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A LOCATION INVARIANT MOMENT-TYPE ESTIMATOR II

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Abstract. The moment estimator (Dekkers et al. (1989)) has been used in extreme value theory to estimate the tail index, but it is not location invariant. The location invariant Hill-type estimator (Fraga Alves (2001)) is only suitable for estimating positive indices. In this paper, a new moment-type estimator is studied, which is location invariant. This new estimator is based on the original moment-type estimator, but it is made location invariant by a random shift. Its asymptotic normality is derived, in a semiparametric setup.

1. Introduction

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with common d.f. $F(x)$, and let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the associated order statistics. If there exist $a_n > 0$, $b_n \in \mathbb{R}$ and some nondegenerate distribution $G(x)$ such that

$$\mathsf{P}(X_{n,n} \le a_n x + b_n) = F^n(a_n x + b_n) \xrightarrow{d} G(x) \quad \text{as} \quad n \to \infty, \tag{1.1}$$

then $G(x)$ must be equivalent to

$$G_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\}, & 1+\gamma x > 0,\ \gamma \ne 0, \\ \exp\{-\exp(-x)\}, & x \in \mathbb{R},\ \gamma = 0. \end{cases}$$

If F satisfies (1.1), we say that $F(x)$ belongs to the domain of attraction of an extreme value of d.f. G_γ, denoted by $F \in D(G_\gamma)$, and γ is referred to as the extreme value index (EVI). In the last two decades many estimators of the extreme value index $\gamma \in \mathbb{R}$ have been proposed that use upper order statistics; see, for example, Hill [11], Pickands [16], Dekkers et al. [4], and Drees [5]. For maximum likelihood estimators of γ, see Hall [10], Smith [20, 21], and Smith and Weissman [22]. For $\gamma > 0$, Hill [11] proposed the estimator given by

$$\hat\gamma_n^H(k) = \frac{1}{k} \sum_{i=0}^{k-1} \log X_{n-i,n} - \log X_{n-k,n} \tag{1.2}$$

for $k = 1, \ldots, n-1$. For $\gamma \in \mathbb{R}$, Dekkers et al. [4] proposed the moment-type estimators

$$\hat\gamma_n^M = M_n^{(1)} + 1 - \frac{1}{2}\left\{1 - \frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}\right\}^{-1}, \tag{1.3}$$

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where

$$M_n^{(j)} = \frac{1}{k}\sum_{i=0}^{k-1}\left(\log\frac{X_{n-i,n}}{X_{n-k,n}}\right)^j$$

for $j = 1, 2$. Pan [12] discussed the asymptotic expansion of the distribution of the Hill and moment estimators. Cheng and Pan [1] and Cuntz et al. [2] obtained penultimate forms of the Hill estimator.

The above estimators are scale invariant but not location invariant. Indeed, there are mathematical as well as practical reasons to require location invariance properties. Since an affine transformation of the r.v.'s X_i merely leads to a change of the normalizing constants a_n and b_n, it influences neither the extreme value index nor the accuracy of the approximation (1.1). Moreover, in practice, the observations (e.g., sea levels, temperatures) depend on an arbitrarily chosen zero-point, which equally should not affect the estimator. In fact, the prominent Pickands [16] estimator given by

$$\hat{\gamma}_n^P = \frac{1}{\log 2}\log\frac{X_{n-k+1,n} - X_{n-2k+1,n}}{X_{n-2k+1,n} - X_{n-4k+1,n}} \tag{1.4}$$

is both scale and location invariant, where $k = k(n)$ is an intermediate integer sequence, i.e., $k = k(n) \to \infty$ and $k/n \to 0$. Dekkers et al. [4], Qi and Cheng [17], and Peng [14] discussed the asymptotic behavior of various Pickands-type estimators. Segers [19] proposed a general Pickands estimator given by

$$\hat{\gamma}_{n,k}(c, v) = \frac{1}{\log v}\log\left(\frac{X_{n-[ck],n} - X_{n-k,n}}{X_{n-[cvk],n} - X_{n-[vk],n}}\right),$$

proved its consistency and asymptotic normality, and discussed the optimal choice of c and v in the sense of minimum square error (MSE). Drees [5] proposed a general class of estimators which have the scale invariant property. For $\gamma > 0$, Fraga Alves [7] established a location invariant Hill estimator given by

$$\hat{\gamma}_n^H(k_0, k) = \frac{1}{k_0}\sum_{i=0}^{k_0-1}\log\left(\frac{X_{n-i,n} - X_{n-k,n}}{X_{n-k_0,n} - X_{n-k,n}}\right), \tag{1.5}$$

where $k \to \infty$, $k_0 \to \infty$, $k/n \to 0$, $k_0/k \to 0$, and discussed its weak consistency, asymptotic expansion and the optimal choice of sample fraction k_0.

Although both (1.4) and (1.5) are location invariant, (1.4) has poor efficiency and it is difficult to decide on the optimal sample fraction k. Also (1.5) is only valid for $\gamma > 0$. In this paper, we propose a general estimator for $\gamma \in \mathbb{R}$ based on the invariant Hill estimator and the moment-type estimator. It is given by

$$\hat{\gamma}_n^M(k_0, k) = M_n^{(1)}(k_0, k) + 1 - \frac{1}{2}\left\{1 - \frac{\left(M_n^{(1)}(k_0, k)\right)^2}{M_n^{(2)}(k_0, k)}\right\}^{-1}, \tag{1.6}$$

where

$$M_n^{(j)}(k_0, k) = \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left(\log\frac{X_{n-i,n} - X_{n-k,n}}{X_{n-k_0,n} - X_{n-k,n}}\right)^j$$

for $j = 1, 2$ and $k = k(n)$, $k_0 = k_0(n)$ are integer sequences that satisfy $0 < k \le n$, $0 < k_0 \le k$. We derive various properties of this new estimator: expansion under the second order regular variation condition (Section 2), necessary and sufficient conditions for asymptotic normality (Section 2), and distribution expansions (Section 2).

2. Asymptotic normality

In order to obtain asymptotic normality of the proposed estimator, one must limit the convergence rate of (1.1). In general, one can obtain asymptotic normality under the following second order conditions.

Second order conditions. Suppose that there exists $b(t) \to 0$ with constant sign near infinity such that

(1) For $\gamma > 0$, $\rho < 0$ and $x > 0$,

$$\frac{1}{b(t)}\left\{\frac{U(tx)}{U(t)} - x^{\gamma}\right\} \to x^{\gamma}\frac{x^{\rho}-1}{\rho} \tag{2.1}$$

as $t \to \infty$.

(2) For $\gamma < 0$ (note that $U(\infty) < \infty$), $\rho < 0$, and $x > 0$,

$$\frac{1}{b(t)}\left\{\frac{U(\infty)-U(tx)}{U(\infty)-U(t)} - x^{\gamma}\right\} \to x^{\gamma}\frac{x^{\rho}-1}{\rho} \tag{2.2}$$

as $t \to \infty$.

(3) For $\gamma = 0$, $\rho < 0$ and $x > 0$,

$$\frac{1}{b(t)}\left\{\frac{U(tx)-U(t)}{a(t)} - \log x\right\} \to \frac{x^{\rho}-1}{\rho} \tag{2.3}$$

with

$$\frac{1}{b(t)}\left\{\frac{a(tx)}{a(t)} - 1\right\} \to 0$$

as $t \to \infty$.

For these three cases, one can easily infer that $|b(t)| \in RV_{\rho}$. We call $b(t)$ the second auxiliary function with the second index parameter ρ. Before stating the theorem and its corollary, we need the following notation: let

$$P_n^0 = \frac{1}{k_0}\sum_{i=0}^{k_0-1} \log Y_i - 1,$$

$$Q_n^0 = \frac{1}{k_0}\sum_{i=0}^{k_0-1} \log^2 Y_i - 2,$$

$$P_n = \frac{1}{k_0}\sum_{i=0}^{k_0-1} (1-Y_i^{\gamma}) + \gamma/(1-\gamma), \qquad \gamma < 0,$$

$$Q_n = \frac{1}{k_0}\sum_{i=0}^{k_0-1} (1-Y_i^{\gamma})^2 - 2\gamma^2/(1-\gamma)(1-2\gamma), \qquad \gamma < 0,$$

$$s^2(\gamma) = \begin{cases} 1+\gamma^2, & \gamma \ge 0, \\ (1-\gamma)^2(1-2\gamma)\left\{4 - 8\frac{1-2\gamma}{1-3\gamma} + \frac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right\}, & \gamma < 0, \end{cases}$$

$$C(k_0,k,\rho) = \begin{cases} \frac{\gamma-\gamma\rho+\rho}{\gamma(1-\rho)^2} b\left(\frac{n}{k}\right)\left(\frac{k_0}{k}\right)^{-\rho} + \left(\frac{\gamma}{1+\gamma}\right)^2\left(\frac{k_0}{k}\right)^{\gamma}, & \gamma > 0, \\ \frac{\rho}{(1-\rho)^2} b\left(\frac{n}{k_0}\right) + \frac{1}{\log\frac{k}{k_0}}, & \gamma = 0, \\ \frac{(1-\gamma)(1-2\gamma)(\gamma+\rho)}{\gamma(1-\gamma-\rho)(1-2\gamma-\rho)} b\left(\frac{n}{k_0}\right) + \frac{-\gamma}{1-\gamma}\left(\frac{k_0}{k}\right)^{-\gamma}, & \gamma < 0, \end{cases}$$

$$e(\gamma) = \begin{cases} -3, & \gamma \ge 0, \\ -\left[(\gamma-1) + \frac{8(1-2\gamma)}{1-3\gamma} - \frac{2(1-\gamma)^2(5-11\gamma)}{(1-3\gamma)(1-4\gamma)}\right], & \gamma < 0, \end{cases}$$

$$W_n = \begin{cases} \frac{1}{\sigma(\gamma)}\sqrt{k_0}\left[(\gamma-2)P_n^0 + \frac{1}{2}Q_n^0\right], & \gamma \geq 0, \\ \frac{(1-\gamma)^2(1-2\gamma)}{\gamma\sigma(\gamma)}\sqrt{k_0}\left[2P_n + \frac{1-2\gamma}{2\gamma}Q_n\right], & \gamma < 0, \end{cases}$$

$$d(\gamma) = \begin{cases} \alpha_1(\gamma), & \gamma \geq 0, \\ \alpha_2(\gamma), & \gamma < 0, \end{cases}$$

$$\alpha_1(\gamma) = \frac{1}{\sigma^3(\gamma)}\left\{\left[(\gamma-2)^3 + 6(\gamma-2)^2\right] + 12(\gamma-2) + 15 \right.$$
$$\left. - (\gamma-1)\left[(\gamma-2)^2 + 3(\gamma-2) + 3\right] + \frac{1}{3}(\gamma-1)^3\right\},$$

$$\alpha_2(\gamma) = \frac{(1-\gamma)^2(1-2\gamma)}{\gamma\sigma(\gamma)}^3$$
$$\times\left\{\frac{4}{3}\left(1+\frac{3(1-2\gamma)}{4(1-\gamma)}\right)\left(1-\frac{3}{1-\gamma}+\frac{3}{1-2\gamma}-\frac{1}{1-3\gamma}\right)\right.$$
$$+\frac{(1-2\gamma)}{\gamma}\left(1-\frac{1-2\gamma}{8(1-\gamma)}\right)\left(1-\frac{4}{1-\gamma}+\frac{6}{1-2\gamma}-\frac{4}{1-3\gamma}+\frac{1}{1-4\gamma}\right)$$
$$+\left(\frac{1-2\gamma}{2\gamma}\right)^2\left(1-\frac{5}{1-\gamma}\frac{10}{1-2\gamma}-\frac{10}{1-3\gamma}+\frac{5}{1-4\gamma}-\frac{1}{1-5\gamma}\right)$$
$$+\frac{1}{6}\left(\frac{1-2\gamma}{2\gamma}\right)^3$$
$$\times\left(1-\frac{6}{1-\gamma}+\frac{15}{1-2\gamma}-\frac{20}{1-3\gamma}+\frac{15}{1-4\gamma}-\frac{6}{1-5\gamma}+\frac{1}{1-6\gamma}\right)$$
$$\left.+\frac{4\gamma^3}{(1-\gamma)^2(1-2\gamma)}-\frac{1}{3}\left(\frac{\gamma}{1-\gamma}\right)^3\right\}$$

$$A_i(\gamma) = \begin{cases} (\gamma-2)\log Y_i + \frac{1}{2}\log^2 Y_i, & \gamma \geq 0, \\ 2\left(1-Y_i^\gamma\right) + \frac{1-2\gamma}{2\gamma}(1-Y_i^\gamma)^2, & \gamma < 0. \end{cases}$$

We also need the following lemmas.

Lemma 2.1. *Let P_n^0, Q_n^0, P_n, and Q_n be as defined above.*

(i) *For $\gamma < 0$,*

$$\sqrt{k_0}\left(P_n^0, Q_n^0\right) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 4\\ 4 & 20\end{pmatrix}\right)$$

as $k_0 \to \infty$.

(ii) *For $\gamma < 0$,*

$$\sqrt{k_0}(P_n, Q_n) \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}, \Sigma\right)$$

as $k_0 \to \infty$, where

$$\Sigma = \frac{\gamma^2}{(1-\gamma)^2(1-2\gamma)}\begin{pmatrix} 1 & -\frac{4\gamma}{1-3\gamma} \\ -\frac{4\gamma}{1-3\gamma} & \frac{4\gamma^2(5-11\gamma)}{(1-2\gamma)(1-3\gamma)(1-4\gamma)}\end{pmatrix}.$$

Proof. See Dekkers et al. [4, Lemma 3.4]. □

Lemma 2.2. *For every integer m,*

$$\mathsf{P}\left(W_n \le x\right) = \Phi(x) + \sum_{v=1}^{m} Q_v(x) k^{-v/2} + o\left(k^{-m/2}\right)$$

holds uniformly for $x \in \mathbb{R}$, where

$$Q_v(x) = -\phi(x) \sum H_{v+2s-1}(x) \prod_{j=1}^{v} \frac{1}{h_j!} \left(\frac{r_{j+2}}{(j+2)!}\right)^{h_j},$$

the sum is over all nonnegative integer solutions of the equations

$$h_1 + 2h_2 + \cdots + vh_v = v \quad \text{and} \quad s = h_1 + \cdots + h_v,$$

H_l is given by

$$H_l(x) = l! \sum_{i=0}^{[l/2]} \frac{(-1)^i x^{l-2i}}{i!\,(l-2i)!\,2^i},$$

and r_j is the jth semi-invariable of $\{\mathrm{Var}(A_1(\gamma))\}^{-1/2}\{A_1(\gamma) - \mathsf{E}\,A_1(\gamma)\}$.

Proof. This follows from Theorem 4 in Petrov [15]. □

Theorem 2.1 concerns the expansion for the proposed estimator under general conditions.

Theorem 2.1. *Suppose* (1.1) *holds and $U(t)$ satisfies one of conditions* (2.1), (2.2), *or* (2.3).

(i) *For $\gamma > 0$,*

$$\hat{\gamma}_n^M(k_0, k) = \gamma + \frac{Q_n^0}{2} + (\gamma - 2)P_n^0 + \frac{\gamma - \gamma\rho + \rho}{\gamma(1-\rho)^2} b\left(\frac{n}{k}\right)\left(\frac{k_0}{k}\right)^{-\rho} + \left(\frac{\gamma}{1+\gamma}\right)^2 \left(\frac{k_0}{k}\right)^{\gamma} - \frac{3}{k_0} + R_1.$$

(ii) *For $\gamma = 0$,*

$$\hat{\gamma}_n^M(k_0, k) = \frac{Q_n^0}{2} - 2P_n^0 + \frac{\rho}{(1-\rho)^2} b\left(\frac{n}{k_0}\right) + \frac{1}{\log Y_{k-k_0,k}} - \frac{3}{k_0} + R_0.$$

(iii) *For $\gamma < 0$,*

$$\begin{aligned}\hat{\gamma}_n^M(k_0, k) = \gamma &+ \frac{(1-\gamma)^2(1-2\gamma)}{\gamma}\left(2P_n + \frac{1-2\gamma}{2\gamma} Q_n\right) \\ &+ \frac{(1-\gamma)(1-2\gamma)(\gamma+\rho)}{\gamma(1-\gamma-\rho)(1-2\gamma-\rho)} b\left(\frac{n}{k_0}\right) + \frac{-\gamma}{1-\gamma}\left(\frac{k_0}{k}\right)^{-\gamma} \\ &+ \left[(\gamma - 1) + \frac{8(1-2\gamma)}{1-3\gamma} - \frac{2(1-\gamma)^2(5-11\gamma)}{(1-3\gamma)(1-4\gamma)}\right]\frac{1}{k_0} + R_{-1}.\end{aligned}$$

Here, the R_i, $i = 1, 0, -1$, can be unified into an R_n^* satisfying

$$|R_n^*| \le k_0^{-1} \left| H\left(\frac{1}{k_0}, b\left(\frac{n}{k_0}\right), \left(\frac{k_0}{k}\right)^{|\gamma|}, \frac{1}{\log Y_{k-k_0,k}}\right)\right|,$$

where $H(w, x, y, z)$ is a polynomial with respect to w, x, y, and z.

Theorem 2.2. *Let $k = k(n) \to \infty$, $k_0 = k_0(n) \to \infty$, $k/n \to 0, k_0/k \to 0$, and $U(t)$ satisfies one of conditions (2.1), (2.2), or (2.3). If there exist a sequence $\{B_n\}$ and a nondegenerate d.f. $\widetilde{G}$ such that*

$$\mathsf{P}\left\{B_n\left(\hat{\gamma}_n^M(k_0,k)-\gamma\right)\le x\right\}\xrightarrow{d}\widetilde{G}(x),$$

then $\widetilde{G}$ must be a normal distribution function.

Theorem 2.3. *Under the assumptions of Theorem 2.2, one has*

$$\widetilde{G}=N\left(\lambda,\sigma^2\right)\Leftrightarrow\begin{cases}\frac{B_n}{\sqrt{k_0}}\to\frac{\sigma}{\sigma(\gamma)},\\ \sqrt{k_0}C\left(k_0,k,\rho\right)\to\frac{\lambda\sigma(\gamma)}{\sigma}.\end{cases}\tag{2.4}$$

Theorem 2.4. *Under the assumptions of Theorem 2.2, one has*

$$\begin{aligned}&\mathsf{P}\left\{\frac{\sqrt{k_0}\left(\hat{\gamma}_n^M(k_0,k)-\gamma\right)}{\sigma(\gamma)}\le x\right\}\\&\qquad=\Phi(x)+k_0^{-1/2}\left\{d(\gamma)\left(1-x^2\right)-\frac{1}{\sigma(\gamma)}\left[\zeta+e(\gamma)\right]\right\}\phi(x)+o\left(k_0^{-1/2}\right)\end{aligned}$$

holding uniformly for $x\in\mathbb{R}$ with k_0 given by $k_0C(k_0,k,\rho)\to\zeta$.

Theorem 2.5. *If we replace $k_0C(k_0,k,\rho)\to\zeta$ in Theorem 2.4 by $\sqrt{k_0}C(k_0,k,\rho)\to\zeta_1$, then*

$$\mathsf{P}\left\{\sqrt{k_0}\left(\hat{\gamma}_n^M(k_0,k)-\gamma\right)\le x\right\}\to N\left(\zeta_1,\sigma^2(\gamma)\right).$$

Proof of Theorem 2.1. We consider the three cases successively.

(i) For $\gamma>0$, (2.1) is equivalent to

$$\frac{U(tx)}{U(t)}=x^\gamma+x^\gamma\frac{x^\rho-1}{\rho}b(t)(1+o(1))$$

as $t\to\infty$, which implies

$$\begin{aligned}&\frac{U\left(Y_{n-i,n}\right)-U\left(Y_{n-k,n}\right)}{U\left(Y_{n-k_0,n}\right)-U\left(Y_{n-k,n}\right)}\\
&\quad=\frac{U\left(Y_{n-i,n}\right)/U\left(Y_{n-k,n}\right)-1}{U\left(Y_{n-k_0,n}\right)/U\left(Y_{n-k,n}\right)-1}\\
&\quad=\frac{\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)^\gamma\left[1-\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)^{-\gamma}+\frac{\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)^\rho-1}{\rho}\ b\left(Y_{n-k,n}\right)\left(1+o_p\left(1\right)\right)\right]}{\left(\frac{Y_{n-k_0,n}}{Y_{n-k,n}}\right)^\gamma\left[1-\left(\frac{Y_{n-k_0,n}}{Y_{n-k,n}}\right)^{-\gamma}+\frac{\left(\frac{Y_{n-k_0,n}}{Y_{n-k,n}}\right)^\rho-1}{\rho}b\left(Y_{n-k,n}\right)\left(1+o_p\left(1\right)\right)\right]}\\
&\quad=Y_{k_0-i,k_0}^\gamma\frac{1-Y_{k_0-i,k_0}^{-\gamma}Y_{k-k_0,k}^{-\gamma}+\frac{Y_{k_0-i,k_0}^\rho Y_{k-k_0,k}^\rho-1}{\rho}b\left(Y_{n-k,n}\right)\left(1+o_p\left(1\right)\right)}{1-Y_{k-k_0,k}^{-\gamma}+\frac{Y_{k-k_0,k}^\rho-1}{\rho}b\left(Y_{n-k,n}\right)\left(1+o_p\left(1\right)\right)}\\
&\quad=Y_{k_0-i,k_0}^\gamma\left[1+\left(1-Y_{k_0-i,k_0}^{-\gamma}\right)Y_{k-k_0,k}^{-\gamma}\left(1+o_p\left(1\right)\right)\right.\\
&\qquad\qquad\left.+\frac{Y_{k_0-i,k_0}^\rho-1}{\rho}Y_{k-k_0,k}^\rho b\left(Y_{n-k,n}\right)\left(1+o_p\left(1\right)\right)\right].\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\gamma}_{n,j}\left(k_0,k\right) &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\log Y_{k_0-i,k_0}^{\gamma}\left[1+\left(1-Y_{k_0-i,k_0}^{-\gamma}\right)Y_{k-k_0,k}^{-\gamma}\right.\right.\\&\qquad\left.\left.+\frac{Y_{k_0-i,k_0}^{\rho}-1}{\rho}Y_{k-k_0,k}^{\rho}b\left(Y_{n-k,n}\right)+r_{n,i}\right]\right\}^{j}\\&= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\gamma\log Y_{k_0-i,k_0}+\left(1-Y_{k_0-i,k_0}^{-\gamma}\right)Y_{k-k_0,k}^{-\gamma}\right.\\&\qquad\left.+\frac{Y_{k_0-i,k_0}^{\rho}-1}{\rho}Y_{k-k_0,k}^{\rho}b\left(Y_{n-k,n}\right)+r_{n,i}\right\}^{j}.\end{aligned}$$

In particular,

$$\begin{aligned}\hat{\gamma}_{n,1}\left(k_0,k\right) &= \gamma+\gamma P_n^0+\frac{\gamma}{\gamma+1}\left(\frac{k_0}{k}\right)^{\gamma}+\frac{1}{1-\rho}\left(\frac{k_0}{k}\right)^{-\rho}b\left(\frac{n}{k}\right)+R_{11},\\\hat{\gamma}_{n,2}\left(k_0,k\right) &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\gamma^2\log^2 Y_{k_0-i,k_0}+2\gamma\left(1-Y_{k_0-i,k_0}^{-\gamma}\right)\log Y_{k_0-i,k_0}Y_{k-k_0,k}^{-\gamma}\right.\\&\qquad\left.+2\gamma\frac{Y_{k_0-i,k_0}^{\rho}-1}{\rho}\log Y_{k_0-i,k_0}Y_{k-k_0,k}^{\rho}b\left(Y_{n-k,n}\right)+r_{n,i}^{*}\right\}\\&= 2\gamma^2+\gamma^2Q_n^0+\frac{2\gamma}{\rho}\left(\frac{1}{\left(1-\rho\right)^2}-1\right)\left(\frac{k_0}{k}\right)^{-\rho}b\left(\frac{n}{k}\right)\\&\qquad-2\gamma\left(\frac{1}{\left(1+\gamma\right)^2}-1\right)\left(\frac{k_0}{k}\right)^{\gamma}+R_{12},\\\hat{\gamma}_{n,1}^2\left(k_0,k\right) &= \gamma^2+2\gamma^2P_n^0+\frac{2\gamma}{1-\rho}\left(\frac{k_0}{k}\right)^{-\rho}b\left(\frac{n}{k}\right)+\frac{2\gamma^2}{1+\gamma}\left(\frac{k_0}{k}\right)^{\gamma}+\gamma^2P_n^{02}+R_{11},\end{aligned}$$

where both R_{11} and R_{12} are $o((k_0/k)^{\gamma}, Y_{k-k_0,k}^{\rho}b(Y_{n-k,n}))$. Hence,

$$\begin{aligned}\hat{\gamma}_{n,2}-\hat{\gamma}_{n,1}^2 &= \gamma^2+\gamma^2\left(Q_n^0-2P_n^0\right)+\frac{2\gamma}{\rho}\left(\frac{1}{\left(1-\rho\right)^2}-\frac{1}{1-\rho}\right)\left(\frac{k_0}{k}\right)^{-\rho}b\left(\frac{n}{k}\right)\\&\qquad-2\gamma\left(\frac{1}{\left(1+\gamma\right)^2}-\frac{1}{1+\gamma}\right)\left(\frac{k_0}{k}\right)^{\gamma}-\gamma^2P_n^{02}+R^{11},\\\hat{\gamma}_{n,2}-2\hat{\gamma}_{n,1}^2 &= \gamma^2\left(Q_n^0-4P_n^0\right)+\frac{2\gamma}{\rho}\left(\frac{1}{1-\rho}-1\right)^2\left(\frac{k_0}{k}\right)^{-\rho}b\left(\frac{n}{k}\right)\\&\qquad-2\gamma\left(\frac{1}{1+\gamma}-1\right)^2\left(\frac{k_0}{k}\right)^{\gamma}-2\gamma^2P_n^{02}+R^{12}.\end{aligned}$$

The result for $\hat{\gamma}_n^M(k_0,k)$ follows by combining these two relations.

(ii) For $\gamma=0$, note that

$$\begin{aligned}&\frac{1}{b\left(t\right)}\left\{\frac{U\left(tx\right)-U\left(t\right)}{a\left(t\right)}-\log x\right\}\to\frac{x^{\rho}-1}{\rho}\\&\iff\frac{U\left(tx\right)-U\left(ty\right)}{U\left(ty\right)-U\left(t\right)}=\frac{\log\frac{x}{y}+\frac{\left(\frac{x}{y}\right)^{\rho}-1}{\rho}b\left(ty\right)\left(1+o\left(1\right)\right)}{\log y-\frac{y^{-\rho}-1}{\rho}b\left(ty\right)\left(1+o\left(1\right)\right)}\end{aligned}$$

and so

$$\begin{aligned}\frac{U\left(Y_{n-i,n}\right)-U\left(Y_{n-k,n}\right)}{U\left(Y_{n-k_0,n}\right)-U\left(Y_{n-k,n}\right)} &= 1+\frac{U\left(Y_{n-i,n}\right)-U\left(Y_{n-k_0,n}\right)}{U\left(Y_{n-k_0,n}\right)-U\left(Y_{n-k,n}\right)}\\ &= 1+\frac{U\left(Y_{k_0-i,k_0}Y_{n-k,n}\right)-U\left(Y_{k-k_0,k}Y_{n-k,n}\right)}{U\left(Y_{k-k_0,k}Y_{n-k,n}\right)-U\left(Y_{n-k,n}\right)}\\ &= 1+\frac{\log Y_{k_0-i,k_0}+\frac{Y^{\rho}_{k_0-i,k_0}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)}{\log Y_{k-k_0,k}-\frac{Y^{\rho}_{k_0-i,k_0}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)}\\ &= 1+\frac{u_1(k_0,k)}{u_2(k_0,k)}.\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\gamma}_{n,j}\left(k_0,k\right) &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\log\left[1+\frac{u_1(k_0,k)}{u_2(k_0,k)}\right]\right\}\\ &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\log^{-1}Y_{k-k_0,k}u_1(k_0,k)\right\}^j\\ &= \log^{-j}Y_{k-k_0,k}\frac{1}{k_0}\sum_{i=0}^{k_0-1}u_1^j(k_0,k).\end{aligned}$$

In particular,

$$\begin{aligned}\hat{\gamma}_{n,1}\left(k_0,k\right) &= \log^{-1}Y_{k-k_0,k}\left[1+P_n^0+\frac{1}{1-\rho}b\left(\frac{n}{k_0}\right)+R_{01}\right],\\ \hat{\gamma}_{n,2}\left(k_0,k\right) &= \log^{-2}Y_{k-k_0,k}\left[2+Q_n^0+\frac{2\left(2-\rho\right)}{\left(1-\rho\right)^2}b\left(\frac{n}{k_0}\right)+R_{02}\right],\\ \hat{\gamma}^2_{n,1}\left(k_0,k\right) &= \log^{-2}Y_{k-k_0,k}\left[1+2P_n^0+\frac{2}{1-\rho}b\left(\frac{n}{k_0}\right)+P_n^{02}+R_{01}\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\hat{\gamma}_{n,2}-\hat{\gamma}^2_{n,1}\\ &\quad=\log^{-2}Y_{k-k_0,k}\left[1+\left(Q_n^0-2P_n^0\right)+2\left(\frac{2-\rho}{\left(1-\rho\right)^2}-\frac{1}{1-\rho}\right)b\left(\frac{n}{k_0}\right)-P_n^{02}+R^{01}\right],\\ &\hat{\gamma}_{n,2}-2\hat{\gamma}^2_{n,1}\\ &\quad=\log^{-2}Y_{k-k_0,k}\left[\left(Q_n^0-4P_n^0\right)+2\left(\frac{2-\rho}{\left(1-\rho\right)^2}-\frac{2}{1-\rho}\right)b\left(\frac{n}{k_0}\right)-2P_n^{02}+R^{02}\right].\end{aligned}$$

The result for $\hat{\gamma}_n^M(k_0,k)$ follows by combining these two relations.

(iii) For $\gamma<0$, note that $V(t)=U(\infty)-U(t)\in\mathbb{R}$ and

$$\begin{aligned}&\left\{\frac{V\left(tx\right)}{V\left(t\right)}-x^{\gamma}\right\}\frac{1}{b\left(t\right)}\to x^{\gamma}\frac{x^{\rho}-1}{\rho}\\ &\qquad\Longleftrightarrow\frac{V\left(tx\right)}{V\left(t\right)}=x^{\gamma}+x^{\gamma}\frac{x^{\rho}-1}{\rho}b\left(t\right)\left(1+o\left(1\right)\right)\end{aligned}$$

and so

$$
\begin{aligned}
&\frac{U\left(Y_{n-i,n}\right)-U\left(Y_{n-k,n}\right)}{U\left(Y_{n-k_0,n}\right)-U\left(Y_{n-k,n}\right)} \\
&\quad = 1+\frac{U\left(Y_{n-i,n}\right)-U\left(Y_{n-k_0,n}\right)}{U\left(Y_{n-k_0,n}\right)-U\left(Y_{n-k,n}\right)} \\
&\quad = 1+\frac{V\left(Y_{k_0-i,k_0}Y_{n-k,n}\right)-V\left(Y_{k-k_0,k}Y_{n-k,n}\right)}{V\left(Y_{k-k_0,k}Y_{n-k,n}\right)-V\left(Y_{n-k,n}\right)} \\
&\quad = 1+\frac{\frac{V\left(Y_{k_0-i,k_0}Y_{n-k,n}\right)}{V\left(Y_{k-k_0,k}Y_{n-k,n}\right)}-1}{1-\frac{V\left(Y_{k-k_0,k}Y_{n-k,n}\right)}{V\left(Y_{n-k,n}\right)}} \\
&\quad = 1+\frac{\left(\frac{Y_{n-i,n}}{Y_{n-k_0,n}}\right)^{\gamma}+\left(\frac{Y_{n-i,n}}{Y_{n-k_0,n}}\right)^{\gamma}\frac{\left(\frac{Y_{n-i,n}}{Y_{n-k_0,n}}\right)^{\rho}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)-1}{1-\left(\frac{Y_{n-k,n}}{Y_{n-k_0,n}}\right)^{\gamma}-\left(\frac{Y_{n-k,n}}{Y_{n-k_0,n}}\right)^{\gamma}\frac{\left(\frac{Y_{n-k,n}}{Y_{n-k_0,n}}\right)^{\rho}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)} \\
&\quad = 1+\frac{Y_{k_0-i,k_0}^{\gamma}+Y_{k_0-i,k_0}^{\gamma}\frac{Y_{k_0-i,k_0}^{\rho}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)-1}{1-Y_{k-k_0,k}^{-\gamma}-Y_{k-k_0,k}^{-\gamma}\frac{Y_{k-k_0,k}^{-\rho}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)} \\
&\quad = 1+Y_{k-k_0,k}^{\gamma}\left[1-Y_{k_0-i,k_0}^{\gamma}-Y_{k_0-i,k_0}^{\gamma}\frac{Y_{k_0-i,k_0}^{\rho}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\hat{\gamma}_{n,j}\left(k_0,k\right) &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\Bigg\{\log\Bigg[1+Y_{k-k_0,k}^{\gamma}\Bigg(1-Y_{k_0-i,k_0}^{\gamma} \\
&\qquad -Y_{k_0-i,k_0}^{\gamma}\frac{Y_{k_0-i,k_0}^{\rho}-1}{\rho} \\
&\qquad \times b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)\Bigg)\Bigg]\Bigg\}^{j} \\
&= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\Bigg\{Y_{k-k_0,k}^{\gamma}\Bigg(1-Y_{k_0-i,k_0}^{\gamma} \\
&\qquad -Y_{k_0-i,k_0}^{\gamma}\frac{Y_{k_0-i,k_0}^{\rho}-1}{\rho}b\left(Y_{n-k_0,n}\right)\left(1+o_p\left(1\right)\right)\Bigg)\Bigg\}^{j}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\hat{\gamma}_{n,1}\left(k_0,k\right) &= Y_{k-k_0,k}^{\gamma}\left[\frac{-\gamma}{1-\gamma}+P_n-\frac{1}{\left(1-\gamma\right)\left(1-\gamma-\rho\right)}b\left(\frac{n}{k_0}\right)+R_{-11}\right], \\
\hat{\gamma}_{n,2}\left(k_0,k\right) &= Y_{k-k_0,k}^{2\gamma}\Bigg[\frac{2\gamma^2}{\left(1-\gamma\right)\left(1-2\gamma\right)}+Q_n+R_{-12} \\
&\qquad -2\left[\frac{1}{\left(1-\gamma\right)\left(1-\gamma-\rho\right)}-\frac{1}{\left(1-2\gamma\right)\left(1-2\gamma-\rho\right)}\right]b\left(\frac{n}{k_0}\right)\Bigg],
\end{aligned}
$$

and

$$\hat{\gamma}_{n,1}^2(k_0,k) = Y_{k-k_0,k}^{2\gamma}\Bigg[\left(\frac{\gamma}{1-\gamma}\right)^2 - \frac{2\gamma}{1-\gamma}P_n + \frac{2\gamma}{(1-\gamma)^2(1-\gamma-\rho)}b\left(\frac{n}{k_0}\right) + P_n^2 + R_{-11}\Bigg].$$

Hence,

$$\begin{aligned}\hat{\gamma}_{n,2} - \hat{\gamma}_{n,1}^2 = Y_{k-k_0,k}^{2\gamma}\Bigg\{&\frac{2\gamma^2}{(1-\gamma)^2(1-2\gamma)} + \left(Q_n + \frac{2\gamma}{1-\gamma}P_n\right)b\left(\frac{n}{k_0}\right) - P_n^2 + R^{-11}\\ &- 2\Bigg[\frac{1}{(1-\gamma)(1-\gamma-\rho)} - \frac{1}{(1-2\gamma)(1-2\gamma-\rho)}\\ &+ \frac{\gamma}{(1-\gamma)^2(1-\gamma-\rho)}\Bigg]\Bigg\},\end{aligned}$$

$$\begin{aligned}\hat{\gamma}_{n,2} - 2\hat{\gamma}_{n,1}^2 = Y_{k-k_0,k}^{2\gamma}\Bigg\{&\frac{2\gamma^2\gamma}{(1-\gamma)^2(1-2\gamma)} + \left(Q_n + \frac{4\gamma}{1-\gamma}P_n\right) - 2P_n^2 + R^{-12}\\ &- 2\Bigg[\frac{1}{(1-\gamma)(1-\gamma-\rho)} - \frac{1}{(1-2\gamma)(1-2\gamma-\rho)}\\ &+ \frac{2\gamma}{(1-\gamma)^2(1-\gamma-\rho)}\Bigg]b\left(\frac{n}{k_0}\right)\Bigg\}.\end{aligned}$$

The result for $\hat{\gamma}_n^M(k_0,k)$ follows by combining these two relations. □

Proof of Theorem 2.2. Since

$$\begin{aligned}&\mathsf{P}\left\{B_n\left(\hat{\gamma}_n^M(k_0,k)-\gamma\right)\le x\right\}\\ &\quad= \mathsf{P}\left\{B_n\left(\frac{\sigma(\gamma)}{\sqrt{k_0}}W_n + C(k_0,k,\rho) + e(\gamma)\frac{1}{k_0} + R_n^*\right)\le x\right\}\\ &\quad= \mathsf{P}\left\{\frac{\sigma(\gamma)}{\sqrt{k_0}}W_n + C(k_0,k,\rho) + e(\gamma)\frac{1}{k_0} + R_n^* \le \frac{x}{B_n}\right\},\end{aligned}$$

by Lemma 2.2, one can bound

$$\begin{aligned}\sup_x\Bigg|&\mathsf{P}\left\{B_n\left(\hat{\gamma}_n^M(k_0,k)-\gamma\right)\le x\right\}\\ &- \mathsf{P}\left\{B_n\left(\frac{\sigma(\gamma)}{\sqrt{k_0}}N + C(k_0,k,\rho) + e(\gamma)\frac{1}{k_0} + R_n^*\right)\le x\right\}\Bigg| \le O\left(k_0^{-1/2}\right).\end{aligned}$$

This means that $B_n(\hat{\gamma}_n^M(k_0,k)-\gamma)$ has a nondegenerate limiting d.f. if and only if

$$\frac{B_n}{\sqrt{k_0}}\sigma(\gamma)N + B_nC(k_0,k,\rho) + B_ne(\gamma)\frac{1}{k_0} + B_nR_n^* \tag{2.5}$$

has the same nondegenerate limiting d.f. Under the assumptions of Theorem 2.2, the limiting d.f. of (2.5) is $\widetilde{G}$, and

$$\limsup_{n\to\infty} \frac{B_n}{\sqrt{k_0}}\sigma(\gamma) < \infty,$$
$$\limsup_{n\to\infty} \left(B_n C(k_0, k, \rho) + B_n e(\gamma) \frac{1}{k_0} + B_n R_n^* \right) < \infty.$$

Thus, there exist a subsequence $\{n'\}$ and constants (λ, σ) such that

$$\lim_{n\to\infty} \frac{B_{n'}}{\sqrt{k_0(n')}}\sigma(\gamma) = \sigma,$$
$$\lim_{n\to\infty} B_{n'} C\left(k_0(n'), k(n'), \rho\right) + B_{n'} e(\gamma) \frac{1}{k_0(n')} + B_{n'} R_{n'}^* = \lambda$$

with $\widetilde{G}$ nondegenerate, which implies $\sigma \neq 0$. Hence, $\widetilde{G}$ is the normal d.f. $N(\lambda, \sigma^2)$. □

Proof of Theorem 2.3. ($\Longleftarrow$) If (2.4) holds, then

$$\lim_{n\to\infty} \frac{B_n}{\sqrt{k_0}}\sigma(\gamma) = \sigma$$

and

$$\lim_{n\to\infty} B_n C\left(k_0, k, \rho\right) + B_n e(\gamma) \frac{1}{k_0} + B_n R_n^* = \lambda \tag{2.6}$$

hold, implying that (2.5) converges to $sN + \lambda$ in distribution, i.e.,

$$B_n \left(\hat{\gamma}_n^M(k_0, k) - \gamma\right) \xrightarrow{d} N\left(\lambda, \sigma^2\right). \tag{2.7}$$

($\Longrightarrow$) If (2.7) holds, then from the proof of Theorem 2.2, one has

$$\lim_{n\to\infty} \frac{B_n}{\sqrt{k_0}}\sigma(\gamma) = \sigma \tag{2.8}$$

and

$$\lim_{n\to\infty} B_n C\left(k_0, k, \rho\right) + B_n e(\gamma) \frac{1}{k_0} + B_n R_n^* = \lambda. \tag{2.9}$$

By the property of R_n^*, (2.9) is equivalent to

$$\lim_{n\to\infty} B_n C(k_0, k, \rho) \to \lambda,$$

which together with (2.8) completes the proof. □

Proof of Theorem 2.4. By Theorem 2.1,

$$\hat{\gamma}_n^M\left(k_0, k\right) = \gamma + \frac{\sigma\left(\gamma\right)}{\sqrt{k_0}} W_n + C\left(k_0, k, \rho\right) + e\left(\gamma\right) \frac{1}{k_0} + R_n^*.$$

Thus, one can write

$$\begin{aligned}\sqrt{k_0}\left(\hat{\gamma}_n^M\left(k_0, k\right) - \gamma\right) &= \sqrt{k_0}\left(\frac{\sigma\left(\gamma\right)}{\sqrt{k_0}} W_n + C\left(k_0, k, \rho\right) + e\left(\gamma\right) \frac{1}{k_0} + R_n^*\right) \\ &= \sigma\left(\gamma\right) W_n + \left[k_0 C\left(k_0, k, \rho\right) + e\left(\gamma\right)\right] k_0^{-1/2} + o\left(k_0^{-1/2}\right) \\ &= \sigma\left(\gamma\right) W_n + \left[\zeta + e\left(\gamma\right)\right] k_0^{-1/2} + o\left(k_0^{-1/2}\right)\end{aligned}$$

and now application of Lemma 2.2 shows that

$$\begin{aligned}
&\mathsf{P}\left\{\frac{\sqrt{k_0}\left(\hat{\gamma}_n^M(k_0,k)-\gamma\right)}{\sigma(\gamma)}\le x\right\}\\
&\quad=\mathsf{P}\left\{W_n\le x-\frac{1}{\sigma(\gamma)}[\zeta+e(\gamma)]k_0^{-1/2}+o\left(k_0^{-1/2}\right)\right\}\\
&\quad=\Phi(x)-k_0^{-1/2}\left\{d(\gamma)\left(1-x^2\right)-\frac{1}{\sigma(\gamma)}[\zeta+e(\gamma)]\right\}\phi(x)+o\left(k_0^{-1/2}\right).
\end{aligned}$$

Note that

$$Q_1(x)=-\phi(x)H_2(x)r_3/6,$$
$$H_2(x)=x^2-1,$$
$$r_3=\{\mathrm{Var}(A_1(\gamma))\}^{-3/2}\,\mathsf{E}\left(A_1(\gamma)-\mathsf{E}\,A_1(\gamma)\right)^3.$$

Furthermore,

$$\mathsf{E}\,A_1(\gamma)=\gamma-1,$$
$$\mathsf{E}\left(A_1(\gamma)\right)^2=2(\gamma-2)^2+6(\gamma-2)+6,$$
$$\mathsf{E}\left(A_1(\gamma)\right)^3=6(\gamma-2)^3+36(\gamma-2)^2+90(\gamma-2)+90$$

for $\gamma\ge 0$ and

$$\mathsf{E}\,A_1(\gamma)=2\,\mathsf{E}(1-R_1)+\frac{1-2\gamma}{2\gamma}\,\mathsf{E}(1-R_1)^2,$$
$$\mathsf{E}\left(A_1(\gamma)\right)^2=4\,\mathsf{E}(1-R_1)^2+\left(\frac{1-2\gamma}{2\gamma}\right)^2\mathsf{E}(1-R_1)^4+2\frac{1-2\gamma}{\gamma}\,\mathsf{E}(1-R_1)^3,$$
$$\begin{aligned}
\mathsf{E}\left(A_1(\gamma)\right)^3&=8\,\mathsf{E}(1-R_1)^3+6\frac{1-2\gamma}{\gamma}\,\mathsf{E}(1-R_1)^4+6\left(\frac{1-2\gamma}{2\gamma}\right)^2\mathsf{E}(1-R_1)^5\\
&\quad+\left(\frac{1-2\gamma}{2\gamma}\right)^3\mathsf{E}(1-R_1)^6
\end{aligned}$$

for $\gamma<0$, where $R_1=Y_1^{\gamma}$. The proof of the theorem is complete upon calculating the necessary parts in Lemma 2.2. □

Proof of Theorem 2.5. The theorem easily follows from Theorems 2.1 and 2.4. □

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