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STRONG STABILITY IN A JACKSON QUEUEING NETWORK

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Abstract. Non-product networks are extremely difficult to analyze, so they are often solved by approximate methods. However, it is indispensable to delimit the domain wherever these approximations are justified. Our goal in this paper is to prove the applicability of the strong stability method to the queueing networks in order to be able to approximate non-product form networks by product ones. Therefore, we established the strong stability of a Jackson network $M/M/1 \to M/M/1$ (ideal model) under perturbations of the service time distribution in the first station of a non-product network $M/GI/1 \to GI/M/1$ (real model).

1. Introduction

Queueing networks comprise a very useful class of models that have been extensively applied in the last decades as a powerful tool for modelling, performance evaluation and prediction of systems as well as production and manufacturing systems, communication networks, computer systems, etc. [10, 15].

The important contribution of queueing networks theory is that, under certain assumptions, it allows one to obtain a simple exact solution for the joint queue length distribution as the product of the distributions of the single queues with appropriate parameters and, for closed networks, with a normalization constant.

This famous product form was introduced by Jackson [17] for opening exponential networks and Gordon–Newell [14] for closed exponential networks. This class of models was then extended to BCMP[1] networks [4], which include open, closed or mixed networks with multiple classes of customers and various disciplines and service time distributions. More recently, further researches have been devoted to extending and characterizing the class of product form networks which result in networks with positive and negative customers [13]. However, the class of this kind of queueing networks is rather small. For a recent contribution in the product networks see, for example, [5, 30].

The non-product form networks are extremely difficult to analyze. A great deal of effort has been devoted to establishing approximate methods for these networks, such as decomposition methods, mean value methods, isolation methods, diffusion methods, aggregation methods, and many other numerical methods [6, 20, 21, 22, 23, 27, 31, 32]. In this paper, we are interested in the method which consists in substituting a non-product form network (real model) by a product one (ideal model). When this substitution is performed, the stability problem arises; then it is indispensable to delimit the domain wherever the ideal queueing networks can be taken as a good approximation.

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[1]Basket, Chandy, Mantz, Palacios.

Stability analysis of queueing networks have received a great deal of attention recently [16, 26, 28, 29]. This is partly due to several examples that demonstrate that the usual conditions "traffic intensities less than one at each station" are not sufficient for stability, even under the well-known FIFO politics. Methods for establishing the stability of queueing networks have been developed by several authors, based on fluid limits [11], Lyapunov functions [12], explicit coupling (renovating event and Harris chains), monotonicity, martingales, large deviations, etc. The actual needs of practice require quantitative estimations in addition to the qualitative analysis (stability), so in the beginning of the 1980s, a quantitative method for studying the stability of stochastic systems, called strong stability method (also called method of operators) was elaborated [3] (for full particulars on this method we refer to [19]). This method is applicable to all operation research models which can be governed by Markov chains. The applicability of this method to queueing systems has been proved considering the perturbation of the following parameters: the arrivals flows [2], the service time [7], the service intensity [1], the retrial flow [8] and demand distribution in an (R, s, S) inventory model [24].

Certainly, numerous efficient approximation methods for queueing networks have therefore been developed over the last decades. However, these methods are not supported by analytic formal errors bounds but just by extensive numerical results. Furthermore, the approximations are usually computationally complex, and therefore many become less practical for engineering.

In the cases of queueing systems, using the strong stability method allows us to obtain simple bounds upon modifying the original complex (real) system by a simple (ideal) one. For the formal proof of these bounds see [18] and practical cases were studied in [9, 25]. Now, our aim is to enlarge the applicability area of this method to the queueing networks in order to exploit the famous product form to study complex queueing networks. As a first attempt, we prove in this paper the strong stability of a Jackson network $M/M/1 \to M/M/1$ under perturbation of the service time distribution in the first station of an $M/GI/1 \to GI/M/1$ network.

2. Preliminaries and notation

Consider a simple Jackson network R_1 consisting of two queues in tandem as is illustrated in Figure 1.

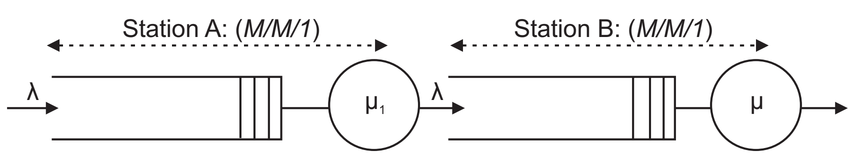

Figure 1. $M/M/1 \to M/M/1$ tandem queues

The arrival flows to the two stations A and B are Poisson with an equal parameter λ. The service time is exponentially distributed with parameter μ_1 (resp. μ) in the station A (resp. B).

We denote by $\overline{X}(t)$ (resp. $\overline{Y}(t)$) the random variable representing the number of customers in the station A (resp. B) at the moment t. The R_1 behavior is described by the two-dimensional process $(\overline{X}_n, \overline{Y}_n)$. Since the external arrivals are Poisson and the service time distributions are exponential, the Chapman–Kolmogorov equations are easily obtained. Thus, if the following geometric ergodicity condition of this queueing network:

$$\lambda < \min(\mu_1, \mu) \tag{1}$$

is satisfied, the stationary distribution will be given by the product form:

$$\overline{\pi}(i,k) = \overline{\pi}^{(A)}(i)\overline{\pi}^{(B)}(k), \tag{2}$$

where $\overline{\pi}^{(A)}(i)$ (resp. $\overline{\pi}^{(B)}(k)$) is the marginal distribution of the station A (resp. B).

Consider the two tandem queues R_2 shown in Figure 2.

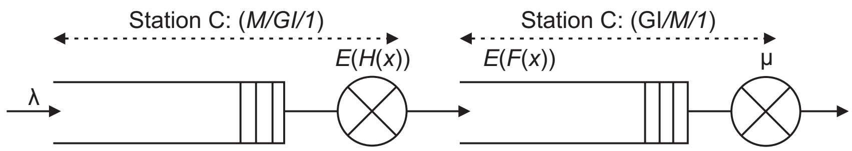

FIGURE 2. $M/GI/1 \to GI/M/1$ tandem queues

The first station of this network has an exponential distribution of the inter-arrivals with the same parameter λ as in R_1 and a general service time distribution H. On the other hand, the second station have a general distribution of inter-arrival F and an exponential distribution of service time E_μ with parameter μ.

The R_2 behavior is described by the two-dimensional process (X_n, Y_n), where $X_n(t)$ (resp. $Y_n(t)$) is the random variable which represents the number of customers in the station C (resp. D) at the moment t.

We use the following notation:

- d_n^+ (resp. $\overline{d}_n^+$): the moment just after departure of an n^{th} customer from the station C (resp. A).
- $X_n = X(d_n^+)$ (resp. $\overline{X}_n = \overline{X}(\overline{d}_n^+)$): the number of customers in the station C (resp. A) at the moment d_n^+ (resp. $\overline{d}_n^+$).
- a_n (resp. $\overline{a}_n$): the arrival time of an n^{th} customer to the station D (resp. B).
- $Y_n = Y(a_n)$ (resp. $\overline{Y}_n = \overline{Y}(\overline{d}_n^+)$): the number of customers in the station D (resp. B) at the moment a_n(resp. $\overline{a}_n$).

Remark 2.1. Since in the two considered networks, the customers head for the second station immediately after their departure from the first station, d_n^+ (resp. $\overline{d}_n^+$) can be regarded as the moment a_n (resp. $\overline{a}_n$). So to unify the notation, we denote this moment I_n (resp. $\overline{I}_n$).

3. FUNDAMENTAL DEFINITIONS AND THEOREMS

In this section, we will survey some fundamental results of the strong stability theory that will be used afterwards in this work. We consider a Markov chain $X = (X_t)_{t\geq 0}$ with a transition kernel $\mathbf{P}$ and a unique stationary probability measure π.

Definition 3.1. The Markov chain X is strongly stable with respect to the norm $\|\cdot\|$ if each neighboring stochastic kernel $\mathbf{Q}$ (i.e. $\|\mathbf{Q}-\mathbf{P}\| < \varepsilon$) admits a unique stationary probability ν and

$$\|\nu - \pi\| \to 0 \quad \text{as} \quad \|\mathbf{Q}-\mathbf{P}\| \to 0.$$

Consider the σ-algebra $\varepsilon = \varepsilon_1 \times \varepsilon_1$, where ε_1 is the σ-algebra generated by the countable partition of $\mathbb{N}$. In the space $m\varepsilon$ of finite measures on ε, we define the following norms:

$$\|\mu\|_v = \sum_{j\geq 0}\sum_{l\geq 0} v(j,l)|\mu(j,l)|, \qquad j,l \in \mathbb{N}, \tag{3}$$

where v is a measurable function on $\mathbb{N}\times\mathbb{N}$, bounded below by a positive constant, not necessarily finite. If a Markov chain X is strongly stable with respects to the norm $\|\cdot\|_v$, defined by formula (3), it is called v-stable.

These norms induce the corresponding norms in the space $f\varepsilon$ of bounded measurable functions on $\mathbb{N}\times\mathbb{N}$:

$$\|f\|_v = \sup_{i\geq 0}\sup_{k\geq 0}\,[v(i,k)]^{-1}\,|f(i,k)| \quad \text{for all } f\in f\varepsilon. \tag{4}$$

Moreover, in the space of linear operators, these norms induce the norms

$$\|\mathbf{Q}\|_v = \sup_{i\geq 0}\sup_{k\geq 0}\,[v(i,k)]^{-1}\sum_{j\geq 0}\sum_{l\geq 0} v(j,l)|\mathbf{Q}_{i,k}(j,l)|. \tag{5}$$

We associate to each transition kernel $\mathbf{Q}$ the linear mapping $\mathbf{Q}\colon f\varepsilon\to f\varepsilon$ defined by

$$\mathbf{Q}f(i,k) = \sum_{j\geq 0}\sum_{l\geq 0}\mathbf{Q}_{i,k}(j,l)f(j,l). \tag{6}$$

Theorem 3.1 ([3, 19])**.** *The Harris Markov chain is strongly v-stable if the following conditions are satisfied:*

1. $\exists\alpha\in\mathcal{M}^+$, $\exists h\in f\varepsilon^+$ *such that:* $\pi h>0$, $\alpha\mathbb{1}=1$, $\alpha h>0$, $\mathcal{M}^+$ *is the space of positive finite measures on* ε, *and* $f\varepsilon^+$ *is the space of positive bounded measurable functions on* $\mathbb{N}\times\mathbb{N}$.
2. $\mathbf{T}=\mathbf{P}-h\circ\alpha\geq 0$, *where* $\circ$ *denotes the sensorial product.*
3. $\exists\rho<1$ *such that*

$$\mathbf{T}v(i,k))\leq \rho v(i,k) \quad \textit{for all } (i,k)\in\mathbb{N}\times\mathbb{N}.$$

4. The v-stability of the $M/M/1\to M/M/1$ network

In this section, we intend to prove the strong stability of a Jackson network with two tandem stations under perturbation of the service law of the first station.

First, we define the corresponding transition kernels in the two queueing networks R_1 and R_2 considered.

Lemma 4.1. *The sequence $V_n=(X_n,Y_n)$ forms a homogeneous Markov chain with state space $\mathbb{N}\times\mathbb{N}$, whose transition kernel $\mathbf{Q}=[\mathbf{Q}_{i,k}(j,l)]_{i,k,j,l\geq 0}$ is defined by:*

$$\begin{aligned}\mathbf{Q}_{i,k}(j,l) &= \mathsf{P}\left[V_{n+1}=(j,l)/V_n=(i,k)\right]\\ &= \begin{cases} P_j q_{kl}, & \textit{if } (i=0, j\geq 0, l\in[1,k+1], k\geq 0),\\ P_j q_{k0}, & \textit{if } (i=0, j\geq 0, l=0, k\geq 0),\\ P_{j-i+1}q_{kl}, & \textit{if } (i\in[1,j+1], j\geq 0, l\in[1,k+1], k\geq 0),\\ P_{j-i+1}q_{k0}, & \textit{if } (i\in[1,j+1], j\geq 0, l=0, k\geq 0),\\ 0, & \textit{if } (i\in[0,j+1], j\geq 0, l>k+1, k\geq 0),\\ 0, & \textit{if } (i>j+1, j\geq 0, l\geq 0, k\geq 0),\end{cases}\end{aligned} \tag{7}$$

where

$$q_{kl} = \int_0^\infty \exp\left(-\mu x\right)\frac{(\mu x)^{k+1-l}}{(k+1-l)!}\,dF(x),$$

$$q_{k0} = 1-\sum_{l=1}^{k+1} q_{kl},$$

and

$$P_r = \int_0^\infty \exp\left(-\lambda x\right)\frac{(\lambda x)^r}{r!}\,dH(x) \quad \textit{for all } r\in\mathbb{N}.$$

Proof. Consider the following random variables:

A_n: the number of arrivals to the networks during the n^{th} service of the first station.

Δ_n: the number of customers served in the first station during the n^{th} service of the first station.

We have

$$X_{n+1} = X_n - \Delta_n + A_{n+1}.$$

We consider the random variable

S_n: the number of customers served in the second station of the network between the two moments I_n and $\overline{I}_n$.

Thus, we obtain

$$Y_{n+1} = Y_n + 1 - S_n.$$

X_{n+1} depends only on the variable X_n, and Y_{n+1} depends just on Y_n. As a result, $V_n = (X_n, Y_n)$ is a Markov chain. We have

$$\begin{aligned}
\mathbf{Q}_{i,k}(j,l) &= \mathsf{P}\left[V_{n+1} = (j,l)/V_n = (i,k)\right] \\
&= \mathsf{P}\left[X_{n+1} = j, Y_{n+1} = l)/X_n = i, Y_n = k)\right] \\
&= \mathsf{P}\left[X_n - \Delta_n + A_{n+1} = j, Y_n + 1 - S_n = l/X_n = i, Y_n = k\right] \\
&= \mathsf{P}\left[A_{n+1} = j + \Delta_n - i, S_n = k + 1 - l\right] \\
&= \mathsf{P}\left[A_{n+1} = j + \Delta_n - i\right] \mathsf{P}\left[S_n = k + 1 - l\right],
\end{aligned}$$

where

$$\mathsf{P}\left[S_n = k - l + 1\right] = \begin{cases} q_{kl}, & \text{if } (l \in [1, k+1], k \geq 0), \\ q_{k0}, & \text{if } (l = 0, k \geq 0), \\ 0, & \text{if } (l > k+1, k \geq 0) \end{cases}$$

and

$$\mathsf{P}\left[A_{n+1} = j + \Delta_n - i\right] = \begin{cases} P_j, & \text{if } (i = 0, j \geq 0), \\ P_{j-i+1}, & \text{if } (i \in [1, j+1], j \geq 0), \\ 0, & \text{if } (i > j+1, j \geq 0). \end{cases} \qquad \square$$

Lemma 4.2. *The double sequence $\overline{V}_n = (\overline{X}_n, \overline{Y}_n)$ forms a homogeneous Markov chain with state space $\mathbb{N} \times \mathbb{N}$, whose transition kernel $\overline{\mathbf{Q}} = \left[\overline{\mathbf{Q}}_{i,k}(j,l)\right]_{i,k,j,l \geq 0}$ is defined by*

$$\overline{\mathbf{Q}}_{i,k}(j,l) = \mathsf{P}\left[\overline{V}_{n+1} = (j,l)/\overline{V}_n = (i,k)\right]$$

$$(8) \qquad = \begin{cases} \overline{P}_j \overline{q}_{kl}, & \textit{if } (i = 0, j \geq 0, l \in [1, k+1], k \geq 0), \\ \overline{P}_j \overline{q}_{k0}, & \textit{if } (i = 0, j \geq 0, l = 0, k \geq 0), \\ \overline{P}_{j-i+1} \overline{q}_{kl}, & \textit{if } (i \in [1, j+1], j \geq 0, l \in [1, k+1], k \geq 0), \\ \overline{P}_{j-i+1} \overline{q}_{k0}, & \textit{if } (i \in [1, j+1], j \geq 0, l = 0, k \geq 0), \\ 0, & \textit{if } (i \in [0, j+1], j \geq 0, l > k+1, k \geq 0), \\ 0, & \textit{if } (i > j+1, j \geq 0, l \geq 0, k \geq 0), \end{cases}$$

where

$$\overline{P}_r = \int_0^\infty \exp(-\lambda x) \frac{(\lambda x)^r}{r!} \, dE_{\mu_1}(x), \qquad r \in \mathbb{N},$$

and

$$\overline{q}_{kl} = \int_0^\infty \exp(-\mu x) \frac{(\mu x)^{k+1-l}}{(k+1-l)!} \, dE_\lambda(x),$$

$$\overline{q}_{k0} = 1 - \sum_{l=1}^{k+1} \overline{q}_{kl}.$$

Proof. Consider the following random variables:

$\overline{A}_n$: the number of arrivals to the networks during the n^{th} service of the first station.

$\overline{\Delta}_n$: the number of customers served in the first station during the n^{th} service of the first station.

It is clear that

$$\overline{X}_{n+1} = \overline{X}_n - \overline{\Delta}_n + \overline{A}_{n+1}.$$

Moreover, we consider the random variable

$\overline{S}_n$: the number of customers served in the second station of the network between the two moments $\overline{I}_n$ and $\overline{I}_{n+1}$.

Thus, we obtain

$$\overline{Y}_{n+1} = \overline{Y}_n + 1 - \overline{S}_n.$$

$\overline{X}_{n+1}$ (resp. $\overline{Y}_{n+1}$) depends only on the variable $\overline{X}_n$ (resp. $\overline{Y}_n$). As a result, $\overline{V}_n = (\overline{X}_n, \overline{Y}_n)$ is a Markov chain. We have

$$\begin{aligned}
\overline{\mathbf{Q}}_{i,k}(j,l) &= \mathsf{P}\left[\overline{V}_{n+1} = (j,l)/\overline{V}_n = (i,k)\right] \\
&= \mathsf{P}\left[\overline{X}_{n+1} = j, \overline{Y}_{n+1} = l)/\overline{X}_n = i, \overline{Y}_n = k)\right] \\
&= \mathsf{P}\left[\overline{X}_n - \overline{\Delta}_n + \overline{A}_{n+1} = j, \overline{Y}_n + 1 - \overline{S}_n = l/\overline{X}_n = i, \overline{Y}_n = k\right] \\
&= \mathsf{P}\left[\overline{A}_{n+1} = j + \overline{\Delta}_n - i, \overline{S}_n = k + 1 - l\right] \\
&= \mathsf{P}\left[\overline{A}_{n+1} = j + \overline{\Delta}_n - i\right] \times \mathsf{P}\left[\overline{S}_n = k + 1 - l\right],
\end{aligned}$$

where

$$\mathsf{P}\left[\overline{S}_n = k - l + 1\right] = \begin{cases} \overline{q}_{kl}, & \text{if } (l \in [1, k+1], k \geq 0), \\ \overline{q}_{k0}, & \text{if } (l = 0, k \geq 0), \\ 0, & \text{if } (l > k+1, k \geq 0), \end{cases}$$

and

$$\mathsf{P}\left[\overline{A}_{n+1} = j + \overline{\Delta}_n - i\right] = \begin{cases} \overline{P}_j, & \text{if } (i = 0, j \geq 0), \\ \overline{P}_{j-i+1}, & \text{if } (i \in [1, j+1], j \geq 0), \\ 0, & \text{if } (i > j+1, j \geq 0). \end{cases}$$ □

We suppose that the service law of the station A in the network R_1 is close to the exponential distribution with parameter μ_1.

Now, in order to established the strong stability of the Markov chain $(\overline{V}_n)$, we apply Theorem 3.1 with the test function:

$$v\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+,$$
$$(i,k) \mapsto \gamma^i \beta^k,$$

where γ and β are constants such that

$$1 < \gamma < \frac{\mu_1}{\lambda} \quad \text{and} \quad \beta > 1.$$

Let α be a sequence of product measures defined by:

$$\alpha\colon \varepsilon \times \varepsilon \to \mathbb{R}^+,$$
$$(\{j\},\{l\}) \mapsto \alpha(\{j\},\{l\}) = \begin{cases} \overline{P}_j \overline{q}_{kl}, & \text{if } 0 \leq l \leq k+1, \\ 0, & \text{otherwise}. \end{cases}$$

Moreover, we consider the measurable function h defined by

$$h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$(i,k) \mapsto h(i,k) = \mathbb{1}_{\{i=0\}}.$$

The sequence of measures and the measurable function introduced above satisfy the conditions required in Theorem 3.1:

$$\begin{aligned}
\alpha \mathbb{1} &= \sum_{j\geq 0}\sum_{l\geq 0} \alpha(\{j\}\times\{l\}) = \sum_{j\geq 0}\sum_{l=0}^{k+1} \overline{P}_j \overline{q}_{kl} = 1,\\
\alpha h &= \sum_{j\geq 0}\sum_{l\geq 0} \alpha(\{j\}\times\{l\}) h(j,l) = \sum_{j\geq 0}\sum_{l=0}^{k+1} \overline{P}_j \overline{q}_{kl} \mathbb{1}_{\{j=0\}}\\
&= \overline{P}_0 \sum_{l=0}^{k+1} \overline{q}_{kl} = \overline{P}_0 = \frac{\mu_1}{\lambda+\mu_1} > 0,\\
\overline{\pi} h &= \sum_{i\geq 0}\sum_{k\geq 0} \overline{\pi}(i,k) h(i,k) = \sum_{k\geq 0} \overline{\pi}(0,k) = \overline{\pi}^{(A)}(0) \sum_{l\geq 0} \overline{\pi}^{(B)}(l)\\
&= \overline{\pi}^{(A)}(0) = \left(1-\frac{\lambda}{\mu_1}\right) > 0,
\end{aligned}$$

where $\overline{\pi}$ is the stationary distribution of the studied Jackson network R_1 and $\overline{\pi}^{(A)}$ (resp. $\overline{\pi}^{(B)}$) is the stationary distribution of the A (resp. B) queue.

Lemma 4.3. *The operator $\overline{\mathbf{T}}$ associated with the Markov chain V_n is non-negative.*

Proof.

$$\overline{\mathbf{T}}_{i,k}(j,l) = \overline{\mathbf{Q}}_{i,k}(j,l) - (h\circ\alpha)_{i,k}(j,l),$$

with

$$\begin{aligned}
(h\circ\alpha)_{i,k}(j,l) &= h(i,k)\alpha(j,l)\\
&= \begin{cases} \overline{P}_j \overline{q}_{kl}, & \text{if } (i=0),\ (0\leq l\leq k+1),\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

So

$$\overline{\mathbf{T}}_{i,k}(j,l) = \begin{cases} \overline{P}_{j-i+1}\overline{q}_{kl}, & \text{if } (i\in[1,j+1], j\geq 0),\ (l\in[1,k+1], k\geq 0),\\ \overline{P}_{j-i+1}\overline{q}_{k0}, & \text{if } (i\in[1,j+1], j\geq 0),\ (l=0, k\geq 0),\\ 0, & \text{otherwise.}\end{cases} \tag{9}$$

Thus, the kernel $\overline{\mathbf{T}}_{i,k}(j,l)$ is non-negative. □

Lemma 4.4. *Suppose the geometric ergodicity condition is satisfied for the Jackson network considered:*

$$\lambda < \min(\mu_1,\mu).$$

Then, for the function $v(i,k) = \gamma^i\beta^k$ with

$$1 < \gamma < \frac{\mu_1}{\lambda}, \qquad \beta \in \left[1,\ \gamma\frac{\lambda+\mu_1-\gamma\lambda}{\mu_1}\right], \tag{10}$$

the following inequality holds:

$$\overline{\mathbf{T}} v(i,k) \leq \gamma^i\beta^k\rho, \tag{11}$$

where

$$\rho = \max\left(\frac{1}{\beta}; \frac{\beta\mu_1}{\gamma(\lambda+\mu_1-\gamma\lambda)}\right) < 1. \tag{12}$$

Proof. In the calculus of $\overline{\mathbf{T}}v(i,k)$, we distinguish four cases.

Case 1 $(i=0,\, k=0)$. We have

$$\overline{\mathbf{T}}v(0,0)=\sum_{j\geq 0}\sum_{l\geq 0}\gamma^j\beta^l\overline{\mathbf{T}}_{0,0}(j,l),$$

where

$$\overline{\mathbf{T}}_{0,0}(j,l)=0 \quad \text{for all } j\geq 0 \quad \text{and for all } l\geq 0.$$

So

$$\overline{\mathbf{T}}v(0,0)=\sum_{j\geq 0}\sum_{l\geq 0}\gamma^j\beta^l\overline{\mathbf{T}}_{0,0}(j,l)=0\leq \gamma^0\beta^0\rho_1,$$

and ρ_1 can be any positive number, particularly a number less than 1.

Case 2 $(i=0,\, k>0)$. We have

$$\left(\overline{\mathbf{T}}v(0,k)\right)_{k>0}=\sum_{j\geq 0}\sum_{l\geq 0}\gamma^j\beta^l\overline{\mathbf{T}}_{0,k}(j,l),$$

where

$$\left(\overline{\mathbf{T}}_{0,k}(j,l)\right)_{k>0}=0 \quad \text{for all } j\geq 0 \quad \text{and for all } l\geq 0.$$

So,

$$\overline{\mathbf{T}}v(0,k)=\sum_{j\geq 0}\sum_{l\geq 0}\gamma^j\beta^l\overline{\mathbf{T}}_{0,k}(j,l)=0<1\leq \beta^k\frac{1}{\beta},$$

because $\beta>1$, by assumption. So it is sufficient to take $\rho_2=1/\beta$, and we will have

$$\overline{\mathbf{T}}v(0,k)<\beta^k\rho_2 \quad \text{with } \rho_2=\frac{1}{\beta}<1.$$

Case 3 $(i>0,\, k=0)$. We have

$$\left(\overline{\mathbf{T}}v(i,0)\right)_{i>0}=\sum_{j\geq 0}\sum_{l\geq 0}\gamma^j\beta^l\overline{\mathbf{T}}_{i,0}(j,l),$$

where

$$\left(\overline{\mathbf{T}}_{i,0}(j,l)\right)_{i>0}=\begin{cases}\overline{P}_{j+1-i}\overline{q}_{01}, & \text{if } (i\in[1,j+1], j\geq 0, l=1),\\ \overline{P}_{j+1-i}\overline{q}_{00}, & \text{if } (i\in[1,j+1], j\geq 0, l=0),\\ 0, & \text{if } (i\in[1,j+1], j\geq 0, l>1),\\ 0, & \text{if } (i>j+1, j\geq 0, l\geq 0).\end{cases}$$

As a result, we obtain

$$\begin{aligned}\left(\overline{\mathbf{T}}v(i,0)\right)_{i>0}&=\sum_{j\geq 0}\sum_{l\geq 0}\gamma^j\beta^l\overline{\mathbf{T}}_{i,0}(j,l)=\sum_{j\geq 0}\gamma^j\sum_{l\geq 0}\beta^l\overline{\mathbf{T}}_{i,0}(j,l)\\ &=\sum_{j\geq i-1}\gamma^j\left[\beta\overline{P}_{j+1-i}\overline{q}_{01}+\overline{P}_{j+1-i}\overline{q}_{00}\right]=\sum_{j\geq i-1}\gamma^j\overline{P}_{j+1-i}\left[\beta\overline{q}_{01}+\overline{q}_{00}\right]\\ &=\sum_{n\geq 0}\gamma^{n+i-1}\overline{P}_n\left[\beta\overline{q}_{01}+\overline{q}_{00}\right]=\gamma^i\sum_{n\geq 0}\gamma^n\overline{P}_n\frac{1}{\gamma}\left[\beta\overline{q}_{01}+\overline{q}_{00}\right]\\ &=\gamma^i\left(\frac{\mu_1}{\lambda+\mu_1-\gamma\lambda}\right)\frac{1}{\gamma}\left[\beta\overline{q}_{01}+\overline{q}_{00}\right]\\ &=\gamma^i\left(\frac{\mu_1}{\lambda+\mu_1-\gamma\lambda}\right)\frac{1}{\gamma}\left[\beta\overline{q}_{01}+1-\overline{q}_{01}\right]\\ &=\gamma^i\left(\frac{\mu_1}{\lambda+\mu_1-\gamma\lambda}\right)\frac{1}{\gamma}\left[\overline{q}_{01}(\beta-1)+1\right].\end{aligned}$$

We put

$$\rho_3 = \left(\frac{\mu_1}{\lambda + \mu_1 - \gamma\lambda}\right) \frac{1}{\gamma} \left[\overline{q}_{01}(\beta - 1) + 1\right].$$

Thus, we have

$$\left(\overline{\mathbf{T}}_v(i,0)\right)_{i>0} = \gamma^i \rho_3.$$

To satisfy the third condition of Theorem 3.1, we must have $\rho_3 < 1$. So, we impose the following condition for β:

$$\begin{aligned}
\rho_3 < 1 &\Leftrightarrow \left(\frac{\mu_1}{\lambda + \mu_1 - \gamma\lambda}\right) \frac{1}{\gamma} \left[\overline{q}_{01}(\beta - 1) + 1\right] < 1 \\
&\Leftrightarrow \left(\frac{\mu_1}{\lambda + \mu_1 - \gamma\lambda}\right) \frac{1}{\gamma} \left[\frac{\lambda}{\lambda + \mu}(\beta - 1) + 1\right] < 1 \\
&\Leftrightarrow \frac{\lambda}{\lambda + \mu}(\beta - 1) < \gamma \left(\frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1}\right) - 1 \\
&\Leftrightarrow (\beta - 1) < \frac{\lambda + \mu}{\lambda} \left[\gamma \left(\frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1}\right) - 1\right] \\
&\Leftrightarrow \beta < 1 + \frac{\lambda + \mu}{\lambda} \left[\gamma \left(\frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1}\right) - 1\right].
\end{aligned}$$

By assumption, $1 < \gamma < \mu_1/\lambda$. So, we show that

$$\left(1 + \frac{\lambda + \mu}{\lambda} \left[\gamma \left(\frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1}\right) - 1\right]\right) > 1.$$

We have $1 < \gamma < \mu_1/\lambda$. Then

$$\begin{aligned}
\frac{1}{\gamma} \left(\frac{\mu_1}{\lambda + \mu_1 - \gamma\lambda}\right) < 1 &\Rightarrow \left(\gamma \frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1}\right) > 1 \\
&\Rightarrow \gamma \left(\frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1}\right) - 1 > 0 \\
&\Rightarrow \frac{\lambda + \mu}{\lambda} \left[\gamma \frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1} - 1\right] > 0 \\
&\Rightarrow 1 + \frac{\lambda + \mu}{\lambda} \left[\gamma \frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1} - 1\right] > 1.
\end{aligned}$$

It is sufficient to take β such that

$$1 < \beta < 1 + \frac{\lambda + \mu}{\lambda} \left[\gamma \frac{\lambda + \mu_1 - \gamma\lambda}{\mu_1} - 1\right];$$

then ρ_3 is less than 1, and we have

$$\overline{\mathbf{T}}v(i,0) \le \gamma^i \rho_3, \quad \text{where} \ \ \rho_3 < 1.$$

Case 4 ($i > 0$, $k > 0$). We have

$$\left(\overline{\mathbf{T}}v(i,k)\right)_{i>0,k>0} = \sum_{j \ge 0} \sum_{l \ge 0} \gamma^j \beta^l \overline{\mathbf{T}}_{i,k}(j,l),$$

where

$$\left(\overline{\mathbf{T}}_{i,k}(j,l)\right)_{i>0,k>0} = \begin{cases} \overline{P}_{j-i+1} \overline{q}_{k,l}, & \text{if } (j \ge 0, l \in [1, k+1], i \in [1, j+1]), \\ \overline{P}_{j-i+1} \overline{q}_{k,0}, & \text{if } (j \ge 0, l = 0, i \in [1, j+1]), \\ 0, & \text{if } (j \ge 0, l > k+1, i \in [1, j+1]), \\ 0, & \text{if } (j \ge 0, l \ge 0, i > j+1). \end{cases}$$

Thus

$$\begin{aligned}
\left(\overline{\mathbf{T}}v(i,k)\right)_{k>0,i>0} &= \sum_{j\geq i-1} \gamma^j \overline{P}_{j+1-i} \sum_{l=0}^{k+1} \beta^l \ \overline{q}_{kl} \\
&= \gamma^i \left(\sum_{n\geq 0} \gamma^n \overline{P}_n \right) \frac{1}{\gamma} \sum_{l=0}^{k+1} \beta^l \overline{q}_{kl} \\
&= \gamma^i \beta^k \frac{1}{\gamma} \left(\frac{\mu_1}{\lambda+\mu_1-\lambda\gamma} \right) \sum_{l=0}^{k+1} \frac{1}{\beta^{k-l}} \overline{q_{kl}} \\
&= \gamma^i \beta^k \frac{\beta}{\gamma} \left(\frac{\mu_1}{\lambda+\mu_1-\lambda\gamma} \right) \sum_{l=0}^{k+1} \int_0^{+\infty} e^{(-\mu x)} \frac{\left(\frac{\mu x}{\beta^{k-l}}\right)^{k+1-l}}{(k+1-l)!} \lambda e^{(-\lambda x)}\, dx \\
&\leq \gamma^i \beta^k \left[\frac{\beta\lambda}{\gamma} \left(\frac{\mu_1}{\lambda+\mu_1-\lambda\gamma} \right) \times \int_0^{+\infty} e^{(-\lambda+\mu-\frac{\mu}{\beta})x}\, dx \right] \\
&\leq \gamma^i \beta^k \left[\frac{\beta}{\gamma} \left(\frac{\mu_1}{\lambda+\mu_1-\lambda\gamma} \right) \left(\frac{\lambda}{\lambda+\mu-\frac{\mu}{\beta}} \right) \right] \\
&\leq \gamma^i \beta^k \left[\frac{\beta}{\gamma} \left(\frac{\mu_1}{\lambda+\mu_1-\lambda\gamma} \right) \right].
\end{aligned}$$

We put

$$\rho_4 = \frac{\beta}{\gamma} \left(\frac{\mu_1}{\lambda+\mu_1-\gamma\lambda} \right).$$

To have $\rho_4 < 1$, it is enough that $\beta < \gamma(\lambda+\mu_1-\gamma\lambda)/\mu_1$. Yet, we have as assumption $1 < \gamma < \mu_1/\lambda$, so $\gamma(\lambda+\mu_1-\gamma\lambda)/\mu_1 > 1$. Therefore, it is sufficient to take

$$1 < \beta < \gamma \frac{\lambda+\mu_1-\gamma\lambda}{\mu_1}.$$

So, the third condition of Theorem 3.1 is satisfied.

From the four cases, we conclude that it is necessary to have

$$1 < \gamma < \frac{\mu_1}{\lambda}$$

and

$$1 < \beta < \min \left(1 + \frac{\lambda+\mu}{\lambda} \left[\gamma \frac{\lambda+\mu_1-\gamma\lambda}{\mu_1} - 1 \right]; \gamma \frac{\lambda+\mu_1-\gamma\lambda}{\mu_1} \right) = \gamma \frac{\lambda+\mu_1-\gamma\lambda}{\mu_1}.$$

So, we obtain

$$\rho = \max(\rho_1, \rho_2, \rho_3, \rho_4) = \max(\rho_2, \rho_4) = \max \left(\frac{1}{\beta}; \frac{\beta\mu_1}{\gamma(\lambda+\mu_1-\gamma\lambda)} \right) < 1. \tag{13}$$

□

Lemma 4.5. *The transition kernel $\overline{\mathbf{Q}}$ of the Markov chain $\overline{V}_n$ is bounded:*

$$\|\overline{\mathbf{Q}}\|_v < +\infty. \tag{14}$$

Proof. We have

$$\begin{aligned}
\overline{\mathbf{T}} = \overline{\mathbf{Q}} - h \circ \alpha &\Rightarrow \overline{\mathbf{Q}} = \overline{\mathbf{T}} + h \circ \alpha \\
&\Rightarrow \|\overline{\mathbf{Q}}\|_v \leq \|\overline{\mathbf{T}}\|_v + \|h \circ \alpha\|_v,
\end{aligned}$$

with

$$\|\overline{\mathbf{T}}\|_v = \sup_{i\geq 0}\sup_{k\geq 0}\frac{1}{v(i,k)}\sum_{j\geq 0}\sum_{l\geq 0} v(j,l)\overline{\mathbf{T}}_{i,k}(j,l) = \sup_{i\geq 0}\sup_{k\geq 0}\frac{1}{\gamma^i\beta^k}\overline{\mathbf{T}}v(i,k)$$
$$\leq \sup_{i\geq 0}\sup_{k\geq 0}\frac{1}{\gamma^i\beta^k}\gamma^i\beta^k\rho \leq \rho$$

and

$$\|h\circ\alpha\|_v = \sup_{i\geq 0}\sup_{k\geq 0}\frac{1}{v(i,k)}\left[\sum_{j\geq 0}\sum_{l\geq 0} v(j,l)(h\circ\alpha)_{i,k}(j,l)\right]$$
$$= \sup\left(0, \sup_{k\geq 0}\frac{1}{v(0,k)}\left[\sum_{j\geq 0}\sum_{l=0}^{k+1} v(j,l)\overline{P}_j\overline{q}_{kl}\right]\right)$$
$$= \sup_{k\geq 0}\frac{1}{v(o,k)}\left[\sum_{j\geq 0}\sum_{l=0}^{k+1} v(j,l)\overline{P}_j\overline{q}_{kl}\right]$$
$$= \sup_{k\geq 0}\frac{1}{\beta^k}\sum_{j\geq 0}\sum_{l=0}^{k+1}\gamma^j\beta^l\overline{P}_j\overline{q}_{kl} = \sup_{k\geq 0}\sum_{j\geq 0}\sum_{l=0}^{k+1}\frac{\gamma^j\beta^l}{\beta^k}\overline{P}_j\overline{q}_{kl}$$
$$\leq \sup_{k\geq 0}\beta\sum_{j\geq 0}\gamma^j\overline{P}_j\sum_{l=0}^{k+1}\overline{q}_{kl} = \beta\sum_{j\geq 0}\gamma^j\overline{P}_j.$$

Thus,

$$\sum_{j\geq 0}\gamma^j\overline{P}_j = \sum_{j\geq 0}\gamma^j\int_0^{+\infty}\exp(-\lambda x)\frac{(\lambda x)^j}{j!}\,dE_{\mu_1}(x)$$
$$= \int_0^{+\infty}\mu_1\exp\left[-(\lambda+\mu_1)\ x\right]\sum_{j\geq 0}\frac{(\lambda\gamma x)^j}{j!}\,dx$$
$$= \int_0^{+\infty}\mu_1\exp\left[-(\lambda+\mu_1-\gamma\lambda)x\right]dx$$
$$= \frac{\mu_1}{\lambda+\mu_1-\gamma\lambda}.$$

So,

$$\|h\circ\alpha\|_v \leq \frac{\beta\mu_1}{\lambda+\mu_1-\gamma\lambda} \leq \left[\gamma\frac{\lambda+\mu_1-\lambda\gamma}{\mu_1}\right]\frac{\mu_1}{\lambda+\mu_1-\lambda\gamma} = \gamma.$$

Thus, we obtain

$$\|\overline{\mathbf{Q}}\|_v \leq \rho+\gamma < +\infty. \tag{15}$$ □

All the conditions required in Theorem 3.1 are satisfied, so we have the following result.

Theorem 4.1. *According to Lemma* 4.5*, the Markov chain*

$$\left(\overline{V}_n\right)_{n\geq 0} = \left(\overline{X}_n, \overline{Y}_n\right)_{n\geq 0}$$

associated with the $M/M/1 \to M/M/1$ *tandem queues is strongly* v*-stable for a function* $v(i,j) = \gamma^i\beta^j$*, where*

$$1 < \gamma < \frac{\mu_1}{\lambda}$$

and

$$1 < \beta < \gamma\frac{\lambda+\mu_1-\gamma\lambda}{\mu_1}.$$

5. Conclusion

This work is a first attempt to prove the applicability of the strong stability method to the queueing networks. We have obtained the conditions under which the imbedded Markov chain of the Jackson networks $M/M/1 \to M/M/1$ is strongly stable under perturbation of the service law of the first station in the $M/GI/1 \to GI/M/1$ network. This means that it is possible to approximate the stationary and non-stationary characteristics of the non-product network $M/GI/1 \to GI/M/1$ by the corresponding characteristics of the studied Jackson network.

The results obtained in this work allow us to view the applicability of the strong stability approach to the whole set of product-networks.

Bibliography

1. D. Aïssani, *Application of the operator methods to obtain inequalities of stability in an $M_2/G_2/1$ system with a relative priority*, Annales Maghrébines de l' Ingénieur, Numéro Hors série **2** (1991), 790–795.
2. D. Aïssani and N. V. Kartashov, *Strong stability of the imbedded Markov chains in an $M/G/1$ system*, Theor. Probability and Math. Statist. **29** (1984), 1–5. MR727097 (85d:60167)
3. D. Aïssani and N. V. Kartashov, *Ergodicity and stability of Markov chains with respect to operator topology in the space of transition kernels*, Dokl. Akad. Nauk Ukr. SSR, ser. A **11** (1983), 1–5. MR728475 (85c:60110)
4. F. Baskett, K. M. Chandy, R. Muntz, and G. F. Palacios, *Open, closed and mixed networks of queueing with different classes of customers*, J. ACM **22** (1975), 248–260. MR0365749 (51:2001)
5. B. Baynat, *Réseaux de files d'attente: Des Chaînes de Markov aux Réseaux à Forme Produit*, Eyrolles Edition, 2000.
6. B. Baynat and Y. Dallery, *A Unified View of Product-form Approximation Techniques for General Closed Queueing Networks*, Technical Report 90.48, Institut Blaise Pascal, Paris, Octobre 1990.
7. M. Benaouicha and D. Aïssani, *Estimate of the strong stability in a $G/M/1$ queueing system*, Theory of Probab. and Math. Statist. **71** (2005), 22–32. MR2144318 (2006a:60171)
8. L. Berdjoudj and D. Aïssani, *Strong stability in retrial queues*, Theor. Probability and Math. Statist. **68** (2004), 11–17. MR2000390 (2004f:60188)
9. L. Bouallouche and D. Aïssani, *Performance analysis approximation in a queueing system of type $M/G/1$*, Mathematical Methods of Operations Research **63** (2006), no. 2. MR2264753 (2007i:90017)
10. Robert B. Cooper, *Fundamentals of Queueing Theory*, 4th ed., Ceep Press Books, 1990.
11. J. G. Dai, *On positive Harris recurrence of multiclass queueing networks: a unified approch via fluid limit models*, Annals of Applied Probability **5** (1993), no. 1, 49–77. MR1325041 (96c:60113)
12. G. Fayolle, V. A. Malyshev, M. V. Menshikov, and A. F. Sidorenko, *Lyaponov Functions for Jackson Networks*, Rapport de Recherche 1380, INRIA, Domaine de Voluceau, LeChenay, Janvier 1991.
13. E. Gelenbe, *Product form queueing networks with positive and negative customers*, J. Appl. Probab. **28** (1991), 656–663. MR1123837 (92k:60210)
14. W. J. Gordon and F. Newell, *Closed queueing systems with exponential servers*, Operations Research **15** (1967).
15. D. Gross and C. M. Harris, *Fundamentals of Queueing Theory*, 3rd ed., Wiley–InterScience, 1998. MR1600527 (98m:60144)
16. C. F. Ipsen and C. D. Meyer, *Uniform stability of Markov chains*, Siam. J. Matrix Anal. Appl. **15** (1994), no. 4, 1061–1074. MR1293904 (95h:65110)
17. J. R. Jackson, *Networks of waiting lines*, Operations Research **5** (1957), 518–521. MR0093061 (19:1203c)
18. N. V. Kartashov, *Strong stability of Markov chains*, Vsesoyuzn. Seminar on Stability Problems for Stochastic Models, VNIISI, Moscow, 1981, pp. 54–59; see also Journal of Soviet Mathematics **34** (1986), 1493–1498. MR668559 (84b:60089)
19. N. V. Kartashov, *Strong Stable Markov Chains*, VSP/TViMS, Utrecht/Kiev, 1996. MR1451375 (99e:60150)

20. H. Kobayashi, *Application of the diffusion approximation to queueing network*, J. Assoc. Comput. Mach. **21** (1974), 316–328. MR0350899 (50:3391)
21. P. R. Kumar, *A tutorial on some new methods for performance evaluation of queueing networks*, IEEE Journal on Selected Areas in Communications **13** (1995), no. 6, 970–980.
22. J. Labetoulle and G. Pujolle, *Isolation method in a network of queues*, IEEE Transaction on Software Engineering **6** (1980), 373–381.
23. M. F. Neuts, *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, 1981. MR618123 (82j:60177)
24. B. Rabta and D. Aïssani, *Stability analysis in an inventory model*, Theory of Stochastic Processes **10(26)** (2004), no. 3–4, 129–135. MR2329786 (2008g:60219)
25. B. Rabta and D. Aïssani, *Strong stability in an* (R, s, S) *inventory model*, International Journal of Production Economics **97** (2005), 159–171.
26. S. T. Rachev, *The problem of stability in queueing theory*, Queueing Systems **4** (1989), 287–318. MR1018523 (91c:60132)
27. P. J. Schweitzer, *A Survey of Mean Value Analysis, its Generalizations, and Applications for Networks of Queues*, Technical report, University of Rochester, Rochester, NY, 1990.
28. K. Sigman, *Notes on the stability of closed queueing networks*, J. Appl. Probab. **27** (1990), 735. MR1067041 (91j:60151)
29. K. Sigman, *The stability of open queueing networks*, Stochastic Processes and their Applications **35** (1990), 11–25. MR1062580 (91m:60178)
30. N. M. van Dijk, *Queueing Networks and Product Forms—A Systems Approach*, Wiley, 1993. MR1266845 (95b:90001)
31. I. Y. Wang and T. G. Robertazzi, *Recursive computation of steady-state probabilities of non-product form queueing networks associated with computer network models*, IEEE Transactions on Communications **38** (January 1990), no. 1.
32. W. Whitt, *The queueing network analyzer*, The Bell System Technical Journal **62** (November 1983), no. 9.

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