AN ESTIMATE FOR THE RATE OF CONVERGENCE
OF THE DISTRIBUTION OF THE NUMBER OF FALSE SOLUTIONS
OF A SYSTEM OF NONLINEAR RANDOM EQUATIONS
IN THE FIELD \( GF(2) \)

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Abstract. We prove a result on the rate of convergence as \( n \to \infty \) of the distribution of the number of false solutions of a system of nonlinear random equations in the field \( GF(2) \) to the Poisson distribution with parameter \( 2^m \). We assume, in particular, that the difference between the number \( n \) of unknowns and the number \( N \) of equations of the system is a constant \( m \).

1. Setting of the problem. Statement of the result

Consider the following system of equations:

\[
g_i(n) \sum_{k=1}^{g_i(n)} \sum_{1 \leq j_1 < \cdots < j_k \leq n} a_{j_1 \ldots j_k}^{(i)} x_{j_1} \cdots x_{j_k} = b_i, \quad i = 1, 2, \ldots, N,
\]

in the field \( GF(2) \). Throughout the paper we assume that the following conditions are satisfied:

- the coefficients \( a_{j_1 \ldots j_k}^{(i)} \), \( 1 \leq j_1 < \cdots < j_k \leq n \), \( k = 1, 2, \ldots, g_i(n) \), \( i = 1, 2, \ldots, N \), are independent random variables such that
  \[ P \{ a_{j_1 \ldots j_k}^{(i)} = 1 \} = 1 - P \{ a_{j_1 \ldots j_k}^{(i)} = 0 \} = p_{ik}; \]
- the elements \( b_i \), \( i = 1, 2, \ldots, N \), are obtained after the substitution of a fixed \( n \)-dimensional \((0, 1)\)-vector \( \bar{x}^0 \) with exactly \( \rho(n) \) nonzero coordinates to the left hand side of system (1) where
  \[ \rho(n) = \rho n, \quad \rho = \text{const}, \quad 0 < \rho < 1; \]
- the functions \( g_i(n) \) are nonrandom, \( g_i(n) \in \{2, 3, \ldots, n\} \), \( i = 1, 2, \ldots, N \).

We denote this set of conditions by (A).

Let \( \nu_n \) be the total number of false solutions of system (1), that is, the total number of solutions of system (1) that do not coincide with \( \bar{x}^0 \). In this paper, we study the rate of convergence of the distribution of the random variable \( \nu_n \) to the limit Poisson distribution with parameter \( 2^m \) if condition (2) holds.

Theorem. Suppose the conditions (A) are satisfied. Assume that

\[ n - N = m, \quad m = \text{const}, \quad -\infty < m < \infty, \]

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121
and that for any \( i = 1, 2, \ldots, N \), there exists a set \( T_i \neq \emptyset \) such that
\[
T_i \subseteq \{2, \ldots, g_i(n)\}, \quad 0 \leq \delta_{it}(n) \leq p_{it} \leq 1 - \delta_{it}(n), \quad t \in T_i,
\]
for sufficiently large \( n \), where \( \delta_{it}(n) \) are some numbers such that \( 0 \leq \delta_{it}(n) \leq \frac{1}{2} \). Furthermore, let a function \( \varphi(n) \) be such that \( \varphi(n) \leq \ln^2 n \). Assume that, given a constant \( \varepsilon_0 \in (0; 1) \) and a fixed integer \( l \geq 0 \), one can find a natural number \( n_0 = n_0(\varepsilon_0, l) \) for which
\[
2^{l+1} B(n) < \varepsilon_0
\]
for all \( n \geq n_0 \), where \( \gamma = [\log_2 n/6] \) for \( n \geq 2^{6l} \),
\[
B(n) = \sum_{i=1}^{N} \exp \left\{ -2 \sum_{t \in T_i} \delta_{it}(n) C_{f(n)}^{t} \right\},
\]
and \( f(n) \) assumes positive integer values and is such that \( f(n) = o(\varphi(n)) \) as \( n \to \infty \).

(Here and in what follows, the symbol \( C_{f(n)}^{t} \) stands for the binomial coefficient \( \binom{f(n)}{t} \).)

If \( k = 0, 1, 2, \ldots \) is fixed, then
\[
\left| P\{\nu_n = k\} - \frac{\lambda^k}{k!} e^{-\lambda} \right| \leq \left( \frac{2e\lambda}{\gamma} \right)^{\gamma} \left\{ 2 + 2^{l+1} B(n) + \Theta_2 (1 + 2^{l+1} B(n)) + 7\Theta_1 \right\} + e^{4\lambda \gamma} B(n) + e^{2\lambda \gamma} \left( \Theta_2 (1 + 2^{l+1} B(n)) + 7\Theta_1 \right),
\]
where \( \lambda = 2^m \),
\[
\Theta_1 = \exp \left\{ -2^{-2\gamma} \sum_{i=1}^{N} \delta_i + \log_2 \sqrt{\varphi(n)} + \sqrt{\varphi(n)} + \ln(\hat{\rho} n) - m \ln 2 \right\},
\]
\[
\Theta_2 = 2^{-n} \exp \left\{ \varepsilon \sqrt{\varphi(n)} \ln 2 n \left( \log_2 \sqrt{\varphi(n)} + \ln \left( \frac{n^{5/6} e}{\varepsilon \ln^2 n} \right) \right) + \sqrt{\varphi(n)} + 2 \ln \left( \varepsilon \sqrt{\varphi(n)} \ln 2 n \right) \right\},
\]
\[
\hat{\rho} = \max \{ \rho, 1 - \rho \}, \quad \varepsilon = \min \left\{ \sum_{t \in T_i} \delta_{it}(n) C_{f(n)}^{t-1}, \frac{2 \ln n}{\sqrt{\varphi(n)}} \right\},
\]
\( r = [\varepsilon \varphi(n)], \, 0 < \varepsilon < 1, \) and \( \varepsilon = \text{const.} \).

Here and in what follows we assume that \( 0^0 = 1 \).

Remark. Fix arbitrary numbers \( \varepsilon_0 \in (0; 1) \) and \( \varepsilon \in (0; 1) \). It is easy to check that, given a number \( \gamma > 0 \), there exists a natural number \( n_1 = n_1(\varepsilon_0, \varepsilon, \gamma) \) such that the right hand side of inequality (5) becomes smaller than \( \gamma \) for all \( n \geq n_1 \).

Example. If \( T_i = \{2\} \), \( \varphi(n) = \ln^2 n \), \( f(n) = \lfloor \ln n \rfloor^{3/2} \), \( \delta_2 = \frac{1}{2} \), \( n = 65 \), \( m = -8 \), \( \rho = 0.9 \), \( \varepsilon = 0.25 \), and \( \varepsilon_0 \leq \varepsilon \), then relations (6)–(8) hold. Applying inequality (8) we obtain
\[
\left| P\{\nu_{65} = k\} - \frac{\lambda^k}{k!} e^{-\lambda} \right| \leq 0.086
\]
for all \( k \geq 0 \).

2. Auxiliary results

Denote by \( \mathbb{E} \nu_n^{[k]} \) the factorial moment of order \( k \) for the random variable \( \nu_n \), \( k = 1, 2, \ldots \). We set \( \mathbb{E} \nu_n^{[0]} = 1 \).
Proposition (I). If the conditions (A) hold, then, for all \( k \geq 1 \),
\[
E_{\nu}^{(k)} = 2^{-kN} S(n, k; Q),
\]
where
\[
S(n, k; Q) = \sum_{s=0}^{n-\rho n} \sum (n - \rho n)! \left( \prod_{i \in I} i! \right)^{-1}
\times \sum_{s' \neq 0} \sum' (\rho n')! \left( \prod_{j \in J} j! \right)^{-1} Q,
\]
and the index of summation in \( \sum' \) runs over all elements \( i \in I \) \((j \in J)\) such that
\[
\sum_{i \in I} i = s, \quad \sum_{j \in J} j = s',
\]
where
\[
I = \{ i_{(u_1, \ldots, u_v)} : 1 \leq u_1 < \ldots < u_v \leq k, \nu = 1, \ldots, k \},
\]
\[
J = \{ j_{(u_1, \ldots, u_v)} : 1 \leq u_1 < \ldots < u_v \leq k, \nu = 1, \ldots, k \}.
\]
(The definition of the numbers \( i_{(u_1, \ldots, u_v)} \) and \( j_{(u_1, \ldots, u_v)} \) is given in \( \text{(8)} \).) The elements \( i \in I \) and \( j \in J \) in inequality \( \text{(8)} \) are such that
\[
\sum_{i \in I_{(u)} \cap J_{(u)}} (i + j) \geq 1, \quad u = 1, \ldots, k,
\]
(the sets \( I_{(u)} \) and \( J_{(u)} \) are defined below) and
\[
\sum_{l=0}^{k-2} \sum_{1 \leq \mu_1 < \cdots < \mu_l \leq k} (i_{(u, \mu_1, \ldots, \mu_l)} + j_{(u, \mu_1, \ldots, \mu_l)} + i_{(u_2, \mu_1, \ldots, \mu_l)} + j_{(u_2, \mu_1, \ldots, \mu_l)}) \geq 1,
\]
for \( 1 \leq u_1 < \cdots < u_v \leq k, \nu \in \{1, \ldots, k\}, \) and \( t \in \{1, \ldots, n\} \). Moreover
\[
\Gamma_{i,j}^{(u_1, \ldots, u_v)} \geq \sum_{(i,j) \in T} (C_i + C_j),
\]
where
\[
T = I_{(u_1, \ldots, u_v)} \times J_{(u_1, \ldots, u_v)}.
\]
Here
\[
I_{(u_1, \ldots, u_v)} = \{ i_{(\sigma_1, \ldots, \sigma_{\psi}, \mu_1, \ldots, \mu_l)} : A(\psi, l, k) \},
\]
\[
J_{(u_1, \ldots, u_v)} = \{ j_{(\sigma_1, \ldots, \sigma_{\psi}, \mu_1, \ldots, \mu_l)} : A(\psi, l, k) \}
\]
are the sets of numbers \( i_{(\sigma_1, \ldots, \sigma_{\psi}, \mu_1, \ldots, \mu_l)} \) and \( j_{(\sigma_1, \ldots, \sigma_{\psi}, \mu_1, \ldots, \mu_l)} \), respectively, satisfying the collection of restrictions \( A(\psi, l, k) \), where
\[
A(\psi, l, k) \text{ means }
\]
\[
1 \leq \sigma_1 < \cdots < \sigma_{\psi} \leq k, \quad \sigma_{\psi} \in \{ u_1, \ldots, u_v \}, \quad z = 1, \ldots, \psi, \quad \psi = 1, \ldots, \nu,
\]
\[
\psi \equiv 1 \pmod{2}, \quad 1 \leq \mu_1 < \cdots < \mu_l \leq k, \quad \mu_1, \ldots, \mu_l \notin \{ u_1, \ldots, u_v \},
\]
\[
l = 0, \ldots, k - \nu.
\]
If
\[
\rho n - s' \geq t,
\]
then
\[
\Gamma_{t,k}^{\{u_1,\ldots,u_\nu\}} \geq C_{p^n - s'}^{r-1} \sum_{(i,j) \in T} (i + j).
\]

The explicit expression for \(\Gamma_{t,k}^{\{u_1,\ldots,u_\nu\}}\) is given in [1] for the case of \(1 \leq u_1 < \cdots < u_\nu \leq k, \nu \in \{1, \ldots, k\}, t = 1, 2, \ldots, g_i(n), i = 1, \ldots, N\).

To prove the theorem of Section 1, we need the following auxiliary result.

**Lemma.** Suppose all the assumptions of the theorem hold for all nonnegative integers \(k\) such that
\[
0 < k \leq \gamma.
\]

Then
\[
E_{\nu}^{[k]} = \lambda^k + \Delta(k, n)
\]
for all sufficiently large \(n\), where
\[
|\Delta(k, n)| \leq 2^{(m+1)k+1}u + 2^{mk} \Theta_2 (1 + 2^{k+1}u) + 7 \left(2^{2^k}\right) 2^{(m+1)k}
\]
\[
\times \exp \left\{ -2^{-2k} \sum_{i=1}^{N} \delta_i + \ln(\rho n) - m \ln 2 \right\},
\]
\[
u = \sum_{i=1}^{N} \exp \left\{ -2 \sum_{t \in T_i} \delta_{it}(n) C_t^r \right\}.
\]

**Proof.** Using equality (7), we represent the factorial moment \(E_{\nu}^{[k]}\) as follows:
\[
E_{\nu}^{[k]} = 2^{-kN} \sum_{\Delta \geq 0} S^{(\Delta)}(n, k; Q),
\]
where \(S^{(\Delta)}(n, k; Q)\) is defined similarly to the term \(S(n, k; Q)\) with additional restrictions imposed on elements \(i \in I\) and \(j \in J\) appearing in definition (5) of \(S(n, k; Q)\), namely, that there are exactly \(\Delta\) pairwise distinct sets \(\omega_\alpha\),
\[
\omega_\alpha = \{u_1^{(\alpha)}, \ldots, u_{\xi_\alpha}^{(\alpha)}\},
\]
\(1 \leq u_1^{(\alpha)} < \cdots < u_{\xi_\alpha}^{(\alpha)} \leq k, \xi_\alpha \in \{1, \ldots, k\}, \alpha = 1, \ldots, \Delta,\)
such that for each of them there exists \(t^{(\alpha)} \in \{2, \ldots, r\}\) that satisfies
\[
\Gamma_{t^{(\alpha)}}^{\omega_\alpha} < C_{t^{(\alpha)}}^r
\]
and
\[
\Gamma_{t,k}^{\{v_1,\ldots,v_\gamma\}} \geq C_{t}^r
\]
for all \(t \in \{2, \ldots, r\}\) and for all sets \(\{v_1, \ldots, v_\gamma\}\), \(1 \leq v_1 < \cdots < v_\gamma \leq k, \gamma = 1, \ldots, k,\)
such that \(\{v_1, \ldots, v_\gamma\} \neq \omega_\alpha, \alpha = 1, \ldots, \Delta.\)

It is worth mentioning that the term corresponding to \(\Delta = 0\) may indeed appear on the right hand side of (15) (see [1]).

Furthermore, we rewrite equality (15) as follows:
\[
E_{\nu}^{[k]} = S_1 + p_1,
\]
where
\[
S_1 = 2^{-kN} S^{(0)}(n, k; Q), \quad p_1 = 2^{-kN} \sum_{\Delta = 1}^{2^k-1} S^{(\Delta)}(n, k; Q).
\]
Now we turn to the estimation of $S_1$. If $\Delta = 0$, we use estimate (17) and condition (4).

Then

$$\prod_{i=1}^{N} \left( 1 - 2^k \exp \left\{ -2 \sum_{t \in T_i} \delta_{it}(n) C_r^t \right\} \right)$$

$$\leq Q \leq \prod_{i=1}^{N} \left( 1 + 2^k \exp \left\{ -2 \sum_{t \in T_i} \delta_{it}(n) C_r^t \right\} \right).$$

Condition (5) and the inequality $r > f(n)$ imply by (12) that

$$2^k u < \varepsilon_0.$$ 

Now we use bounds (19) and (20) and the elementary inequalities $1 + u_0 \leq e^{u_0}$ and $e^{u_0} \leq 1 + 2u_0$. Then

$$\prod_{i=1}^{N} (1 - u_i) \geq 1 - \sum_{i=1}^{N} u_i, \quad 0 < u_i < 1, \quad i = 0, 1, \ldots, N,$$

and

$$Q_* \leq Q \leq Q^*,$$

where

$$Q^* = 1 + 2^{k+1} u, \quad Q_* = 1 - 2^k u.$$

If

$$\Gamma_{t,k}^{\{u_1, \ldots, u_\nu\}} < C_r^t$$

for some set $\{u_1, \ldots, u_\nu\}$, $1 \leq u_1 < \cdots < u_\nu \leq k$, $\nu = 1, \ldots, k$, and some $t \in \{2, \ldots, r\}$, then inequality (10) holds, whence we get

$$0 \leq i < r, \quad i \in I_{\{u_1, \ldots, u_\nu\}}, \quad 0 \leq j < r, \quad j \in J_{\{u_1, \ldots, u_\nu\}}.$$

Applying the polynomial theorem and relation (21) we get

$$2^{-kN} \left( 2^{nk} - \sigma_0 \right) Q_* \leq S_1 \leq 2^{-kN} \left( 2^{nk} - \sigma_0 \right) Q^*,$$

where

$$\sigma_0 = 1 + \sum_{d=1}^{2^k-1} S_d^{(0)}(n, k; 1).$$

The definition of $S_d^{(0)}(n, k; 1)$ differs from that of the term $S(n, k; 1)$ in that the elements $i \in I$ and $j \in J$ on the right hand side of equality (8) satisfy an extra restriction, namely that there are exactly $d$ elements of the set

$$\left\{ \Gamma_{t,k}^{\{u_1, \ldots, u_\nu\}}, \ 1 \leq u_1 < \cdots < u_\nu \leq k, \nu = 1, \ldots, k \right\}$$

for which relation (22) holds, $d = 1, 2, \ldots, 2^k - 1$.

Let all the expressions

$$\Gamma_{t,k}^{\{u_1, \ldots, u_\nu\}}, \quad 1 \leq u_1 < \cdots < u_\nu \leq k, \nu = 1, \ldots, k,$$

be labeled with the numbers $1, 2, \ldots, 2^k - 1$. Assume that this numbering is a one-to-one correspondence between the expressions and the numbers. Then the sum $S_d^{(0)}(n, k; 1)$ can be represented as follows:

$$S_d^{(0)}(n, k; 1) = \sum_{1 \leq \zeta_1 < \cdots < \zeta_d \leq 2^k-1} S_d^{(0)}(\zeta_1, \ldots, \zeta_d)(n, k; 1).$$
where the definition of $S_d^{(0)}(n, k; 1)$ differs from that of $S_d^{(0)}(n, k; 1)$ by the restriction that relation (22) holds for those expressions $\Gamma_{t,k}\{u_1, \ldots, u_n\}$ that correspond to the numbers $\zeta_1, \ldots, \zeta_d$. Denote by $A(\zeta_1, \ldots, \zeta_d) \ (B(\zeta_1, \ldots, \zeta_d))$ the set of all $i \in I \ (j \in J)$ that are used in the bound (10) for all $\zeta_1, \ldots, \zeta_d$. By inequality (22), the number of elements of the set $A(\zeta_1, \ldots, \zeta_d) \ (B(\zeta_1, \ldots, \zeta_d))$ is at least $2^{k-1}$:

\[
|A(\zeta_1, \ldots, \zeta_d)| \geq 2^{k-1}, \quad |B(\zeta_1, \ldots, \zeta_d)| \geq 2^{k-1}.
\]

The sum $S_d^{(0)}(n, k; 1)$ can be represented as follows:

\[
S_d^{(0)}(n, k; 1) = \sum_{1 \leq \zeta_1 < \cdots < \zeta_d \leq 2^{k-1}} \sum_{s=0}^{n-\rho n} C_s \sum_{s' + s'' = s} C_{s'} s! \prod_{i \in A} \frac{s_i!}{i!} \prod_{j \in B} \frac{s'_i!}{i!} \prod_{j \in J \setminus B} \frac{s''_i!}{i!},
\]

where $s! \prod_{i \in A} \frac{s_i!}{i!}$ means the sum over all $i \in A(\zeta_1, \ldots, \zeta_d)$ such that $\sum i = s_1$, $s! \prod_{j \in B} \frac{s'_i!}{i!}$ means the sum over all $i \in I \setminus A(\zeta_1, \ldots, \zeta_d)$ such that $\sum i = s_2$, $s! \prod_{j \in J \setminus B} \frac{s''_i!}{i!}$ means the sum over all $j \in B(\zeta_1, \ldots, \zeta_d)$ such that $\sum j = s'_1$, and $s! \prod_{j \in J \setminus B} \frac{s''_i!}{i!}$ means the sum over all $j \in J \setminus B(\zeta_1, \ldots, \zeta_d)$ such that $\sum j = s'_2$.

Using the polynomial theorem and relations (25)–(27) we obtain the following bound:

\[
\sigma_0 \leq 1 + 2^{k-1} \sum_{s=0}^{n-\rho n} C_s \left( \sum_{s_1 \geq 0} C_{s_1} \left( 2^{k-1} - 1 \right)^{s_1} \right) \prod_{i \in A} \frac{s_i!}{i!} \prod_{j \in B} \frac{s'_i!}{i!} \prod_{j \in J \setminus B} \frac{s''_i!}{i!}.
\]

Taking into account the inequalities $s_1 \leq \lfloor r2^k \rfloor$ and $s'_1 \leq \lfloor r2^k \rfloor$, we conclude from (28) that

\[
\sigma_0 \leq 2^{2^k} \left( 2^{k-1} \right)^n \left( 2^k \right)^{r2^k} \left( \sum_{s_1 = 0}^{\lfloor r2^k \rfloor} C_{s_1} \right) \left( \sum_{s'_1 = 0}^{\lfloor r2^k \rfloor} C_{s'_1} \right).
\]

This implies

\[
\sigma_0 \leq 2^{2^k} \left( 2^{k-1} \right)^n \left( 2^k \right)^{r2^k} \left( \sum_{s'_1 = 0}^{\lfloor r2^k \rfloor} C_{s'_1} \right)^2.
\]

Then

\[
0 \leq \sigma_0 \leq 2^{nk} \Theta_2
\]

by inequality (29), condition (12), and by the lower bound (31)

\[
n! > n^n e^{-n} \sqrt{2\pi ne^{1/12n+1}}
\]

proved in [2].
Considering condition (33) and relations (21), (24), and (30), we get the following bounds:

\[
\lambda^k - \left\{2^{(m+1)}k + 2^{m+k} \Theta_2 (1 + 2^k u)\right\} \\
\leq S_1 \leq \lambda^k + \left\{2^{(m+1)}k + 2^{m+k} \Theta_2 (1 + 2^{k+1} u)\right\}.
\]

Using restrictions (12) we show that

\[
p_1 \leq 7 \left(2^{2k}\right)^{(m+1)k} \exp \left\{-2^{2-k} \sum_{i=1}^{N_i} \delta_i + \ln(\tilde{\rho}n) - m \ln 2\right\}
\]

for \(\Delta \geq 1\). Indeed, put

\[
p_2 = p_1 - S_2,
\]

where

\[
S_2 = 2^{-(kN - kN - 1)} \sum_{\Delta=1}^{2^k - 1} S_{\Delta}^{(\Delta)}(n, k; Q).
\]

The definition of \(S_{\Delta}^{(\Delta)}(n, k; Q)\) differs from that of the term \(S_{\Delta}^{(\Delta)}(n, k; Q)\) in that the index of summation \(s'\) in \(S_{\Delta}^{(\Delta)}(n, k; Q)\) satisfies the additional condition, called \(G_0\):

\[
\rho n - r + 1 \leq s' \leq \rho n.
\]

Now we find a bound for \(S_2\). Denote by \(M_1 (\tilde{M}_1)\) the family of all \(i \in I (j \in J)\) that do not belong to \(I_{\omega_n} (J_{\omega_n})\), \(\alpha = 1, \ldots, \Delta\). Also we put

\[
M_2 = I \setminus M_1, \quad \tilde{M}_2 = J \setminus \tilde{M}_1.
\]

Let \(R_1 (\tilde{R}_1)\) denote the number of elements of the set \(M_1 (\tilde{M}_1)\). Let \(z\) be the minimum number such that

\[
\Delta \leq 2^z - 1, \quad 1 \leq z \leq k.
\]

According to Proposition 2.1 of [1] we obtain

\[
R_1 \leq 2^{k-z} - 1, \quad \tilde{R}_1 \leq 2^{k-z} - 1.
\]

If lower bound (17) holds, we take into account (4) and get the following inequality for \(Q\) defined in (34):

\[
Q \leq 2^{zN} \left(1 + 2^{-z} (2^k - \Delta - 1) u\right).
\]

Relation (16) implies that

\[
0 \leq i < r \quad (0 \leq j < r)
\]

for all \(i \in M_2 (j \in \tilde{M}_2)\) by condition (22) and (23). Using (36)–(38) and condition \(G_0\), we prove that

\[
S_2 \leq 2^{2k} 2^{(k-1)m} \exp \left\{-\rho n 2^{-k} + 2^{k} \ln^2 n \ln \left(\frac{\tilde{\rho} n e}{2^k \ln^2 n}\right) + 2^k u\right\}.
\]

Now we introduce condition \(G_1\): let

\[
s' \leq \rho n - r
\]

and let there exist \(i \in M_2\) and (or) \(j \in \tilde{M}_2\) such that \(i \in (r/E_n, r]\) and (or) \(j \in (r/E_n, r]\), where

\[
E_n > 3, \quad E_n = o(\ln n), \quad n \to \infty.
\]

Let

\[
p_3 = p_2 - S_3,
\]
where
\[ S_3 = 2^{-kn} \sum_{\Delta=1}^{2^k-1} S^{(\Delta)}(G_1)(n,k;Q). \]

The definition of \( S^{(\Delta)}(G_1)(n,k;Q) \) differs from that of the term \( S^{(\Delta)}(n,k;Q) \) in that the index of summation \( s' \) in \( S \) satisfies the additional condition \( G_1 \).

We show that
\[ S_3 \leq 2^{2^k 2^{mk}} \frac{2^m}{2^m} \exp \left\{ -2^{-k} N \left( 1 - N^{-\Delta} \right) + 2^{k} \varepsilon n \ln \left( \frac{\hat{\rho} \varepsilon \ln^2 n}{2^{k} \varepsilon \ln^2 n} \right) \right\} \]
where \( A_n = 2\varepsilon/E_n \). If condition \( G_1 \) holds, we get
\[ \Gamma_{i,k}^{\omega} \geq C_r^{t-1} \frac{r}{E_n} \]
for all \( t \in \{2, \ldots, r\} \) and some \( \alpha = 1, \ldots, \Delta \) by inequality (11). Using bound (42) and condition (3) we obtain
\[ \left| \prod_{i=1}^{g(n)} (1 - 2p_{it}) \Gamma_{i,k}^{\omega} \right| \leq \exp \left\{ -2 \sum_{i \in T_i} \delta_{it}(n) C_r^{t-1} \frac{r}{E_n} \right\}, \quad i = 1, \ldots, N. \]
The latter bound implies
\[ Q \leq 2^{-N} \exp \left\{ -2^{-z} \left( N - \sum_{i=1}^{N} \exp \left\{ -2 \sum_{i \in T_i} \delta_{it}(n) C_r^{t-1} \frac{r}{E_n} \right\} \right) \right\}. \]
Relation (43) yields
\[ Q \leq \hat{Q} \]
by Hölder’s inequality and relation (20), where
\[ \hat{Q} = 2^{-N} \exp \left\{ -2^{-z} \left( N - N^{1-\Delta} \right) \right\}. \]
Now we derive from condition \( G_1 \) that
\[ S_3 \leq 2^{2^k 2^{mk}} \frac{2^m}{2^m} \exp \left\{ -2^{-k} N \left( 1 - N^{-\Delta} \right) + 2^{k} \varepsilon n \ln \left( \frac{\hat{\rho} \varepsilon \ln^2 n}{2^{k} \varepsilon \ln^2 n} \right) \right\} \]
\[ \times \sum_{s=0}^{\rho_n-r} C_{\rho,n-s}^{s'} \sum_{s_1+s_2=s} C_{s_1}^{s_1} \left( \sum_{\sigma_1=\Gamma_1} s_1! \prod_{i \in M_2} \Gamma_i^{s_i} \right) \left( \sum_{\sigma_2=\Gamma_1} s_2! \prod_{i \in M_1} \Gamma_i^{s_i} \right) Q. \]

Relations (36), (38), (41), and (45) prove (41).
Now we introduce condition \( G_2 \): let inequality (40) hold and let there exist \( i \in M_2 \) and (or) \( j \in M_2 \) such that \( i \in \lfloor r/\ln n, r/E_n \rfloor \) and (or) \( j \in \lfloor r/\ln n, r/E_n \rfloor \).
Put
\[ p_4 = p_3 - S_4, \]
where
\[ S_4 = 2^{-kn} \sum_{\Delta=1}^{2^k-1} S^{(\Delta)}(G_2)(n,k;Q). \]
The definition of \( S^{(\Delta)}(G_2)(n,k;Q) \) differs from that of the term \( S^{(\Delta)}(n,k;Q) \) in that the index of summation \( s' \) in the sum \( S \) satisfies condition \( G_2 \).
We show that
\[(46) \quad S_4 \leq \frac{2^{2^k}2^{mk}}{2^m} \exp \left\{ -2^{-k} \left( 1 - e^{-2\varepsilon} \right) N + \frac{2^k\varepsilon \ln^2 n}{E_n} \ln \left( \frac{\tilde{p}nE_n}{2^{k\varepsilon \ln^2 n}} \right) \right\}. \]

Similarly to the proof of \[(44)\] we obtain
\[(47) \quad Q \leq 2^{zN} \exp \left\{ -2^{-k} \left( 1 - e^{-2\varepsilon} \right) N \right\}
if condition \(G_2\) holds. Note that the constant \(\tilde{A}_n = 2\varepsilon/\ln n\) substitutes the constant \(A_n = 2\varepsilon/E_n\) in the proof.

If the indices \(i\) and \(j\) satisfy condition \(G_2\), then bound \((46)\) follows from \((36)\) and \((47)\) similarly to the proof of the corresponding bound for \(S_3\).

The following condition is called \(G_3\): let inequality \((40)\) hold and let
\[(48) \quad 0 \leq i \leq \frac{r}{\ln n} \quad \text{and} \quad 0 \leq j \leq \frac{r}{\ln n}
for all \(i \in M_2\) and \(j \in \tilde{M}_2\). Put
\[(49) \quad p_5 = p_4 - S_5,
where
\[S_5 = 2^{-kN} \sum_{\Delta = 1}^{2^k - 1} S^{(\Delta)}_{(G_3,2^z-2)}(n,k;Q).\]
The definition of \(S^{(\Delta)}_{(G_3,2^z-2)}(n,k;Q)\) differs from that of the term \(S^{(\Delta)}(n,k;Q)\) in that the index of summation \(s'\) in \([8]\) satisfies condition \(G_3\) and that \(\Delta < 2^z - 1\).

We show that
\[(50) \quad S_5 \leq \frac{2^{2^k}2^{mk}}{2^m} \exp \left\{ -2^{-k}N + 2^k\varepsilon \ln^2 n + 2^k u \right\}.
Using \((40)\) and inequality \((10)\), we get
\[(51) \quad \Gamma_{t,k}^{(s)} \geq C_r^{-1} \left( s^{(\alpha)} + \tilde{s}^{(\alpha)} \right)
for all \(t \in \{2, \ldots, r\}\) and \(\alpha = 1, \ldots, \Delta\), where
\[s^{(\alpha)} = \sum_{i \in I_{\alpha}} i, \quad \tilde{s}^{(\alpha)} = \sum_{j \in J_{\alpha}} j.\]
According to \((4)\),
\[(52) \quad \prod_{t=1}^{T} \left( 1 - 2p_{it} \right)^{\Gamma_{t,k}^{(s)}} \leq \prod_{t \in T_i} \left( 1 - 2\delta_{it}(n) \right)^{\Gamma_{t,k}^{(s)}},
for \(i = 1, \ldots, N\) and \(\alpha = 1, \ldots, \Delta\).

Now we apply \((51)\) to the right hand side of \((52)\). Then
\[(53) \quad \prod_{t \in T_i} \left( 1 - 2\delta_{it}(n) \right)^{\Gamma_{t,k}^{(s)}} \leq \exp \left\{ -\frac{2\delta_i}{\sqrt{n}} \left( s^{(\alpha)} + \tilde{s}^{(\alpha)} \right) \right\}.
The inequality \(e^{-y} \leq 1 - y/2, 0 \leq y < 1\), implies that
\[(54) \quad \exp \left\{ -\frac{2\delta_i}{\sqrt{n}} \left( s^{(\alpha)} + \tilde{s}^{(\alpha)} \right) \right\} \leq 1 - \frac{\delta_i}{\sqrt{n}} \left( s^{(\alpha)} + \tilde{s}^{(\alpha)} \right)\]
for $i = 1, \ldots, N$ and $\alpha = 1, \ldots, \Delta$. In turn, inequality (51) yields

$$2^{-kN} \sum_{\Delta=1}^{2^k-1} S^{(\Delta)}_{(G_3)} (n; k; Q) \leq 2^{-kN} 2 \Delta (\Delta + 1)^N$$

(55)

where the definition of $S^{(\Delta)}_{(G_3)} (n; k; Q)$ differs from that of $S^{(\Delta)} (n; k; Q)$ in the restriction that the indices in the sum (58) satisfy condition $G_3$. If $\Delta < 2^z - 1$, then (55) implies bound (50) by condition (3), inequalities (36),

$$\max \{s_*, \tilde{s}_*\} \leq 2^k \varepsilon \ln n,$$

and

(56)

$$\sum_{\alpha=1}^\Delta (s^{(\alpha)} + \tilde{s}^{(\alpha)}) \geq s_* + \tilde{s}_*,$$

where

$$s_* = \sum_{i \in M_2} i, \quad \tilde{s}_* = \sum_{j \in M_2} j.$$

Now let $\Delta = 2^z - 1$. Put

$$p_6 = p_5 - S_6,$$

where

$$S_6 = 2^{-kN} \sum_{\Delta=1}^{2^k-1} S^{(\Delta)}_{(G_3, 2^z-1)} (n; k; Q).$$

The definition of $S^{(\Delta)}_{(G_3, 2^z-1)} (n; k; Q)$ differs from that of the term $S^{(\Delta)} (n; k; Q)$ in that the index of summation $s'$ in (58) satisfies condition $G_3$ and $\Delta = 2^z - 1$. If condition $G_3$ holds and $\Delta = 2^z - 1$, then we use condition (3) and relations (56) and (55) together with inequality (50) to find a bound for $S_6$:

(57)

$$S_6 \leq 2^{2k} 2^{mk} \exp \left\{ -2^{-2k} \sum_{i=1}^N \delta_i + k + \ln(\tilde{\rho} n) - m \ln 2 \right\}$$

for the case of

(58)

$$s_* + \tilde{s}_* \geq 1.$$

Now we show that there exists $\alpha \in \{1, 2, \ldots, \Delta\}$ such that $\xi_\alpha \leq 2$ if $\Delta = 2^z - 1$, $1 \leq z \leq k$, and either $z \in \{k, k-1\}$ or $k \in \{1, 2\}$. Indeed, if either $z = k$ or $k \in \{1, 2\}$, then this property is obvious. If $z = k - 1$, then we derive this property from Remark 2.2 in [1].
Consider condition \( G_4 \); let inequality (59) hold and let
\[
\xi_\alpha \geq 3, \quad \alpha = 1, \ldots, \Delta, \quad \Delta = 2^z - 1, \quad 1 \leq z \leq k - 2, \quad 3 \leq k < \infty.
\]
Put
\[
p_7 = p_6 - S_7
\]
and
\[
S_7 = 2^{-kN} \sum_{\Delta=1}^{2^k-1} S^{(\Delta)}(n, k; Q),
\]
where the definition of \( S^{(\Delta)}(n, k; Q) \) differs from that of the term \( S^{(\Delta)}(n, k; Q) \) in that the index of summation \( s' \) in (55) satisfies conditions \( G_4 \).

Now we obtain a bound for \( S_7 \). If (59) holds, \( \Delta = 2^z - 1 \), and \( \tilde{R}_1 < 2^{k-z} - 1 \), then we rewrite bound (55) as follows:
\[
S_7 \leq 2^{2^k+1+zN-kN} \sum_{s=0}^{n-\rho n} C_{n-\rho n}^s |M_1|^s \sum_{s' = 0}^{\rho n} C_{\rho n}^{s'} |M_1|^{s'} \\
\leq \frac{2^{2^k+1+m}}{2^m} (1 - 2^{1-k})^{\rho n}
\]
(here we used restrictions (36)).

It remains to check that
\[
S_8 \leq \frac{2^{k}2^{mk}}{2^m} \exp \left\{-\rho n 2^{-k+1} + \varepsilon \ln^2 n \ln \left( \frac{\rho n e}{\varepsilon \ln^2 n} \right) \right\}
\]
if the conditions \( G_4 \) hold and
\[
\tilde{R}_1 = 2^{k-z} - 1,
\]
where
\[
p_7 = S_8 = 2^{-kN} \sum_{\Delta=1}^{2^k-1} S^{(\Delta)}(G_4, \tilde{R}_1)(n, k; Q).
\]

Here the definition of \( S^{(\Delta)}(G_4, \tilde{R}_1)(n, k; Q) \) differs from that of the term \( S^{(\Delta)}(n, k; Q) \) in that the index of summation \( s' \) in (55) satisfies conditions \( G_4 \) and (63).

Similarly to the proof in [1] and according to Proposition 2.2 in [1] we conclude from (63) and \( G_4 \) that there exists at least one element \( j_* \in M_1 \) such that \( j_* \leq r \). Therefore, under the assumptions of the theorem and conditions \( G_4 \) and (63), we get
\[
S_8 \leq 2^{2^k} 2^{-kN} 2^{zN} 2^{(k-z)(n-\rho n)} \sum_{s' = 0}^{\rho n} C_{\rho n}^{s'} \sum_{s \geq 1} s! \prod_{j \in M_1 \setminus j_*} s'_{j} \prod_{j \in M_1 \setminus j_*} j!
\]
\[
= 2^{2^k} 2^{-kN} 2^{zN} 2^{(k-z)(n-\rho n)} \sum_{s' = 0}^{\rho n} C_{\rho n}^{s'} \sum_{s \geq 1} s! \prod_{j \in M_1 \setminus j_*} s'_{j} \prod_{j \in M_1 \setminus j_*} j!
\]
\[
\leq 2^{2^k} 2^{(k-z)m} \left( 1 - \frac{1}{2^{k-z}} \right) \sum_{j_* = 0}^{\rho n} C_{\rho n}^{j_*}.
\]
Applying inequality (31) we prove bound (62).
Considering conditions $G_0–G_4$, we make sure that they exhaust all possible cases of summation in (8) with respect to the parameters $s, s', i, j, i \in I$ and $j \in J$ for which inequality (16) holds if $\Delta \geq 1$.

Therefore relations (39), (41), (46), (50), (57), (61), and (62) prove (33) under the assumptions of the lemma. Then, by (18), (32), and (33), we find that

$$E_{\nu_n[k]} = \lambda^k + \Delta(k, n),$$

where

$$\Delta(k, n) = \psi(k, n) + p_1,$$

and

$$- \left\{ 2^{(m+1)k}u + 2^{mk}\Theta_2 \left( 1 + 2^k u \right) \right\} \leq \psi(k, n) \leq 2^{(m+1)k+1}u + 2^{mk}\Theta_2 \left( 1 + 2^{k+1} u \right).$$

Using relations (33) and (65), we complete the proof of (13) and (14).

3. PROOF OF THE THEOREM

Fix an integer $q \geq 0$. Consider the following inequality:

$$\left| P\{\nu_n = q\} - \frac{\lambda^q}{q!}e^{-\lambda} \right| \leq R_1 + R_2 + R_3,$$

where

$$R_1 = \left| P\{\nu_n = q\} - \sum_{k=q}^{q+2\nu-1} (-1)^{k-q}C_k^q B_{kn} \right|,$$

$$R_2 = \sum_{k=q}^{q+2\nu-1} (-1)^{k-q}C_k^q \left\{ B_{kn} - \frac{\lambda k}{k!} \right\} \left| \right|,$$

$$R_3 = \sum_{k=q}^{q+2\nu-1} (-1)^{k-q}C_k^q \left( \frac{\lambda k}{k!} - \frac{\lambda^q}{q!} \right) e^{-\lambda} \left| \right|,$$

and $B_{kn}$ denotes the binomial moment of order $k$ for the random variable $\nu_n$. Choose $n$ such that

$$\frac{\lambda^{q+2\nu}}{q!(2\nu)!} < \left( \frac{2e\lambda}{\gamma} \right)^\gamma,$$

where $2\nu = \gamma - q \geq 0$. Such a number $n$ exists in view of $n \geq 2^{6q}$. The inequality

$$R_3 < \frac{\lambda^{q+2\nu}}{q!(2\nu)!}$$

together with (67) implies that

$$R_3 < \left( \frac{2e\lambda}{\gamma} \right)^\gamma.$$
Applying (13) we prove that

\[
\left| B_{q+2\nu,n} - \frac{\lambda^{q+2\nu}}{(q+2\nu)!} \right| = \frac{1}{(q+2\nu)!} |\Delta(q+2\nu,n)|
\]

(70)

\[
\leq \frac{2^{q+2\nu+1} B(n) + \Theta_2 \left(1 + 2^{q+2\nu+1} B(n)\right)}{(q+2\nu)!} \left(2^{q+2\nu+1} B(n) + \Theta_2 \left(1 + 2^{q+2\nu+1} B(n)\right) + 7\Theta_1 \right).
\]

Taking into account (12), we get

(71)

\[
\left| B_{q+2\nu,n} - \frac{\lambda^{q+2\nu}}{(q+2\nu)!} \right| \leq \frac{2^{q+2\nu+1} B(n) + \Theta_2 \left(1 + 2^{q+2\nu+1} B(n)\right)}{(q+2\nu)!} \left(2^{q+2\nu+1} B(n) + \Theta_2 \left(1 + 2^{q+2\nu+1} B(n)\right) + 7\Theta_1 \right).
\]

By Bonferroni’s inequality [3],

(72)

\[
0 \leq P\{\nu_n = q\} = \sum_{k=q}^{q+2\nu-1} (-1)^{k-q} C_k^q B_k n \leq C^q_{q+2\nu} B_{q+2\nu,n}.
\]

Using (67) and (72), we derive from (71) that

(73)

\[
R_1 < \left(\frac{2e^{\lambda}}{\gamma}\right) \gamma \left(1 + 2^{q+1} B(n) + \Theta_2 \left(1 + 2^{q+1} B(n)\right) + 7\Theta_1 \right).
\]

Consider

\[
R_2 = \left| \sum_{k=q}^{q+2\nu-1} (-1)^{k-q} C_k^q \left[ B_k n - \frac{\lambda^k}{k!} \right] \right|.
\]

It is easy to show that

(74)

\[
\sup_{q \leq k \leq q+2\nu-1} C_k^q \left| B_k n - \frac{\lambda^k}{k!} \right| \leq e^{4\lambda} B(n) + e^{2\lambda} \left(\Theta_2 \left(1 + 2^{q+1} B(n)\right) + 7\Theta_1 \right)
\]

by (12)–(14). Inequality (74) implies

(75)

\[
R_2 < \sum_{k=q}^{q+2\nu-1} C_k^q \left| B_k n - \frac{\lambda^k}{k!} \right| \leq e^{4\lambda} B(n) + e^{2\lambda} \gamma \left(\Theta_2 \left(1 + 2^{q+1} B(n)\right) + 7\Theta_1 \right).
\]

Thus (60), (69), (73), and (75) imply (6). The theorem is proved.
BIBLIOGRAPHY


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