(\frac{m+k-1}{2} + l\right)^{1/2} \left(\frac{m+k-n-1}{2} + l\right)^{(m+k-n)/2} b^{(m+k-n-1)/2} z^l}{l^{1/2} \left(\frac{n}{2} + m + l - 1\right)^{(m+n-k)/2} a^{(n+m-k-1)/2}} \\
 & \leq C \sum_{l=1}^{\infty} \frac{C_1^m z^l}{l^{1/2} m^{n-k-1/2}}.
 \end{aligned}$$

After some algebra in (7) and (8) we complete the proof of the lemma. □

Theorem 4.1. For $|t| < r$,

$$(9) \quad \sum_{m=N}^{\infty} d_m(n, r, a, t) = O\left(\sum_{m=N}^{\infty} \frac{C^m}{m^{m-n/2+2}}\right), \quad N \rightarrow \infty,$$

$$(10) \quad \sum_{m=N}^{\infty} d_m(n, r, a, t) = O\left(\frac{C^N}{N^{N(1-\varepsilon)}}\right), \quad N \rightarrow \infty,$$

where $\varepsilon > 0$ is arbitrary.

Proof. Consider the asymptotic behavior of the factors in representation (4) for the function $g_{n,r,a}(|t|)$:

$$(11) \quad \binom{n+m-3}{m} \sim \frac{m^{n-3}}{(n-3)!}.$$

Applying formula [5, (22)] we obtain

$$(12) \quad J_{n/2+m-1}(|t|a) = O\left(\frac{C^m}{m^{n/2+m-1/2}}\right), \quad m \rightarrow \infty.$$

Consider

$$\begin{aligned}
 & d_k^*(m, n, r, a, t) \\
 & := \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \frac{|t|^{m+1} {}_2F_1\left(\frac{m+k+1}{2}, \frac{m+k+1-n}{2}; \frac{n}{2} + m; \left(\frac{|t|}{r}\right)^2\right)}{r^{m+k+1} \Gamma\left(\frac{n}{2} + m\right) \Gamma\left(\frac{n-m-k+1}{2}\right)}.
 \end{aligned}$$

We use Stirling's formula and follow the lines of the above proof:

$$(13) \quad d_k^*(m, n, r, a, t) = \sum_{k=0}^{n-1} O\left(\frac{C^m}{m^{n-k-1/2}}\right) = O\left(\frac{C^m}{m^{1/2}}\right), \quad m \rightarrow \infty.$$

The asymptotic formula (10) follows from

$$\begin{aligned}
 \sum_{m=N}^{\infty} \frac{C^m}{m^{(m-n/2+2)}} & \leq \sum_{m=N}^{\infty} \frac{C^m}{m^{(m-\varepsilon m)}} \leq \sum_{m=N}^{\infty} \left(\frac{C}{N^{1-\varepsilon}}\right)^m = \frac{C^N N^{1-\varepsilon}}{N^{N(1-\varepsilon)}(N^{1-\varepsilon} - C)} \\
 & = O\left(\frac{C^N}{N^{N(1-\varepsilon)}}\right), \quad N \rightarrow \infty,
 \end{aligned}$$

for all $\varepsilon > 0$. □

Now we study the asymptotic behavior of

$${}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2} + 1; \left(\frac{r}{|t|}\right)^2\right) \quad \text{as } m \rightarrow \infty.$$

We apply the result of [8, §2.3.2] again. Similarly to the above-considered case we get

$${}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right) \sim O\left(\frac{C^m}{m^{(n+1)/2}}\right), \quad m \rightarrow \infty.$$

This result holds in the domain $z \in \mathbf{C} \setminus (-\infty, 1)$ and cannot be used directly in our case. Nevertheless we prove that the asymptotic behavior is of order $O(C^m/m^{(n+1)/2})$.

Lemma 4.2. For $|t| > r$,

$${}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}, \frac{n}{2}+1; \left(\frac{r}{|t|}\right)^2\right) = O\left(\frac{C^m}{m^{n/2+1/2}}\right),$$

$m \rightarrow \infty.$

Proof. First we consider the case where $(m+n-k-3)$ is an even number. Let

$$m+n-k-3 = 2p.$$

Then

$$\begin{aligned} \left| {}_2F_1\left(\frac{m+k+1}{2}, -p, \frac{n}{2}+1; z\right) \right| &= \left| \sum_{l=0}^p \frac{\Gamma\left(\frac{m+k+1}{2}+l\right) (-1)^l p! \Gamma\left(\frac{n}{2}+1\right) z^l}{\Gamma\left(\frac{m+k+1}{2}\right) \Gamma\left(\frac{n}{2}+1+l\right) (p-l)! l!} \right| \\ &\leq \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{m+k+1}{2}\right)} \sum_{l=0}^p \binom{p}{l} z^l \frac{\Gamma\left(\frac{m+k+1}{2}+l\right)}{\Gamma\left(\frac{n}{2}+1+l\right)} \\ &\leq \frac{\Gamma\left(\frac{m+k+1}{2}+p\right) \Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1+p\right) \Gamma\left(\frac{m+k+1}{2}\right)} (1+z)^p, \end{aligned}$$

since

$$\frac{\Gamma\left(\frac{m+k+1}{2}+l\right)}{\Gamma\left(\frac{n}{2}+1+l\right)} \leq \frac{\Gamma\left(\frac{m+k+1}{2}+p\right)}{\Gamma\left(\frac{n}{2}+1+p\right)}$$

for all $l = 0, \dots, p$. By Stirling's formula,

$$\begin{aligned} \frac{\Gamma\left(\frac{m+k+1}{2}+p\right)}{\Gamma\left(\frac{n}{2}+1+p\right)} &= \frac{\Gamma\left(m-2+\frac{n}{2}+1\right)}{\Gamma\left(n+\frac{m-k-3}{2}+1\right)} \sim C^m \frac{m^{m+n/2-3/2}}{m^{m/2+n-k/2-1}} = C^m m^{(m-n+k-1)/2}, \\ {}_2F_1\left(\frac{m+k+1}{2}+l, -p, \frac{n}{2}+1; z\right) &= O\left(\frac{C^m}{m^{(n+1)/2}}\right). \end{aligned}$$

Now let $(m+n-k-3)$ be an odd number. Then

$$\begin{aligned} (14) \quad &{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}, \frac{n}{2}+1; z\right) \\ &= \sum_{l=0}^{\lfloor \frac{m+n-k-3}{2} \rfloor + 1} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!} z^l \\ &\quad + \sum_{l=\lfloor \frac{m+n-k-3}{2} \rfloor + 2}^{\infty} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2}+1\right)_l l!} z^l. \end{aligned}$$

We have

$$\begin{aligned} & \left| \left(\frac{m+k+1}{2} \right)_l \left(\frac{3+k-m-n}{2} \right)_l \right| \\ &= \left| \left(\frac{m+k+1}{2} \right) \dots \left(\frac{m+k+1}{2} + l - 1 \right) \right. \\ & \quad \left. \times \left(\frac{m+n-k-3}{2} \right) \dots \left(\frac{m+n-k-3}{2} - l + 1 \right) \right| \\ & \leq \left| \left(\frac{m+k+1}{2} + \frac{1}{2} \right) \dots \left(\frac{m+k+1}{2} + l - 1 + \frac{1}{2} \right) \right. \\ & \quad \left. \times \left(\frac{m+n-k-3}{2} + \frac{1}{2} \right) \dots \left(\frac{m+n-k-3}{2} - l + 1 + \frac{1}{2} \right) \right| \\ &= \left| \left(\frac{m+k}{2} + 1 \right)_l \left(\frac{2+k-m-n}{2} \right)_l \right|. \end{aligned}$$

Thus

$$\begin{aligned} (15) \quad & \left| \sum_{l=0}^{\lfloor \frac{m+n-k-3}{2} \rfloor + 1} \frac{\left(\frac{m+k+1}{2} \right)_l \left(\frac{3+k-m-n}{2} \right)_l}{\left(\frac{n}{2} + 1 \right)_l l!} z^l \right| \leq \sum_{l=0}^{\frac{m+n-k-2}{2}} \frac{\left(\frac{m+k+2}{2} \right)_l \left(\frac{2+k-m-n}{2} \right)_l}{\left(\frac{n}{2} + 1 \right)_l l!} z^l \\ &= O\left(\frac{C^m}{m^{(n+1)/2}} \right), \quad m \rightarrow \infty. \end{aligned}$$

Now we estimate the second term in (14):

$$\begin{aligned} & \sum_{l=\lfloor \frac{m+n-k-3}{2} \rfloor + 2}^{\infty} \frac{\left(\frac{m+k+1}{2} \right)_l \left(\frac{3+k-m-n}{2} \right)_l}{\left(\frac{n}{2} + 1 \right)_l l!} z^l \\ &= \sum_{l=\lfloor \frac{m+n-k-3}{2} \rfloor + 2}^{\infty} \frac{\Gamma\left(\frac{m+k+1}{2} + l\right) \Gamma\left(\frac{n}{2} + 1\right) \Gamma\left(-\frac{m+n-k-3}{2}\right) \Gamma\left(-\frac{m+n-k-3}{2} + 1\right) \dots}{\Gamma\left(\frac{m+k+1}{2}\right) \Gamma\left(\frac{n}{2} + 1 + l\right) \Gamma(l + 1)} \\ & \quad \times \left(-\frac{1}{2}\right) \left(\frac{1}{2}\right) \dots \left(-\frac{m+n-k-3}{2} + l - 1\right) z^l \\ &= \left| \begin{array}{l} s = l - \lfloor \frac{m+n-k-3}{2} \rfloor - 1 \\ l = s + \frac{m+n-k-2}{2} \end{array} \right| \\ &= \frac{(-1)^{\frac{m+n-k-2}{2}} \Gamma\left(\frac{m+n-k-1}{2}\right) \Gamma\left(\frac{n}{2} + 1\right) z^{\frac{m+n-k-2}{2}}}{\pi \Gamma\left(\frac{m+k+1}{2}\right)} \\ & \quad \times \sum_{s=1}^{\infty} \frac{\Gamma\left(s + m + \frac{n-1}{2}\right) \Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(s + n + \frac{m-k}{2}\right) \Gamma\left(s + \frac{m+n-k}{2}\right)} \end{aligned}$$

and

$$\begin{aligned} (16) \quad & \sum_{s=1}^{\infty} \frac{\Gamma\left(s + m + \frac{n-1}{2}\right) \Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(s + n + \frac{m-k}{2}\right) \Gamma\left(s + \frac{m+n-k}{2}\right)} \leq C^m \sum_{s=1}^{\infty} \frac{b_{s + \frac{m+n-k-2}{2}}^{\frac{m-n+k-1}{2} (s + m + \frac{n}{2} - 1)}}{\left(s + \frac{m+n-k-2}{2}\right)^{n-k}} z^s \\ & \leq \frac{C^m}{\left(\frac{m+n-k-2}{2}\right)^{n-k}} \sum_{s=1}^{\infty} z^s. \end{aligned}$$

Representation (16) and Stirling's formula imply that

$$(17) \quad \left| \sum_{l=\lceil \frac{m+n-k-3}{2} \rceil + 2}^{\infty} \frac{\left(\frac{m+k+1}{2}\right)_l \left(\frac{3+k-m-n}{2}\right)_l}{\left(\frac{n}{2} + 1\right)_l l!} z^l \right| \\ = O\left(\frac{C^m \Gamma\left(\frac{m+n-k-1}{2}\right)}{\left(\frac{m+n-k-2}{2}\right)^{n-k} \Gamma\left(\frac{m+k+1}{2}\right)}\right) = O\left(\frac{C^m}{m^{(n+1)/2}}\right), \quad m \rightarrow \infty.$$

Substituting (15) and (17) in (14) we complete the proof of the lemma. \square

Theorem 4.2. *If $|t| > r$, then*

$$(18) \quad \sum_{m=N}^{\infty} s_m(n, r, a, t) = O\left(\sum_{m=N}^{\infty} \frac{C^m}{m^{m-n/2+2}}\right), \quad N \rightarrow \infty,$$

$$(19) \quad \sum_{m=N}^{\infty} s_m(n, r, a, t) = O\left(\frac{C^N}{N^{N(1-\varepsilon)}}\right), \quad N \rightarrow \infty,$$

for all $\varepsilon > 0$.

Proof. First we consider the asymptotic behavior of the expression for $s_m(n, r, a, t)$ in formula (6). Put

$$s_m^*(n, r, a, t) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{2}{a}\right)^k \Gamma\left(\frac{m+k+1}{2}\right) \frac{{}_2F_1\left(\frac{m+k+1}{2}, \frac{3+k-m-n}{2}; \frac{n}{2} + 1; \left(\frac{r}{|t|}\right)^2\right)}{|t|^k \Gamma\left(\frac{n}{2} + 1\right) \Gamma\left(\frac{m+n-k-1}{2}\right)}.$$

Using Stirling's formula and Lemma 4.2 we obtain

$$(20) \quad s_m^*(n, r, a, t) = \sum_{k=0}^{n-1} O\left(\frac{C^m}{m^{n-k-1/2}}\right) = O\left(\frac{C^m}{m^{1/2}}\right).$$

Applying bounds (11), (12), and (20) we prove relation (18).

Equality (19) is proved similarly to the proof of Theorem 4.1. \square

5. RECURRENCE RELATIONS FOR THE WEIGHT FUNCTIONS $g_{n,r,a}(t)$

Recalling the well-known identities for the derivatives of the Bessel functions (see §3.2 in [7]), we get

$$\frac{d}{dz} (z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z), \quad \frac{d}{dz} \left(\frac{J_\nu(z)}{z^\nu}\right) = -\frac{J_{\nu+1}(z)}{z^\nu}.$$

Integrating by parts the integral in representation (9) (see [1]) for the weight function $g_{n,r,a}(t)$ and considering the asymptotic behavior of the Bessel function at the origin and

at infinity, we get

$$\begin{aligned}
g_{n,r,a}(|t|) &= \frac{1}{|t|^{n/2-1}} \int_0^\infty (\lambda+a)^{n/2} J_{n/2-1}(|t|(\lambda+a)) \frac{J_{n/2}(r\lambda)}{(r\lambda)^{n/2}} d\lambda \\
&= \frac{J_{n/2}(r\lambda)}{|t|^n (r\lambda)^{n/2}} \left(|t|(\lambda+a)^{n/2} J_{n/2}(|t|(\lambda+a)) \right) \Big|_0^\infty \\
&\quad + \frac{r}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2}} d\lambda \\
&= - \left(\frac{a}{2|t|} \right)^{n/2} \cdot \frac{J_{n/2}(|t|a)}{\Gamma(\frac{n}{2}+1)} \\
&\quad + \frac{r}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2}} d\lambda.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
g_{n,r,a}(|t|) &= - \left(\frac{a}{2|t|} \right)^{n/2} \cdot \frac{J_{n/2}(|t|a)}{\Gamma(\frac{n}{2}+1)} \\
&\quad + \frac{r^2}{|t|^{n+1}} \int_0^\infty (|t|(\lambda+a))^{n/2+1} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}} d\lambda \\
&\quad - \frac{ar^2}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}} d\lambda \\
&= -h_{n,a}(|t|) + r^2 g_{n+2,r,a}(|t|) - r^2 G_n(r, a, |t|),
\end{aligned}$$

where

$$(21) \quad h_{n,a}(|t|) = \left(\frac{a}{2|t|} \right)^{n/2} \cdot \frac{J_{\frac{n}{2}}(|t|a)}{\Gamma(\frac{n}{2}+1)},$$

$$(22) \quad G_n(r, a, |t|) = \frac{a}{|t|^n} \int_0^\infty (|t|(\lambda+a))^{n/2} J_{n/2}(|t|(\lambda+a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}} d\lambda.$$

The recurrence relation implies the following result.

Lemma 5.1. *For an arbitrary $n \geq 1$,*

$$g_{n+2,r,a}(|t|) = \frac{g_{n,r,a}(|t|) + h_{n,a}(|t|)}{r^2} + G_n(r, a, |t|),$$

where $h_{n,a}(|t|)$ and $G_n(r, a, |t|)$ are defined in (21) and (22), respectively.

Corollary 5.1. *Let $n > m \geq 1$. Then*

(i) *if n and m are even numbers,*

$$g_{n,r,a}(|t|) = \frac{g_{m,r,a}(|t|)}{r^{n-m}} + \sum_{k=m/2+1}^{n/2} \frac{G_{2k-2}(r, a, |t|)}{r^{n-2k}} + \sum_{k=m/2+1}^{n/2} \frac{h_{2k-2,a}(|t|)}{r^{n-2k+2}};$$

(ii) *if n and m are odd numbers,*

$$g_{n,r,a}(|t|) = \frac{g_{m,r,a}(|t|)}{r^{n-m}} + \sum_{k=(m+1)/2}^{(n-1)/2} \frac{G_{2k-1}(r, a, |t|)}{r^{n-2k-1}} + \sum_{k=(m+1)/2}^{(n-1)/2} \frac{h_{2k-1,a}(|t|)}{r^{n-2k+1}}.$$

6. EVALUATION OF $g_{1,r,a}(|t|)$

Lemma 6.1. *The function $g_{1,r,a}(|t|)$ is given by*

$$(23) \quad g_{1,r,a}(|t|) = \frac{\log\left(\frac{|t|-r}{|t|+r}\right) \sin(a|t|)}{\pi r}$$

for $|t| > r$ and by

$$(24) \quad g_{1,r,a}(|t|) = \frac{\pi \cos(a|t|) + \log\left(\frac{r-|t|}{r+|t|}\right) \sin(a|t|)}{\pi r}$$

for $|t| < r$.

Proof. Applying [7, §3.4] we get

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z), \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos(z).$$

Hence

$$g_{1,r,a}(|t|) = \int_0^\infty \frac{J_{\frac{1}{2}}(r\lambda)}{\sqrt{r\lambda}} \sqrt{|t|(\lambda+a)} J_{-\frac{1}{2}}(|t|(\lambda+a)) d\lambda = \frac{2}{\pi} \int_0^\infty \frac{\cos(|t|(\lambda+a)) \sin(r\lambda)}{r\lambda} d\lambda.$$

Now we use *Mathematica 5.0* and [9] to obtain equalities (23) and (24). \square

7. EVALUATION OF $g_{2,r,a}(|t|)$

Lemma 7.1. *If $|t| \neq r$, then*

$$(25) \quad g_{2,r,a}(|t|) = \frac{2}{\pi r^2} \int_0^{\min(1,r/|t|)} \frac{a(r^2 - |t|^2 x^2) \cos(a|t|x) - |t|x \sin(a|t|x)}{\sqrt{1-x^2} \sqrt{r^2 - |t|^2 x^2}} dx \\ + \frac{J_0(a|t|) - a|t|J_1(a|t|)}{r^2} \\ + \frac{2}{\pi r^2} \int_{\min(1,r/|t|)}^1 \frac{a(|t|^2 x^2 - r^2) \sin(a|t|x) - |t|x \cos(a|t|x)}{\sqrt{1-x^2} \sqrt{|t|^2 x^2 - r^2}} dx.$$

Proof. We use the Poisson integral (3) to evaluate $g_{2,r,a}(|t|)$:

$$(26) \quad g_{2,r,a}(|t|) = \int_0^\infty \frac{\lambda+a}{r\lambda} J_0(|t|(\lambda+a)) J_1(r\lambda) d\lambda \\ = \frac{1}{r\pi} \int_0^\infty \frac{\lambda+a}{\lambda} J_1(r\lambda) \int_{-1}^1 \frac{\cos(|t|(\lambda+a)x)}{\sqrt{1-x^2}} dx d\lambda \\ (27) \quad = \frac{1}{r\pi} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \int_0^\infty \frac{\lambda+a}{\lambda} J_1(r\lambda) \cos(|t|(\lambda+a)x) d\lambda dx.$$

Now we justify the change of order of integration. The expression (27) can be rewritten as follows:

$$(28) \quad \int_0^\infty \int_{-1}^1 Q_{|t|,r,a}(x, \lambda) dx d\lambda \\ = \int_0^C \int_{-1}^1 Q_{|t|,r,a}(x, \lambda) dx d\lambda + \int_C^\infty \int_{-1}^1 Q_{|t|,r,a}(x, \lambda) dx d\lambda,$$

where

$$Q_{|t|,r,a}(x, \lambda) := \frac{\lambda+a}{\lambda\sqrt{1-x^2}} J_1(r\lambda) \cos(|t|(\lambda+a)x).$$

The domain of integration for the integral (28) is a compact set, and the integral converges absolutely. Thus the change of order is justified for this integral.

Now we show that

$$(29) \quad \int_C \int_{-1}^1 Q_{|t|,r,a}(x, \lambda) dx d\lambda \rightarrow 0, \quad \int_{-1}^1 \int_C Q_{|t|,r,a}(x, \lambda) d\lambda dx \rightarrow 0$$

as $C \rightarrow +\infty$. The first part of (29) follows from representation (26):

$$\int_C \int_{-1}^1 Q_{|t|,r,a}(x, \lambda) dx d\lambda = \pi \int_C \frac{\lambda + a}{\lambda} J_0(|t|(\lambda + a)) J_1(r\lambda) d\lambda$$

and from the known asymptotic behavior of the Bessel function (see [5, (2)]). We introduce the notation

$$\begin{aligned} & \int_C \frac{\lambda + a}{\lambda} J_1(r\lambda) \cos(|t|(\lambda + a)x) d\lambda \\ &= \cos(|t|ax) \underbrace{\int_C \frac{\lambda + a}{\lambda} J_1(r\lambda) \cos(|t|\lambda x) d\lambda}_{I_1} - \sin(|t|ax) \underbrace{\int_C \frac{\lambda + a}{\lambda} J_1(r\lambda) \sin(|t|\lambda x) d\lambda}_{I_2}. \end{aligned}$$

to prove the second part of (29).

It is known that

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2},$$

where $H_\nu^{(k)}(z)$, $k = 1, 2$, are Bessel functions of the third kind (see [7, (1), §3.61]).

Now we use the asymptotic representations of the Bessel functions (3) and (4) for $\nu = 1$ and $p = 1$ (see [7, §7.2]). Then

$$H_1^{(k)}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i(z - 3\pi/4)} \left(1 + O\left(\frac{1}{z}\right) \right)$$

for $k = 1$ and 2 , respectively.

Consider the following bound:

$$\begin{aligned} |I_1| &\leq \left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} e^{ir\lambda} \cos(|t|\lambda x) d\lambda \right| + \left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} e^{ir\lambda} \cos(|t|\lambda x) O\left(\frac{1}{\lambda}\right) d\lambda \right| \\ &\quad + \left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} e^{-ir\lambda} \cos(|t|\lambda x) d\lambda \right| + \left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} e^{-ir\lambda} \cos(|t|\lambda x) O\left(\frac{1}{\lambda}\right) d\lambda \right| \\ &\leq \left| \int_C \frac{\sqrt{2}(\lambda + a)}{\lambda\sqrt{\pi r\lambda}} \cos(r\lambda) \cos(|t|\lambda x) d\lambda \right| + \left| \int_C \frac{\sqrt{2}(\lambda + a)}{\lambda\sqrt{\pi r\lambda}} \sin(r\lambda) \cos(|t|\lambda x) d\lambda \right| \\ &\quad + \int_C \frac{C_1 d\lambda}{2\lambda^{3/2}} \\ &\leq \underbrace{\left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} \cos(\lambda(r + |t|x)) d\lambda \right|}_{I_1} + \underbrace{\left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} \cos(\lambda(r - |t|x)) d\lambda \right|}_{I_2} \\ &\quad + \underbrace{\left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} \sin(\lambda(r + |t|x)) d\lambda \right|}_{I_3} + \underbrace{\left| \int_C \frac{\lambda + a}{\lambda\sqrt{2\pi r\lambda}} \sin(\lambda(r - |t|x)) d\lambda \right|}_{I_4} + \frac{C_1}{\sqrt{C}}. \end{aligned}$$

Then

$$\left| \int_{-1}^1 \frac{\cos(|t|ax) I_1}{\sqrt{1-x^2}} dx \right| \leq \int_{-1}^1 \left(|I_1| + |I_2| + |I_3| + |I_4| + \frac{C_1}{\sqrt{C}} \right) \frac{dx}{\sqrt{1-x^2}}.$$

Now

$$\int_{-1}^1 \frac{|\mathcal{I}_1|}{\sqrt{1-x^2}} dx = \int_{-1}^1 \left| \int_{C|r+|t|x}^{\infty} \frac{z+a(r+|t|x)}{\sqrt{2\pi rz|r+|t|x}} \cdot \frac{\cos(z)}{z} dz \right| \frac{dx}{\sqrt{1-x^2}}.$$

We have

$$\frac{1}{\sqrt{1-x^2}} \left| \int_{C|r+|t|x}^{\infty} \frac{z+a(r+|t|x)}{\sqrt{2\pi rz|r+|t|x}} \cdot \frac{\cos(z)}{z} dz \right| \rightarrow 0 \quad \text{as } C \rightarrow +\infty$$

and

$$\begin{aligned} & \frac{1}{\sqrt{1-x^2}} \left| \int_{C|r+|t|x}^{\infty} \frac{z+a(r+|t|x)}{\sqrt{2\pi rz|r+|t|x}} \cdot \frac{\cos(z)}{z} dz \right| \\ & \leq \frac{1}{\sqrt{1-x^2} \sqrt{2\pi rz|r+|t|x}} \cdot \sup_{C \geq 0} \left| \int_C^{\infty} \frac{z+a|r+|t|x|}{z} \cdot \frac{\cos(z)}{\sqrt{z}} dz \right| \in L_1([-1, 1]) \end{aligned}$$

for all $|t| \neq r$ and $x \in (-1, 1)$. By the Lebesgue dominated convergence theorem,

$$\int_{-1}^1 \frac{|\mathcal{I}_1|}{\sqrt{1-x^2}} dx \rightarrow 0 \quad \text{as } C \rightarrow \infty.$$

The proof of

$$\int_{-1}^1 \frac{|\mathcal{I}_k|}{\sqrt{1-x^2}} dx \rightarrow 0 \quad \text{as } C \rightarrow \infty$$

for $k = 2, 3, 4$ is similar. Thus the second relation in (29) holds. Running *Mathematica 5.0*, we get expression (25) for the function $g_{2,r,a}(|t|)$. \square

8. THE MAIN REPRESENTATION

Theorem 8.1. *If n is an odd number, then*

$$g_{n,r,a}(|t|) = \frac{g_{1,r,a}(|t|)}{r^{n-1}} + \sum_{k=1}^{\frac{n-1}{2}} \frac{G_{2k-1}(r, a, |t|)}{r^{n-2k-1}} + \sum_{k=1}^{\frac{n-1}{2}} \frac{h_{2k-1,a}(|t|)}{r^{n-2k+1}},$$

where $g_{1,r,a}(|t|)$ is defined in Lemma 6.1.

If n is an even number, then

$$g_{n,r,a}(|t|) = \frac{g_{2,r,a}(|t|)}{r^{n-2}} + \sum_{k=2}^{n/2} \frac{G_{2k-2}(r, a, |t|)}{r^{n-2k}} + \sum_{k=2}^{n/2} \frac{h_{2k-2,a}(|t|)}{r^{n-2k+2}},$$

where $g_{2,r,a}(|t|)$ is defined in Lemma 7.1.

The functions $h_{k,a}(|t|)$ and $G_k(r, a, |t|)$ are given by equalities (21) and (22), respectively.

Remark. In contrast to representation (1) of Theorem 8.1, the function $g_{n,r,a}(|t|)$ is expressed in terms of absolutely convergent integrals.

9. PROPERTIES OF WEIGHT FUNCTIONS

We study some properties of weight functions $g_{n,r,a}(|t|)$.

Theorem 9.1. *If $n \geq 1$, then*

$$\lim_{|t| \rightarrow +\infty} g_{n,r,a}(|t|) = 0.$$

Proof. Representation (23) obtained in Lemma 6.1 implies that

$$|g_{1,r,a}(|t|)| \leq \frac{1}{\pi r} \left| \log \left(\frac{|t| - r}{|t| + r} \right) \right| \rightarrow 0, \quad |t| \rightarrow +\infty,$$

for $|t| > r$. The function $g_{2,r,a}(|t|)$ defined by equality (26) can be rewritten as a sum of two integrals, namely

$$(30) \quad \begin{aligned} g_{2,r,a}(|t|) &= \int_0^1 \frac{\lambda + a}{r\lambda} J_0(|t|(\lambda + a)) J_1(r\lambda) d\lambda \\ &+ \int_1^\infty \frac{\lambda + a}{r\lambda} J_0(|t|(\lambda + a)) J_1(r\lambda) d\lambda. \end{aligned}$$

Consider the first integral in equality (30). The representation

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos(z \cos \theta) \sin^{2\nu} \theta d\theta$$

(see [7, §3.1]) implies that

$$(31) \quad \left| \frac{J_\nu(z)}{z^\nu} \right| \leq \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu + \frac{1}{2})}.$$

Thus

$$\left| \frac{J_1(r\lambda)}{r\lambda} \right| \leq 1, \quad |J_0(|t|(\lambda + a))| \leq 1$$

by inequality (31) for all $\lambda \in [0, 1]$, $r > 0$, and $|t| \geq 0$. Now we use the asymptotic expansion (2) and obtain

$$J_0(|t|(\lambda + a)) \rightarrow 0, \quad |t| \rightarrow \infty,$$

for all $\lambda \in [0, 1]$. By the Lebesgue dominated convergence theorem,

$$\left| \int_0^1 \frac{\lambda + a}{r\lambda} J_0(|t|(\lambda + a)) J_1(r\lambda) d\lambda \right| \rightarrow 0, \quad |t| \rightarrow \infty.$$

Consider the second integral in representation (30). Using the bound obtained in Theorem 4.1 of the paper [6], we show that

$$(32) \quad \left| J_\nu(x) - \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\pi\nu}{2} - \frac{\pi}{4} \right) \right| \leq \frac{d_\nu}{x^{3/2}},$$

where $\nu \geq -1/2$, $x > 0$, and d_ν are some constants.

Applying inequality (32) to the second integral on the right-hand side of equality (30), we get

$$\begin{aligned}
& \left| \int_1^\infty \frac{\lambda+a}{r\lambda} J_0(|t|(\lambda+a)) J_1(r\lambda) d\lambda \right| \\
&= \left| \int_1^\infty \frac{\lambda+a}{r\lambda} \left(\sqrt{\frac{2}{\pi|t|(\lambda+a)}} \cos(|t|(\lambda+a) - \pi/4) + \varepsilon_0(|t|, a, \lambda) \right) \right. \\
&\quad \left. \times \left(\sqrt{\frac{2}{\pi r\lambda}} \cos(r\lambda - 3\pi/4) + \varepsilon_1(r, \lambda) \right) d\lambda \right| \\
&\leq \underbrace{\left| \frac{2}{\pi r \sqrt{r|t|}} \int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a) - \pi/4) \cos(r\lambda - 3\pi/4) d\lambda \right|}_{\mathcal{J}_1} \\
&\quad + \underbrace{\left| \frac{\sqrt{2}}{r\sqrt{\pi|t|}} \int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda} \cos(|t|(\lambda+a) - \pi/4) \varepsilon_1(r, \lambda) d\lambda \right|}_{\mathcal{J}_2} \\
&\quad + \underbrace{\left| \frac{\sqrt{2}}{\sqrt{\pi r^{3/2}}} \int_1^\infty \frac{\lambda+a}{\lambda^{3/2}} \cos(r\lambda - 3\pi/4) \varepsilon_0(|t|, a, \lambda) d\lambda \right|}_{\mathcal{J}_3} \\
&\quad + \underbrace{\left| \frac{1}{r} \int_1^\infty \frac{\lambda+a}{\lambda} \varepsilon_0(|t|, a, \lambda) \varepsilon_1(r, \lambda) d\lambda \right|}_{\mathcal{J}_4},
\end{aligned}$$

where

$$|\varepsilon_0(|t|, a, \lambda)| \leq \frac{d_0}{(|t|(\lambda+a))^{3/2}}, \quad |\varepsilon_1(r, \lambda)| \leq \frac{d_1}{(r\lambda)^{3/2}}.$$

Now we study the asymptotic behavior of the integrals \mathcal{J}_i , $i = 1, \dots, 4$, as $|t| \rightarrow \infty$. It is clear that

$$\begin{aligned}
|\mathcal{J}_1| &= \left| \frac{1}{\pi r \sqrt{r|t|}} \int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a) + r\lambda - \pi) d\lambda \right. \\
&\quad \left. + \frac{1}{\pi r \sqrt{r|t|}} \int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a) - r\lambda + \pi/2) d\lambda \right| \\
&\leq \underbrace{\left| \frac{1}{\pi r \sqrt{r|t|}} \int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda+a) + r\lambda) d\lambda \right|}_{\mathcal{K}_1} \\
&\quad + \underbrace{\left| \frac{1}{\pi r \sqrt{r|t|}} \int_1^\infty \frac{\sqrt{\lambda+a}}{\lambda\sqrt{\lambda}} \sin(|t|(\lambda+a) - r\lambda) d\lambda \right|}_{\mathcal{K}_2}.
\end{aligned}$$

The integrand in \mathcal{K}_1 changes its sign and assumes zero values at the points

$$(33) \quad \lambda = \frac{\pi/2 + \pi k - |t|a}{|t| + r}, \quad k \in \mathbf{N} \cup 0.$$

Since the lower limit of the integral \mathcal{K}_1 equals 1, we derive from (33) that $k \geq s$, where

$$s := \frac{r + |t|(1 + a) - \pi/2}{\pi}.$$

Then

$$(34) \quad \left| \int_1^\infty \frac{\sqrt{\lambda + a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda + a) + r\lambda) d\lambda \right| \\ = \left| \int_1^{\frac{3\pi/2 + \pi[s] - |t|a}{|t| + r}} \frac{\sqrt{\lambda + a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda + a) + r\lambda) d\lambda \right. \\ \left. + \sum_{k \geq [s] + 1} \int_{\frac{\pi/2 + \pi k - |t|a}{|t| + r}}^{\frac{\pi/2 + \pi(k+1) - |t|a}{|t| + r}} \frac{\sqrt{\lambda + a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda + a) + r\lambda) d\lambda \right|.$$

The series in representation (34) is sign alternating and the absolute value of every term is less than or equal to the absolute value of the preceding term, since $\sqrt{\lambda + a}/(\lambda\sqrt{\lambda})$ is a decreasing function of λ for $\lambda > 1$. By Leibniz' theorem

$$(35) \quad \left| \sum_{k \geq [s] + 1} \int_{\frac{\pi/2 + \pi k - |t|a}{|t| + r}}^{\frac{\pi/2 + \pi(k+1) - |t|a}{|t| + r}} \frac{\sqrt{\lambda + a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda + a) + r\lambda) d\lambda \right| \\ \leq \left| \int_{\frac{3\pi/2 + \pi[s] - |t|a}{|t| + r}}^{\frac{5\pi/2 + \pi[s] - |t|a}{|t| + r}} \frac{\sqrt{\lambda + a}}{\lambda\sqrt{\lambda}} \cos(|t|(\lambda + a) + r\lambda) d\lambda \right|.$$

We derive from (34) and (35) that

$$|\mathcal{K}_1| \leq \frac{1}{\pi r \sqrt{r|t|}} \int_1^{\frac{5\pi/2 + \pi[s] - |t|a}{|t| + r}} \frac{\sqrt{\lambda + a}}{\lambda\sqrt{\lambda}} d\lambda \\ \leq \frac{1}{\pi r \sqrt{r|t|}} \int_1^{1 + \frac{2\pi}{|t| + r}} \frac{\sqrt{\lambda + a}}{\lambda\sqrt{\lambda}} d\lambda \rightarrow 0, \quad |t| \rightarrow \infty.$$

A similar reasoning shows that $|\mathcal{K}_2| \rightarrow 0$ as $|t| \rightarrow \infty$. Therefore $|\mathcal{J}_1| \rightarrow 0$ as $|t| \rightarrow \infty$.

Estimating the second integral \mathcal{J}_2 we prove that

$$|\mathcal{J}_2| \leq \frac{\sqrt{2}d_1}{r^2 \sqrt{\pi r|t|}} \int_1^\infty \frac{\sqrt{\lambda + a}}{\lambda^{5/2}} d\lambda \rightarrow 0, \quad |t| \rightarrow \infty,$$

since the integral $\int_1^\infty \sqrt{\lambda + a}/\lambda^{5/2} d\lambda$ converges. Similarly we show that

$$|\mathcal{J}_i| \rightarrow 0, \quad |t| \rightarrow \infty,$$

$i = 3, 4$. Thus $|g_{2,r,a}(|t|)| \rightarrow 0$ as $|t| \rightarrow \infty$.

It is clear that

$$\lim_{|t| \rightarrow +\infty} |h_{n,a}(|t|)| = \frac{a^n}{2^{n/2} \Gamma(\frac{n}{2} + 1)} \lim_{|t| \rightarrow +\infty} \frac{|J_{n/2}(|t|a)|}{(|t|a)^{n/2}} = 0.$$

Now we show that

$$(36) \quad |G_n(r, a, |t|)| = \frac{a}{|t|^n} \left| \int_0^\infty \underbrace{(|t|(\lambda + a))^{n/2} J_{n/2}(|t|(\lambda + a)) \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}}}_{S(n,r,a,|t|)} d\lambda \right| \rightarrow 0$$

as $|t| \rightarrow +\infty$. To check relation (36) we use the Lebesgue dominated convergence theorem again. In view of the asymptotic behavior of the Bessel function at infinity and at zero,

$$(37) \quad \frac{a}{|t|^n} |S(n, r, a, |t|)| \leq \frac{C}{|t|^{n/2}} \frac{|J_{n/2+1}(r\lambda)|}{(r\lambda)^{n/2+1}} (\lambda + a)^{n/2} \in L_1([0, +\infty]),$$

$$\frac{a}{|t|^{n/2}} \left| \frac{J_{n/2+1}(r\lambda)}{(r\lambda)^{n/2+1}} (\lambda + a)^{n/2} J_{n/2}(|t|(\lambda + a)) \right| \rightarrow 0$$

as $|t| \rightarrow +\infty$. Therefore relation (36) holds and the proof of Theorem 9.1 is complete. \square

Theorem 9.2. *If $n \geq 1$, then*

$$\lim_{r \rightarrow \infty} g_{n,r,a}(|t|) = 0.$$

Proof. Representation (24) of Lemma 6.1 implies that

$$|g_{1,r,a}(|t|)| = \frac{1}{\pi r} \left| \pi \cos(a|t|) + \log \left(\frac{r - |t|}{r + |t|} \right) \sin(a|t|) \right| \rightarrow 0$$

as $r \rightarrow \infty$.

By (3) and [9] we know that

$$J_0(a|t|) = \frac{2}{\pi} \int_0^1 \frac{\cos(a|t|x)}{\sqrt{1-x^2}} dx \quad ia|t|J_1(a|t|) = \frac{2}{\pi} \int_0^1 \frac{a|t|x}{\sqrt{1-x^2}} \sin(a|t|x) dx,$$

$$(38) \quad |g_{2,r,a}(|t|)| = \frac{2}{\pi r^2} \left| \int_0^1 \underbrace{\left((1 + a\sqrt{r^2 - |t|^2 x^2}) \cos(a|t|x) \right)}_{R_1(|t|, r, a, x)} \right. \\ \left. - \underbrace{|t|x \left(a + \frac{1}{\sqrt{r^2 - |t|^2 x^2}} \right) \sin(a|t|x)}_{R_2(|t|, r, a, x)} \frac{dx}{\sqrt{1-x^2}} \right| \rightarrow 0$$

as $r \rightarrow \infty$. Indeed,

$$\frac{1}{r^2} \left| \frac{R_1(|t|, r, a, x) - R_2(|t|, r, a, x)}{\sqrt{1-x^2}} \right| \\ \leq \frac{1}{\sqrt{1-x^2}} \left| \frac{1 + a\sqrt{r^2 - |t|^2 x^2}}{r^2} \right| \cdot \left| 1 + \frac{|t|x}{\sqrt{r^2 - |t|^2 x^2}} \right| \in L_1([0, 1])$$

and

$$\frac{1}{r^2} \left| \frac{R_1(|t|, r, a, x) - R_2(|t|, r, a, x)}{\sqrt{1-x^2}} \right| \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

This proves relation (38) by the Lebesgue dominated convergence theorem. Similarly to the proof of (36) (but for the case of $r \rightarrow \infty$) we apply the Lebesgue dominated convergence theorem and obtain

$$\frac{a}{|t|^n} |S(n, r, a, |t|)| \leq \frac{C}{|t|^{n/2}} (\lambda + a)^{n/2} |J_{n/2}(|t|(\lambda + a))| \in L_1([0, C_1]),$$

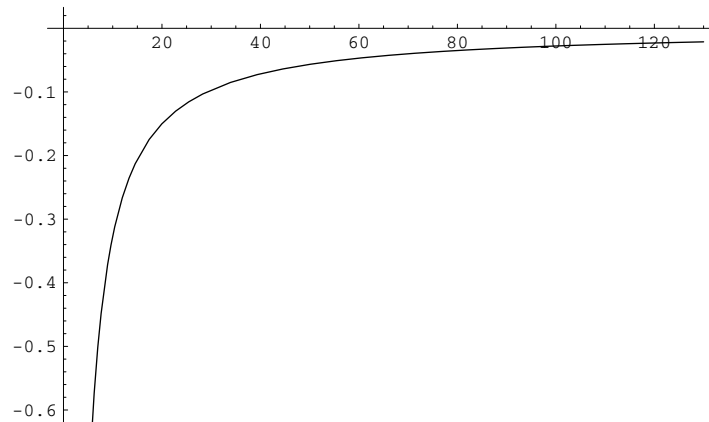
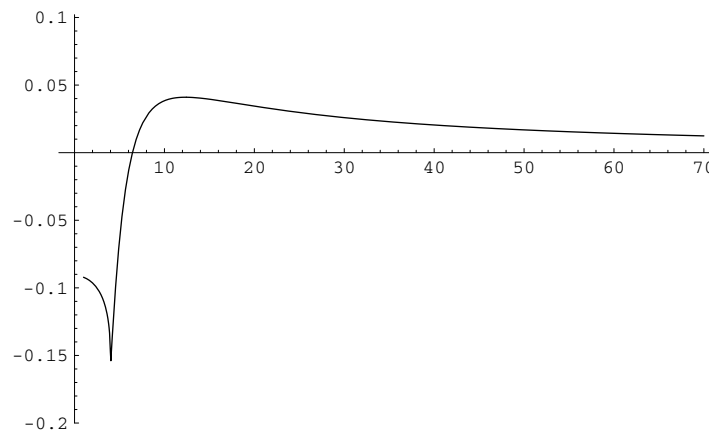
$$\frac{a}{|t|^n} |S(n, r, a, |t|)| \leq \frac{C}{|t|^{n/2}} \frac{(\lambda + a)^{n/2}}{\lambda^{n/2+1}} |J_{n/2}(|t|(\lambda + a))| \in L_1([C_1, +\infty]),$$

whence

$$|S(n, r, a, |t|)| \rightarrow 0 \quad \text{as } r \rightarrow +\infty.$$

Hence

$$|G_n(r, a, |t|)| \rightarrow \infty \quad \text{as } r \rightarrow \infty.$$

FIGURE 1. $g_{3,r,1.2}(1)$ FIGURE 2. $g_{3,r,1.2}(5)$

Every term in the representation of $g_{n,r,a}(|t|)$ considered in Theorem 8.1 approaches zero as $r \rightarrow \infty$. The theorem is proved. \square

Example 1. Let $n = 3$, $a = 10$, $|t| = 1$, and $|t| = 4$. Then the result of Theorem 9.2 is shown in Figures 1 and 2.

Theorem 9.3. *The weight function $g_{n,r,a}(|t|)$, $n \geq 1$, is discontinuous if $|t| = r$. This is a discontinuity of the first kind if $ar/\pi \in \mathbf{N}$; otherwise the discontinuity is of the second kind.*

Proof. The continuity of $G_n(r, a, |t|)$ as a function of $|t|$ follows from representation (22) and bound (37). The functions $g_{1,r,a}(|t|)$ and $g_{2,r,a}(|t|)$ are continuous for $|t| \neq r$. Consider the behavior of these functions for $|t| = r$. Lemmas 6.1 and 7.1 imply that

$$\begin{aligned} \lim_{|t| \rightarrow r+} g_{1,r,a}(|t|) &= \frac{1}{\pi r} \lim_{|t| \rightarrow r+} \sin(a|t|) \log \left(\frac{|t| - r}{|t| + r} \right) \\ &= \begin{cases} 0, & \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k + 1), \quad k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k + 1, 2k + 2), \quad k \in \mathbf{Z}_+, \end{cases} \\ \lim_{|t| \rightarrow r-} g_{1,r,a}(|t|) &= \frac{1}{\pi r} \lim_{|t| \rightarrow r-} \left(\pi \cos(a|t|) + \log \left(\frac{r - |t|}{r + |t|} \right) \sin(a|t|) \right) \\ &= \begin{cases} \frac{(-1)^{ar/\pi}}{r}, & \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k + 1), \quad k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k + 1, 2k + 2), \quad k \in \mathbf{Z}_+. \end{cases} \end{aligned}$$

Running *Mathematica 5.0* and using [9] we obtain

$$\begin{aligned} \lim_{|t| \rightarrow r-} g_{2,r,a}(|t|) &= \frac{J_0(ar) - arJ_1(ar)}{r^2} \\ &+ \frac{2}{\pi r^2} \lim_{|t| \rightarrow r-} \int_0^1 \frac{a(r^2 - |t|^2 x^2) \cos(a|t|x) - tx \sin(a|t|x)}{\sqrt{1 - x^2} \sqrt{r^2 - |t|^2 x^2}} dx \\ &= \frac{J_0(ar) - arJ_1(ar)}{r^2} + \frac{2}{\pi r^2} \sin(ar) \\ &+ \begin{cases} \frac{(-1)^{ar/\pi} \text{Si}(2ar\pi)}{\pi r^2}, & \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k + 1), \quad k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k + 1, 2k + 2), \quad k \in \mathbf{Z}_+, \end{cases} \end{aligned}$$

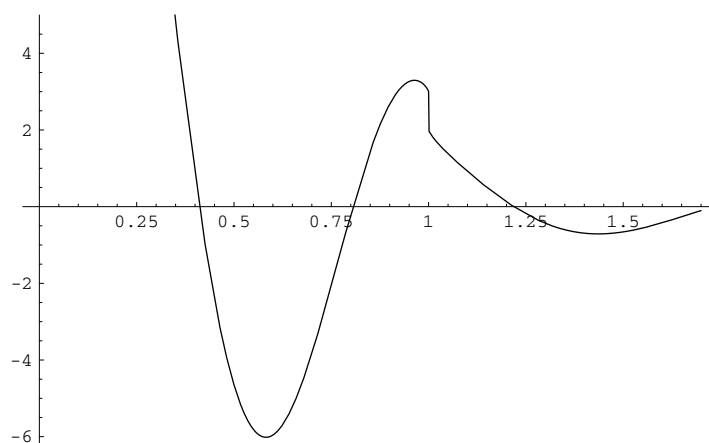
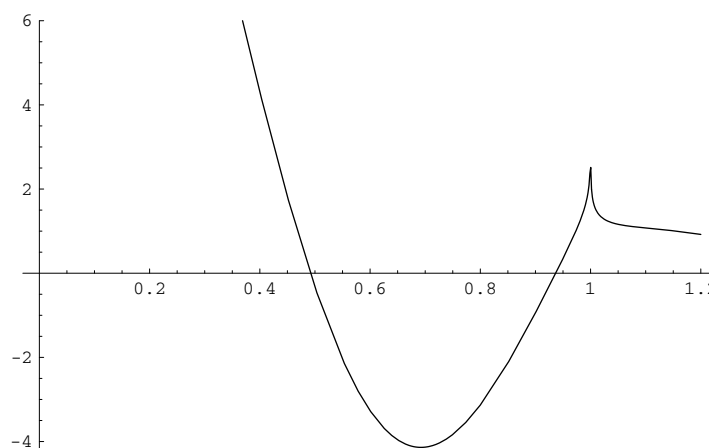
where $\text{Si}(\cdot)$ is the integral sinus; see [2].

Furthermore,

$$\begin{aligned} \lim_{|t| \rightarrow r+} g_{2,r,a}(|t|) &= \frac{J_0(ar) - arJ_1(ar)}{r^2} \\ &+ \frac{2}{\pi r^2} \lim_{|t| \rightarrow r+} \int_0^{r/|t|} \frac{a(r^2 - |t|^2 x^2) \cos(a|t|x) - |t|x \sin(a|t|x)}{\sqrt{1 - x^2} \sqrt{|t|^2 - r^2 x^2}} dx \\ &+ \frac{2}{\pi r^2} \lim_{|t| \rightarrow r+} \int_{r/|t|}^1 \frac{a(|t|^2 x^2 - r^2) \sin(a|t|x) - |t|x \cos(a|t|x)}{\sqrt{1 - x^2} \sqrt{|t|^2 x^2 - r^2}} dx \\ &= \frac{J_0(ar) - arJ_1(ar)}{r^2} + \frac{2}{\pi r^2} \sin(ar) + \frac{\cos(ar)}{r^2} \\ &+ \begin{cases} \frac{(-1)^{ar/\pi} \text{Si}(2ar\pi)}{\pi r^2}, & \frac{ar}{\pi} \in \mathbf{N}, \\ -\infty, & \frac{ar}{\pi} \in (2k, 2k + 1), \quad k \in \mathbf{Z}_+, \\ +\infty, & \frac{ar}{\pi} \in (2k + 1, 2k + 2), \quad k \in \mathbf{Z}_+. \end{cases} \end{aligned}$$

These calculations allow one to determine the kind of discontinuity of the function $g_n(r, a, |t|)$ for $|t| = r$. This completes the proof of the theorem. \square

Example 2. Let $n = 3$, $r = 1$, $a = 2\pi$ and $a = 5$. The result of Theorem 9.3 is shown in Figures 3 and 4.

FIGURE 3. $g_{3,1,2\pi}(|t|)$ FIGURE 4. $g_{3,1,5}(|t|)$

10. CONCLUDING REMARKS

We obtained the rate of convergence of function series in the representation of weight functions in Tauberian theorems. We constructed the recurrence relations for weight functions. We studied the asymptotic behavior of weight functions.

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