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LIMITING BEHAVIOUR OF MOVING AVERAGE PROCESSES UNDER NEGATIVE ASSOCIATION ASSUMPTION

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P. CHEN, T.-C. HU, AND A. VOLODIN

Abstract. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed negatively associated random variables, and $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we prove the complete convergence and complete moment convergence of the maximum partial sums of moving average processes $\{\sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$. We improve the results of Baek *et al.* (2003) and Li and Zhang (2005).

1. Introduction

We assume that $\{Y_i, -\infty < i < +\infty\}$ is a doubly infinite sequence of identically distributed random variables. Let $\{a_i, -\infty < i < +\infty\}$ be an absolutely summable sequence of real numbers, and let

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, \qquad n \geq 1,$$

be the *moving average process based on the sequence* $\{Y_i, -\infty < i < +\infty\}$.

For the moving average process $\{X_n, n \geq 1\}$, many limiting results have been obtained. For example, under the independence assumption of the base sequence

$$\{Y_i, -\infty < i < +\infty\},$$

Burton and Dehling [3] obtained a large deviation principle, and Li *et al.* [11] obtained the complete convergence result. Under different dependence assumptions of the base sequence $\{Y_i, -\infty < i < +\infty\}$, Zhang [17] obtained the complete convergence result when the base sequence $\{Y_i, -\infty < i < +\infty\}$ consists of φ-mixing random variables, and Baek *et al.* [2] and Liang *et al.* [13] obtained the complete convergence result when the base sequence $\{Y_i, -\infty < i < +\infty\}$ consists of negatively associated random variables. For the Banach space generalizations we refer to the papers Ahmed *et al.* [1], Chen *et al.* [5, 6].

Recall that a finite family of random variables $\{Y_i, 1 \leq i \leq n\}$ is said to be *negatively associated* (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and

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any real coordinatewise nondecreasing functions f on R^A and g on R^B,

$$\mathrm{Cov}\,(f(Y_i, i \in A), g(Y_j, j \in B)) \le 0$$

whenever the covariance exists. An infinite family of random variables

$$\{Y_i, < -\infty \le i < \infty\}$$

is NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [10].

In what follows, we let $S_n = \sum_{k=1}^n X_k$, $n \ge 1$, be the partial sums of the sequence $\{X_i, i \ge 1\}$, and $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers, that is, $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Furthermore, for a real number x, let $x_+ = \max\{0, x\}$, and for any number q, we define $x_+^q = (x_+)^q$. As usual, C represents a positive constant although its value may change from one appearance to the next.

First we discuss the previous results connected with complete convergence. The following was proved in Hsu and Robbins [9] and Erdös [8].

Theorem A. *Suppose that* $\{X_n, n \ge 1\}$ *is a sequence of independent identically distributed random variables. Then* $\mathsf{E}\, X_1 = 0$, $\mathsf{E}\,|X_1|^2 < \infty$ *if and only if*

$$\sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \ge \varepsilon n\} < \infty$$

for all $\varepsilon > 0$.

Hsu–Robbins–Erdös result was generalized by Li *et al.* [11] for a moving average process based on a sequence of i.i.d. random variables $\{Y_i, -\infty < i < +\infty\}$.

Theorem B. *Suppose* $\{X_n, n \ge 1\}$ *is the moving average process based on a sequence of independent identically distributed random variables* $\{Y_i, -\infty < i < \infty\}$ *with* $\mathsf{E}\, Y_1 = 0$, $\mathsf{E}\,|Y_1|^{2t} < \infty$, $1 \le t < 2$. *Then* $\sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \ge \varepsilon n^{1/t}\} < \infty$ *for all* $\varepsilon > 0$.

The result of Li *et al.* was generalized for a moving average process based on a sequence of NA random variables $\{Y_i, -\infty < i < +\infty\}$ by Baek *et al.* [2] and Liang *et al.* [13]. If we omit some insignificant details connected with slowly varying functions and stochastic domination condition, their result could be formulated in the following way.

Theorem C. *Suppose that* $\{X_n, n \ge 1\}$ *is the moving average process based on a sequence of NA identically distributed random variables* $\{Y_i, -\infty < i < \infty\}$ *with* $\mathsf{E}\, Y_1 = 0$, $\mathsf{E}\,|Y_1|^{2t} < \infty$, $1 \le t < 2$. *Then* $\sum_{n=1}^{\infty} \mathsf{P}\{|S_n| \ge \varepsilon n^{1/t}\} < \infty$ *for all* $\varepsilon > 0$.

Next, we discuss the previous results connected with complete moment convergence. The notion was introduced in Chow [7], where the following result was also proved.

Theorem D. *Suppose that* $\{X_n, n \ge 1\}$ *is a sequence of independent and identically distributed random variables with* $\mathsf{E}\, X_1 = 0$ *and* $1 \le p < 2$, $r > p$. *If*

$$\mathsf{E}\{|X_1|^{rp} + |X_1| \log(1 + |X_1|)\} < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{r-2-1/p}\, \mathsf{E}\left\{|S_n| - \varepsilon n^{1/p}\right\}_+ < \infty \quad \textit{for all } \varepsilon > 0.$$

We refer the interested reader to the papers by Chen [4] and Rosalsky *et al.* [14] for the generalizations of Theorem D on the Banach space setting.

Li and Zhang [12] extended Theorem D to the case of moving average process based on NA random variables. Their result can be formulated in the following way, where we again omit some insignificant details connected with slowly varying functions.

Theorem E. *Suppose $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of independent identically distributed random variables with $\mathsf{E}\, Y_1 = 0$, $\mathsf{E}\, Y_i^2 < \infty$. Let $1 \leq p < 2$, $r > 1 + p/2$. Then $\mathsf{E}\, |Y_1|^{rp} < \infty$ implies that*

$$\sum_{n=1}^{\infty} n^{r-2-1/p}\, \mathsf{E}\left\{|S_n| - \varepsilon n^{1/p}\right\}_+ < \infty \quad \text{for all } \varepsilon > 0.$$

2. Formulation of the main results

The purpose of this paper is to improve the results of Baek *et al.* [2] (stated above as Theorem C) to the maximum partial sums, and extend the results of Li and Zhang [12] (Theorem E) to the maximum partial sums of a moving average process based on a sequence of NA random variables $\{Y_i, -\infty < i < +\infty\}$ under more optimal moment conditions.

Our main results are as follows.

Theorem 1. *Let $1 \leq p < 2$, $r \geq 1$, $rp \neq 1$. Suppose that $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of independent identically distributed random variables $\{Y_i, -\infty < i < \infty\}$. If $\mathsf{E}\, Y_1 = 0$ and $\mathsf{E}\, |Y_1|^{rp} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2}\, \mathsf{P}\left\{\max_{1 \leq k \leq n} |S_k| > \varepsilon n^{1/p}\right\} < \infty \quad \text{for all } \varepsilon > 0.$$

Theorem 2. *Let $q > 0$, $1 \leq p < 2$, $r \geq 1$, $rp \neq 1$. Suppose that $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of independent identically distributed random variables $\{Y_i, -\infty < i < \infty\}$. If $\mathsf{E}\, Y_1 = 0$ and*

$$\begin{gathered}\mathsf{E}\, |Y_1|^{rp} < \infty, \quad q < rp,\\ \mathsf{E}\, |Y_1|^{rp} \log(1 + |Y_1|) < \infty, \quad q = rp,\\ \mathsf{E}\, |Y_1|^{q} < \infty, \quad q > rp,\end{gathered}$$

then

$$\sum_{n=1}^{\infty} n^{r-2-q/p}\, \mathsf{E}\left\{\max_{1 \leq k \leq n} |S_k| - \varepsilon n^{1/p}\right\}_+^q < \infty \quad \text{for all } \varepsilon > 0.$$

3. Two technical lemmas

The following lemma plays a crucial role in our proofs; see Shao [16, Theorem 2].

Lemma 1. *Let $q \geq 2$, and let $\{X_j, 1 \leq i \leq n\}$ be a sequence of NA random variables with zero mean and $\mathsf{E}\, |X_j|^q < \infty$ for every $1 \leq j \leq n$. Then*

$$\mathsf{E} \max_{1 \leq k \leq n} \left|\sum_{j=1}^{k} X_i\right|^q \leq C \left\{ \left(\sum_{j=1}^{n} \mathsf{E}\, |X_j|^2\right)^{q/2} + \sum_{j=1}^{n} \mathsf{E}\, |X_j|^q \right\}.$$

The next lemma is pure technical.

Lemma 2. *Let Y be a random variable with $\mathsf{E}\, |Y|^{rp} \psi(1 + |Y|) < \infty$, where $r \geq 1$, $p \geq 1$, and $\psi(x) = 1$ or $\psi(x) = \log(x)$, $x \geq 1$. Then*

(i) $\sum_{n=1}^{\infty} n^{r-1}\, \mathsf{P}\{|Y| > n^{1/p}\} \leq C\, \mathsf{E}\, |Y|^{rp}$.

(ii) *For $q > rp$,*

$$\sum_{n=1}^{\infty} n^{r-1-q/p} \psi(n)\, \mathsf{E}\, |Y|^q I\left\{|Y| \leq n^{1/p}\right\} \leq C\, \mathsf{E}\, |Y|^{rp} \psi(|Y|).$$

(iii) *For* $s \geq 1$, $v > 0$, $sp > v$ *and* $\mathsf{E}\,|Y|^{sp}\psi(1+|Y|) < \infty$,

$$\sum_{n=1}^{\infty} n^{s-1-\ v/p}\psi(n)\,\mathsf{E}\,|Y|^{v} I\left\{|Y| > n^{1/p}\right\} \leq C\,\mathsf{E}\,|Y|^{sp}\psi(1+|Y|).$$

Proof. First, we mention that statement (i) is well known, so we will prove only (ii) and (iii). Note that the function ψ has the following properties:

(a) for any $m \geq 1$,

$$\sum_{n=1}^{m} n^{u-1}\psi(n) \leq Cm^{u}\psi(m) \quad \text{if } u > 0$$

and

$$\sum_{n=m}^{\infty} n^{u-1}\psi(n) \leq Cm^{u}\psi(m) \quad \text{if } u < 0;$$

(b) for any $p > 0$, $\psi(|x|^{p}) = C\psi(|x|) \leq C\psi(1+|x|)$.

(ii) Since $r - q/p < 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1-q/p}\psi(n)\,\mathsf{E}\,|Y|^{q} I\left\{|Y| \leq n^{1/p}\right\} \\
&\quad = C\sum_{n=1}^{\infty} n^{r-1-q/p}\psi(n)\sum_{m=1}^{n}\mathsf{E}\,|Y|^{q} I\{m-1 < |Y|^{p} \leq m\} \\
&\quad = C\sum_{m=1}^{\infty}\mathsf{E}\,|Y|^{q} I\{m-1 < |Y|^{p} \leq m\}\sum_{n=m}^{\infty} n^{r-1-q/p}\psi(n) \\
&\quad \leq C\sum_{m=1}^{\infty} m^{r-q/p}\psi(m)\,\mathsf{E}\,|Y|^{q} I\{m-1 < |Y|^{p} \leq m\} \qquad \text{(by (a))} \\
&\quad \leq C\sum_{m=1}^{\infty}\mathsf{E}\, m^{r-q/p}\psi(m)|Y|^{q} I\{m-1 < |Y|^{p} \leq m\} \\
&\quad \leq C\sum_{m=1}^{\infty}\mathsf{E}(|Y|^{p})^{r-q/p}\psi(1+|Y|)|Y|^{q} I\{m-1 < |Y|^{p} \leq m\} \\
&\quad \leq C\sum_{m=1}^{\infty}\mathsf{E}\,|Y|^{rp}\psi(1+|Y|) I\{m-1 < |Y|^{p} \leq m\} \\
&\quad \leq C\,\mathsf{E}\,|Y|^{rp}\psi(1+|Y|).
\end{aligned}$$

(iii) We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{s-1-v/p}\psi(n)\,\mathsf{E}\,|Y|^{v} I\left\{|Y| > n^{1/p}\right\} \\
&\quad = C\sum_{n=1}^{\infty} n^{s-1-v/p}\psi(n)\sum_{m=n}^{\infty}\mathsf{E}\,|Y|^{v} I\{m < |Y|^{p} \leq m+1\} \\
&\quad = C\sum_{m=1}^{\infty}\mathsf{E}\,|Y|^{v} I\{m < |Y|^{p} \leq m+1\}\sum_{n=1}^{m} n^{s-1-v/p}\psi(n) \\
&\quad \leq C\sum_{m=1}^{\infty} m^{s-v/p}\psi(m)\,\mathsf{E}\,|Y|^{v} I\{m < |Y|^{p} \leq m+1\} \qquad \text{(by (a))}
\end{aligned}$$

$$
\begin{aligned}
&= C \sum_{m=1}^{\infty} \mathsf{E}\, m^{s-v/p} \psi(m) |Y|^v I\{m < |Y|^p \le m+1\} \\
&\le C \sum_{m=1}^{\infty} \mathsf{E} (|Y|^p)^{s-v/p} \psi(|Y|^p) |Y|^v I\{m < |Y|^p \le m+1\} \\
&\le C\, \mathsf{E}\, |Y|^{sp} \psi(1+|Y|) \qquad \text{(by (b))}. \qquad \square
\end{aligned}
$$

4. Proof of the main results

First we prove Theorem 1.

Proof. Let

$$Y_{nj}^{(1)} = -n^{1/p} I\left\{Y_j < -n^{1/p}\right\} + Y_j I\left\{|Y_j| \le n^{1/p}\right\} + n^{1/p} I\left\{Y_j > n^{1/p}\right\},$$

and let $Y_{nj}^{(2)} = Y_j - Y_{nj}^{(1)}$ be *monotone truncations* of $Y_j, -\infty < j < \infty$. Then, by Joag-Dev and Proschan [10], for any $n \ge 1$,

$$\left\{Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)}, -\infty < j < \infty\right\} \quad \text{and} \quad \left\{Y_{nj}^{(2)}, -\infty < j < \infty\right\}$$

are two sequences of NA random variables. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and

$$
\begin{aligned}
n^{-1/p} \max_{1\le k\le n} \left| \mathsf{E} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(1)} \right| &\le n^{-1/p} \sum_{i=-\infty}^{\infty} |a_i| \max_{1\le k\le n} \left| \mathsf{E} \sum_{j=i+1}^{i+k} Y_{nj}^{(1)} \right| \\
&\le C n^{-1/p} n \left(\left| \mathsf{E}\, Y_1 I\left\{|Y_1| \le n^{1/p}\right\} \right| + n^{1/p}\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \right) \\
&\le C n^{1-1/p}\, \mathsf{E}\, |Y_1| I\left\{|Y_1| > n^{1/p}\right\} + C n\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \\
&\le C\, \mathsf{E}\, |Y_1|^p I\left\{|Y_1| > n^{1/p}\right\} + C n\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence, for any $\varepsilon > 0$ there exists n large enough that

$$n^{-1/p} \max_{1\le k\le n} \left| \mathsf{E} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(1)} \right| < \varepsilon/4.$$

Therefore, in order to prove Theorem 1, it is enough to prove that

$$I := \sum_{n=1}^{\infty} n^{r-2}\, \mathsf{P}\left\{ \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \left(Y_{nj}^{(1)} - \mathsf{E}\, Y_{nj}^{(1)} \right) \right| > \varepsilon n^{1/p} \Big/ 4 \right\} < \infty$$

and

$$J := \sum_{n=1}^{\infty} n^{r-2}\, \mathsf{P}\left\{ \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(2)} \right| > \varepsilon n^{1/p} \Big/ 2 \right\} < \infty.$$

For J, by Markov's inequality, we have

$$\begin{aligned}
J &\le C\sum_{n=1}^{\infty} n^{r-2}n^{-1/p}\,\mathsf{E}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}^{(2)}\right| \\
&\le C\sum_{n=1}^{\infty} n^{r-1-1/p}\left(\mathsf{E}\,|Y_1| I\left\{|Y_1|>n^{1/p}\right\} + n^{1/p}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\}\right) \\
&\le C\sum_{n=1}^{\infty} n^{r-1-1/p}\,\mathsf{E}\,|Y_1| I\left\{|Y_1|>n^{1/p}\right\} + C\,\mathsf{E}\,|Y_1|^{rp} \qquad \text{(by Lemma 2(i))} \\
&\le C\,\mathsf{E}\,|Y_1|^{rp}<\infty \qquad \text{(by Lemma 2(iii) with } \psi(x)=1, v=1, \text{ and } s=r).
\end{aligned}$$

For I, fix any $q\ge 2$ (to be specified later). Then

$$\begin{aligned}
I &\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\,\mathsf{E}\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-\mathsf{E}\,Y_{nj}^{(1)}\right)\right|^q \\
&\quad \text{(by Markov's inequality)} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\,\mathsf{E}\left(\sum_{i=-\infty}^{\infty}|a_i|\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-\mathsf{E}\,Y_{nj}^{(1)}\right)\right|\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\,\mathsf{E}\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{1-1/q}\right)\left(|a_i|^{1/q}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-\mathsf{E}\,Y_{nj}^{(1)}\right)\right|\right)\right)^q \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{q-1}\sum_{i=-\infty}^{\infty}|a_i|\,\mathsf{E}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-\mathsf{E}\,Y_{nj}^{(1)}\right)\right|^q \\
&\quad \text{(by Hölder's inequality)} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\sum_{i=-\infty}^{\infty}|a_i|\,\mathsf{E}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-\mathsf{E}\,Y_{nj}^{(1)}\right)\right|^q \\
&\quad \text{(since } \textstyle\sum_{i=-\infty}^{\infty}|a_i|<\infty) \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\sum_{i=-\infty}^{\infty}|a_i|\left\{\sum_{j=i+1}^{i+n}\left(\mathsf{E}\left(Y_{nj}^{(1)}-\mathsf{E}\,Y_{nj}^{(1)}\right)^2\right)^{q/2}\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.+\sum_{j=i+1}^{i+n}\mathsf{E}\left|Y_{nj}^{(1)}-\mathsf{E}\,Y_{nj}^{(1)}\right|^q\right\} \\
&\quad \text{(by Lemma 1)} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\sum_{i=-\infty}^{\infty}|a_i|\left\{\left(n\left(\mathsf{E}\,Y_1^2 I\left\{|Y_1|\le n^{1/p}\right\}+n^{2/p}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\}\right)\right)^{q/2}\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.+\,n\left(\mathsf{E}\,|Y_1|^q I\left\{|Y_1|\le n^{1/p}\right\}+n^{q/p}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\}\right)\right\} \\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}\left\{\left(n\left(\mathsf{E}\,Y_1^2 I\left\{|Y_1|\le n^{1/p}\right\}+n^{2/p}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\}\right)\right)^{q/2}\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.+\,n\left(\mathsf{E}\,|Y_1|^q I\left\{|Y_1|\le n^{1/p}\right\}+n^{q/p}\,\mathsf{P}\left\{|Y_1|>n^{1/p}\right\}\right)\right\} \\
&\quad \text{(since } \textstyle\sum_{i=-\infty}^{\infty}|a_i|<\infty).
\end{aligned}$$

We consider two separate cases. If $rp < 2$, let $q = 2$. We have

$$\begin{aligned}
I &\le C \sum_{n=1}^{\infty} n^{r-1-2/p} \left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1| \le n^{1/p}\right\} + n^{2/p}\,\mathsf{P}\left\{|Y_1| > n^{1/p}\right\}\right) \\
&\le C \sum_{n=1}^{\infty} n^{r-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1| \le n^{1/p}\right\} + C\,\mathsf{E}\,|Y_1|^{rp} \qquad \text{(by Lemma 2(i))} \\
&\le C\,\mathsf{E}\,|Y_1|^{rp} < \infty \qquad \text{(by Lemma 2(ii) with } q = 2 \text{ and } \psi(x) = 1).
\end{aligned}$$

If $rp \ge 2$, let $q > 2p(r-1)/(2-p) \ge 2$. Note that in this case

$$\mathsf{E}\,|Y_1|^2 I\left\{|Y_1| \le n^{1/p}\right\} \le \mathsf{E}\,|Y_1|^2 \le (\mathsf{E}\,|Y_1|^{rp})^{2/(rp)} < \infty$$

and by Markov's inequality,

$$n^{2/p}\,\mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \le \mathsf{E}\,|Y_1|^2 < \infty.$$

Hence

$$\begin{aligned}
I &\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \Bigg\{ \left(n \left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1| \le n^{1/p}\right\} + n^{2/p}\,\mathsf{P}\left\{|Y_1| > n^{1/p}\right\}\right)\right)^{q/2} \\
&\qquad\qquad + n \left(\mathsf{E}\,|Y_1|^q I\left\{|Y_1| \le n^{1/p}\right\} + n^{q/p}\,\mathsf{P}\left\{|Y_1| > n^{1/p}\right\}\right)\Bigg\} \\
&\le C \sum_{n=1}^{\infty} n^{r-2-q/p+q/2} + C \sum_{n=1}^{\infty} n^{r-1-q/p}\,\mathsf{E}\,|Y_1|^q I\left\{|Y_1| \le n^{1/p}\right\} \\
&\quad + C \sum_{n=1}^{\infty} n^{r-1}\,\mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \\
&\le C + C\,\mathsf{E}\,|Y_1|^{rp} < \infty
\end{aligned}$$

by Lemma 2(i) and (ii) with $\psi(x) = 1$. □

Next, we prove Theorem 2.

Proof. For every $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-q/p}\,\mathsf{E}\left\{\max_{1\le k\le n} |S_k| - \varepsilon n^{1/p}\right\}_+^q \\
&\quad = \sum_{n=1}^{\infty} n^{r-2-q/p} \int_0^{\infty} \mathsf{P}\left\{\max_{1\le k\le n} |S_k| - \varepsilon n^{1/p} > t^{1/q}\right\} dt \\
&\quad = \sum_{n=1}^{\infty} n^{r-2-q/p} \int_0^{n^{q/p}} \mathsf{P}\left\{\max_{1\le k\le n} |S_k| - \varepsilon n^{1/p} > t^{1/q}\right\} dt \\
&\qquad + \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{\max_{1\le k\le n} |S_k| - \varepsilon n^{1/p} > t^{1/q}\right\} dt \\
&\quad \le \sum_{n=1}^{\infty} n^{r-2}\,\mathsf{P}\left\{\max_{1\le k\le n} |S_k| > \varepsilon n^{1/p}\right\} \\
&\qquad + \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{\max_{1\le k\le n} |S_k| > t^{1/q}\right\} dt.
\end{aligned}$$

Hence, by Theorem 1, in order to prove Theorem 2, it is enough to show that

$$\sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{\max_{1\le k\le n} |S_k| > t^{1/q}\right\} dt < \infty.$$

Let

$$Y_j^{(t,1)} = -t^{1/q} I\left\{Y_j < -t^{1/q}\right\} + Y_i I\left\{|Y_j| \le t^{1/q}\right\} + t^{1/q} I\left\{Y_j > t^{1/q}\right\},$$

and let $Y_j^{(t,2)} = Y_j - Y_j^{(t,1)}$ be monotone truncations of Y_j, $-\infty < j < \infty$. Then by Joag-Dev and Proschan [10], for any $t > 0$,

$$\left\{Y_j^{(t,1)} - \mathsf{E}\, Y_j^{(t,1)}, -\infty < j < \infty\right\}$$

and

$$\left\{Y_j^{(t,2)}, -\infty < j < \infty\right\}$$

are two sequences of NA random variables. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and

$$\begin{aligned}
&\sup_{t\ge n^{q/p}} t^{-1/q} \max_{1\le k\le n}\left|\mathsf{E} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,1)}\right| \\
&\quad\le \sup_{t\ge n^{q/p}} t^{-1/q} \sum_{i=-\infty}^{\infty} |a_i| \max_{1\le k\le n}\left|\mathsf{E} \sum_{j=i+1}^{i+k} Y_j^{(t,1)}\right| \\
&\quad\le C \sup_{t\ge n^{q/p}} t^{-1/q} n \left(\left|\mathsf{E}\, Y_1 I\left\{|Y_1| \le t^{1/q}\right\}\right| + t^{1/q}\, \mathsf{P}\left\{|Y_1| > t^{1/q}\right\}\right) \\
&\quad\le C \sup_{t\ge n^{q/p}} t^{-1/q} n \left(\mathsf{E}\, |Y_1| I\left\{|Y_1| > t^{1/q}\right\} + t^{1/q}\, \mathsf{P}\left\{|Y_1| > t^{1/q}\right\}\right) \\
&\quad\le C\big(n^{q/p}\big)^{-1/q} n\, \mathsf{E}\, |Y_1| I\left\{|Y_1| > \big(n^{q/p}\big)^{1/q}\right\} + Cn\, \mathsf{P}\left\{|Y_1| > \big(n^{q/p}\big)^{1/q}\right\} \\
&\quad\le C\, \mathsf{E}\, n^{1-1/p} |Y_1| I\left\{|Y_1| > n^{1/p}\right\} + Cn\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \\
&\quad\le C\, \mathsf{E}\, |Y_1|^p I\left\{|Y_1| > n^{1/p}\right\} + Cn\, \mathsf{P}\left\{|Y_1| > n^{1/p}\right\} \to 0
\end{aligned}$$

as $n \to \infty$. Hence for n large enough we have

$$\sup_{t\ge n^{q/p}} t^{-1/q} \max_{1\le k\le n}\left|\mathsf{E} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,1)}\right| < 1/4.$$

Therefore, in order to prove Theorem 2, it is enough to show that

$$I := \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \left(Y_j^{(t,1)} - \mathsf{E}\, Y_j^{(t,1)}\right)\right| > t^{1/q}/4\right\} dt < \infty$$

and

$$J := \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} \mathsf{P}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,2)}\right| > t^{1/q}/2\right\} dt < \infty.$$

We first show that $J < \infty$. By Markov's inequality,

$$
\begin{aligned}
J &\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} t^{-1/q}\, \mathsf{E} \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^{(t,2)} \right| dt \\
&\le C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} t^{-1/q} \left(\mathsf{E}\, |Y_1| I\left\{ |Y_1| > t^{1/q} \right\} + t^{1/q}\, \mathsf{P}\left\{ |Y_1| > t^{1/q} \right\} \right) dt \\
&\le C \sum_{n=1}^{\infty} n^{r-1-q/p} \int_{n^{q/p}}^{\infty} t^{-1/q}\, \mathsf{E}\, |Y_1| I\left\{ |Y_1| > t^{1/q} \right\} dt \\
&= C \sum_{n=1}^{\infty} n^{r-1-q/p} \sum_{m=n}^{\infty} \int_{m^{q/p}}^{(m+1)^{q/p}} t^{-1/q}\, \mathsf{E}\, |Y_1| I\left\{ |Y_1| > t^{1/q} \right\} dt \\
&\le C \sum_{n=1}^{\infty} n^{r-1-q/p} \sum_{m=n}^{\infty} m^{q/p-1-1/p}\, \mathsf{E}\, |Y_1| I\left\{ |Y_1| > m^{1/p} \right\} \\
&= C \sum_{m=1}^{\infty} m^{q/p-1-1/p}\, \mathsf{E}\, |Y_1| I\left\{ |Y_1| > m^{1/p} \right\} \sum_{n=1}^{m} n^{r-1-q/p} \\
&\le \begin{cases} C \sum_{m=1}^{\infty} m^{r-1-1/p}\, \mathsf{E}\, |Y_1| I\{ |Y_1| > m^{1/p} \} & \text{if } q < rp, \\ C \sum_{m=1}^{\infty} m^{r-1-1/p} \log(1+m)\, \mathsf{E}\, |Y_1| I\{ |Y_1| > m^{1/p} \} & \text{if } q = rp, \\ C \sum_{m=1}^{\infty} m^{q/p-1-1/p}\, \mathsf{E}\, |Y_1| I\{ |Y_1| > m^{1/p} \} & \text{if } q > rp \end{cases} \\
&\le \begin{cases} C\, \mathsf{E}\, |Y_1|^{rp} & \text{if } q < rp, \\ C\, \mathsf{E}\, |Y_1|^{rp} \log(1+|Y_1|) & \text{if } q = rp, \\ C\, \mathsf{E}\, |Y_1|^{q} & \text{if } q > rp \end{cases} \qquad \text{(by Lemma 2(iii) with } v = 1) \\
&< \infty \qquad \text{(by the assumptions of Theorem 2).}
\end{aligned}
$$

We now prove that $I < \infty$. Fix any $s \ge 2$ (to be specified later). By Markov's inequality,

$$
\begin{aligned}
I &\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} t^{-s/q}\, \mathsf{E} \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} \left(Y_j^{(t,1)} - \mathsf{E}\, Y_j^{(t,1)} \right) \right|^s dt \\
&\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} t^{-s/q}\, \mathsf{E} \left(\sum_{i=-\infty}^{\infty} |a_i| \max_{1\le k\le n} \left| \sum_{j=i+1}^{i+k} \left(Y_j^{(t,1)} - \mathsf{E}\, Y_j^{(t,1)} \right) \right| \right)^s dt \\
&\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \int_{n^{q/p}}^{\infty} t^{-s/q}\, \mathsf{E} \Bigg(\sum_{i=-\infty}^{\infty} \left(|a_i|^{1-1/s} \right) \\
&\qquad\qquad \times \left(|a_i|^{1/s} \max_{1\le k\le n} \left| \sum_{j=i+1}^{i+k} \left(Y_j^{(t,1)} - \mathsf{E}\, Y_j^{(t,1)} \right) \right| \right) \Bigg)^s dt \\
&\le C \sum_{n=1}^{\infty} n^{r-2-q/p} \\
&\qquad \times \int_{n^{q/p}}^{\infty} t^{-s/q} \left(\sum_{i=-\infty}^{\infty} |a_i| \right)^{s-1} \sum_{i=-\infty}^{\infty} |a_i|\, \mathsf{E} \max_{1\le k\le n} \left| \sum_{j=i+1}^{i+k} \left(Y_j^{(t,1)} - \mathsf{E}\, Y_j^{(t,1)} \right) \right|^s dt
\end{aligned}
$$

(by Hölder's inequality)

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\sum_{i=-\infty}^{\infty}|a_i|\,\mathsf{E}\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k}\left(Y_j^{(t,1)}-\mathsf{E}\,Y_j^{(t,1)}\right)\right|^s dt$$

(since $\sum_{i=-\infty}^{\infty}|a_i|<\infty$)

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\sum_{i=-\infty}^{\infty}|a_i|\left\{\left(\sum_{j=i+1}^{i+n}\mathsf{E}\left(Y_j^{(t,1)}-\mathsf{E}\,Y_j^{(t,1)}\right)^2\right)^{s/2} + \sum_{j=i+1}^{i+n}\mathsf{E}\left|Y_j^{(t,1)}-\mathsf{E}\,Y_j^{(t,1)}\right|^s\right\}dt$$

(by Lemma 1)

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p} \times\int_{n^{q/p}}^{\infty} t^{-s/q}\sum_{i=-\infty}^{\infty}|a_i|\left\{n^{s/2}\left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\leq t^{1/q}\right\}+t^{2/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right)^{s/2} + n\left(\mathsf{E}\,|Y_1|^s I\left\{|Y_1|\leq t^{1/q}\right\}+t^{s/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right)\right\}dt$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\left\{n^{s/2}\left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\leq t^{1/q}\right\}+t^{2/q}\left\{|Y_1|>t^{1/q}\right\}\right)^{s/2} + n\left(\mathsf{E}\,|Y_1|^s I\left\{|Y_1|\leq t^{1/q}\right\}+t^{s/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right)\right\}dt$$

(since $\sum_{i=-\infty}^{\infty}|a_i|<\infty$).

Consider two separate cases. If $\max\{q, rp\}<2$, let $s=2$. We have

$$I\leq C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} t^{-2/q}\left(\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\leq t^{1/q}\right\}+t^{2/q}\,\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}\right)dt$$

$$\leq C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} t^{-2/q}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\leq t^{1/q}\right\}dt + C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty}\mathsf{P}\left\{|Y_1|>t^{1/q}\right\}dt$$

$$= C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n+1}^{\infty}\int_{(m-1)^{q/p}}^{m^{q/p}} t^{-2/q}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\leq t^{1/q}\right\}dt + C\sum_{n=1}^{\infty} n^{r-1-q/p}\,\mathsf{E}\,|Y_1|^q I\left\{|Y_1|>n^{1/p}\right\}$$

$$\leq C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n+1}^{\infty}\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\leq m^{1/q}\right\}\int_{(m-1)^{q/p}}^{m^{q/p}} t^{-2/q}\,dt + C\,\mathsf{E}\,|Y_1|^{rp}$$

(by Lemma 2(iii) with $s=r$ and $v=q$)

$$\leq C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n+1}^{\infty}(m-1)^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\leq m^{1/p}\right\} + C\,\mathsf{E}\,|Y_1|^{rp}$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n}^{\infty} m^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le m^{1/p}\right\} + C\,\mathsf{E}\,|Y_1|^{rp}\\
&= C\sum_{m=1}^{\infty} m^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le m^{1/p})\right\}\sum_{n=1}^{m} n^{r-1-q/p} + C\,\mathsf{E}\,|Y_1|^{rp}\\
&\le \begin{cases} C\sum_{m=1}^{\infty} m^{r-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\{|Y_1|\le m^{1/p}\} + C\,\mathsf{E}\,|Y_1|^{rp} & \text{if } q<rp,\\ C\sum_{m=1}^{\infty} m^{r-1-2/p}\log(1+m)\,\mathsf{E}\,|Y_1|^2 I\{|Y_1|\le m^{1/p}\} + C\,\mathsf{E}\,|Y_1|^{rp} & \text{if } q=rp,\\ C\sum_{m=1}^{\infty} m^{q/p-1-2/p}\,\mathsf{E}\,|Y_1|^2 I\{|Y_1|\le m^{1/p}\} + C\,\mathsf{E}\,|Y_1|^{rp} & \text{if } q>rp\end{cases}\\
&\le \begin{cases} C\,\mathsf{E}\,|Y_1|^{rp} & \text{if } q<rp,\\ C\,\mathsf{E}\,|Y_1|^{rp}\log(1+|Y_1|) + C\,\mathsf{E}\,|Y_1|^{rp} & \text{if } q=rp, \qquad \text{(by Lemma 2(ii))}\\ C\,\mathsf{E}\,|Y_1|^{q} + C\,\mathsf{E}\,|Y_1|^{rp} & \text{if } q>rp\end{cases}\\
&<\infty.
\end{aligned}
$$

If $\max\{q, rp\} \ge 2$, let $s > \max\{q, 2p(r-1)/(2-p)\}$. Note that in this case

$$\mathsf{E}\,|Y_1|^2 I\left\{|Y_1|\le t^{1/q}\right\} \le \mathsf{E}\,|Y_1|^2 < \infty$$

and by Markov's inequality,

$$t^{2/q}\,\mathsf{P}\left\{|Y_1| > t^{1/q}\right\} \le \mathsf{E}\,|Y_1|^2 < \infty.$$

Hence

$$
\begin{aligned}
I &\le C\sum_{n=1}^{\infty} n^{r-2-q/p+s/2}\int_{n^{q/p}}^{\infty} t^{-s/q}\,dt\\
&\quad + C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\,\mathsf{E}\,|Y_1|^s I\left\{|Y_1|\le t^{1/q}\right\}dt\\
&\quad + C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} \mathsf{P}\left\{|Y_1| > t^{1/q}\right\}dt\\
&\le C\sum_{n=1}^{\infty} n^{r-2-s/p+s/2} + C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\,\mathsf{E}\,|Y_1|^s I\left\{|Y_1|\le t^{1/q}\right\}dt\\
&\quad + C\,\mathsf{E}\,|Y_1|^{rp}\\
&\le C + C\,\mathsf{E}\,|Y_1|^{rp} + C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} t^{-s/q}\,\mathsf{E}\,|Y_1|^s I\left\{|Y_1|\le t^{1/q}\right\}dt\\
&<\infty. \qquad \square
\end{aligned}
$$

Remark. The key point of the proofs of Theorems 1 and 2 is the application of Hölder's and the Rosenthal type inequalities for maximum partial sums of the NA sequence presented in Lemma 1. Note that the Rosenthal type inequality for maximum partial sums also holds for ρ- and ρ^*-mixing random variables (cf., for example, Shao [15]). Hence Theorems 1 and 2 remain true for ρ and ρ^*-mixing random variables.

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Department of Mathematics, Jinan University, Guangzhou, 510630, People's Republic of China

E-mail address: chenpingyan@yahoo.com.cn

Department of Mathematics, National Tsing Hua University Hsinchu 300, Taiwan, Republic of China

E-mail address: tchu@math.nthu.edu.tw

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, S4S0A2, Canada

E-mail address: andrei@math.uregina.ca

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