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ANALYTICAL PROBLEMS OF THE ASYMPTOTIC BEHAVIOR OF MARKOV FUNCTIONALS. II

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Abstract. The asymptotic behavior of Markov functionals of a homogeneous ergodic Markov process is studied in this paper.

This paper is a continuation of [1]. We number the displayed formula continuously after the last number in [1].

Let us turn to the proof of the theorem. First we show that

$$h^i_\varepsilon(x, t\cdot s) \underset{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow} \delta^{il} e^{c_{ii}us}\langle \pi, \varphi\rangle. \tag{48}$$

We recall that $\langle \pi, \varphi\rangle = \int_E \pi(dy)\varphi(y)$ and the convergence is uniform with respect to $x \in D$ and $t \in [\sigma, T]$ for all $T > \sigma > 0$.

To prove the latter relation we use a result similar to the Markov renewal theorem. Let a family of nonnegative semihomogeneous kernels $G_\varepsilon(x, dy\times dt)$ depending on a small parameter $\varepsilon > 0$ converge as $\varepsilon \to 0$ to a nonnegative stochastic kernel $G(x, dy \times dt)$ in the sense that

$$\sup_{A\in\mathcal{A}} \left\| \int_0^\infty G_\varepsilon(\cdot, A\times dt)\varphi(t) - \int_0^\infty G(\cdot, A\times dt)\varphi(t) \right\| \underset{\varepsilon\to 0}{\longrightarrow} 0 \tag{49}$$

for all continuous bounded functions $\varphi(t)$, $t \ge 0$, where

$$\|f\| = \inf\{c\colon |f(x)| \le ch(x)\} \quad \pi\text{-almost everywhere.}$$

Denote by $G_\varepsilon(x, dy)$ and $G(x, dy)$ the bases of the kernels $G_\varepsilon(x, dy\times dt)$ and $G(x, dy\times dt)$, respectively. Let $U_\varepsilon(x, dy \times dt)$ denote the potential of the kernel $G_\varepsilon(x, dy \times dt)$.

We assume that the kernel $G(x, A)$ does not depend on x, that is,

$$G(x, A) = \pi(A) \quad \text{and} \quad h(x) = 1, \tag{50}$$

where $\pi(A)$ is a probability distribution on $(E, \mathcal{B})$.

We denote by λ_ε the spectral radius of the kernel $G_\varepsilon(x, A) = G_\varepsilon(x, A \times [0, \infty))$ and by π_ε the eigenprobability measure of the kernel $G_\varepsilon(x, A)$ corresponding to the eigenvalue λ_ε. Put $\gamma_\varepsilon = (1 - \lambda_\varepsilon)/m$.

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Theorem 1. *Assume that conditions* (49) *and* (50) *hold. We also assume that*

a) *the kernel $G(x, dy \times dt)$ is nonlattice and*

$$\sup_{\varepsilon>0}\sup_{x\in E}\int_0^\infty G_\varepsilon(x, E\times dt)t < \infty, \tag{51}$$

$$\sup_{\varepsilon>0}\int_E\int_T^\infty \pi(dx)G_\varepsilon(x, E\times dt)t \underset{T\to\infty}{\longrightarrow} 0; \tag{52}$$

b) *a family of $\mathcal{B}\times\mathcal{B}_+$-measurable functions $g_\varepsilon(x,t)$ is such that $g_\varepsilon(x,t)\ge 0$ and*

$$\sup_{\varepsilon>0}\sup_{x\in E}\int_0^\infty g_\varepsilon(x,t)\,dt < \infty, \tag{53}$$

$$\sup_{\varepsilon>0}\sup_{x\in E}\sup_{t\ge 0} U_\varepsilon * g_\varepsilon(x,t) < \infty, \tag{54}$$

$$\lim_{N\to\infty}\sup_{\varepsilon>0}\sup_{x\in E}\sum_{k=N}^\infty \sup_{k\le t\le k+1} g_\varepsilon(x,t) = 0, \tag{55}$$

$$\sup_{\varepsilon>0}\delta\int_E \pi(dx)\sum_{k=0}^\infty\left\{\sup_{k\delta\le t\le k\delta+\delta} g_\varepsilon(x,t) - \inf_{k\delta\le t\le k\delta+\delta} g_\varepsilon(x,t)\right\}\underset{\delta\to 0}{\longrightarrow} 0; \tag{56}$$

c) *there exists a $\mathcal{B}_+$-measurable function $g(t)$ such that*

$$\int_0^\infty\left|g(t) - \int_E \pi(dx)g_\varepsilon(x,t)\right|\,dt \underset{\varepsilon\to 0}{\longrightarrow} 0; \tag{57}$$

d) *there exists a number $\delta>0$ such that*

$$\inf_{\varepsilon>0}\inf_{x\in E} G_\varepsilon(x, E\times[\delta,\infty)) \ge \delta, \tag{58}$$

$$\int_E \pi_\varepsilon(dx)G_\varepsilon(x, A\times[\delta,\infty)) \ge \delta\pi_\varepsilon(A). \tag{59}$$

Then

$$\lim_{\substack{\gamma_\varepsilon t\to c\\ \varepsilon\to 0\\ t\to\infty}} U_\varepsilon \times g_\varepsilon(x,t) = e^{-c}\frac{1}{m}\int_0^\infty g(s)\,ds$$

uniformly with respect to $x\in E$, where

$$m = \int_E\int_0^\infty \pi(dx)G(x, E\times dt)t < \infty.$$

Conditions (12)–(15) imply that the kernels Q_ε^i satisfy all the assumptions of Theorem 2. Now we show that the functions $g_\varepsilon^i(x,t)$ satisfy conditions (55)–(57). Without loss of generality, we assume that $0\le\varphi(x)\le 1$. Then

$$\sup_{k\le t\le k+1} g_\varepsilon^i(x,t) = \sup_{k\le t\le k+1}\mathbb{P}_{x,i}\{\varphi(X(t)), \xi_\varepsilon(t) = l, t\le\tau\} \le \mathbb{P}_x\{k<\tau\} \le \int_{k-1}^k \mathbb{P}_x\{\tau>t\}\,dt.$$

Thus

$$\sup_{x\in D}\sum_{k=N+1}^\infty \sup_{k\le t\le k+1} g_\varepsilon^i(x,t) = \sup_{x\in D}\sum_{k=N+1}^\infty\int_{k-1}^k \mathbb{P}_x\{\tau>t\}\,dt = \sup_{x\in D}\int_N^\infty \mathbb{P}_x\{\tau>t\}\,dt \le \sup_{x\in D}[\mathbb{P}_x\{\tau>N\} + \mathbb{P}_x\{\tau,\tau>N\}].$$

In view of (14), the latter expression approaches zero as $N \to \infty$, and this proves the uniform convergence of series (55) with respect to $n \ge 1$ and $x \in D$.

Now we check condition (56). It follows from (16) that the process $\{\varphi(X(t)), \xi_\varepsilon(t)\}$ is stochastically continuous uniformly in $\varepsilon > 0$ and $t \in [0,T]$ for an arbitrary initial distribution of the process $X(t)$. In particular,

$$\sup_{a\le s\le t\le b} \mathbb{P}_{x,i}|\varphi(X(t)) - \varphi(X(s))| \underset{\substack{b-a\to 0\\ 0\le a\le b\le T}}{\longrightarrow} 0,$$
$$\sup_{a\le s\le t\le b}\sup_{\varepsilon>0} \mathbb{P}_{x,i}\{\xi_\varepsilon(t) = l, \xi_\varepsilon(s) \ne l\} \underset{\substack{b-a\to 0\\ 0\le a\le b\le T}}{\longrightarrow} 0$$

for all $T > 0$, $i, l \in I$, and $x \in E$.

Next we apply the inequality

$$\begin{aligned}&\sup_{a\le t\le b} g_\varepsilon^i(x,t) - \inf_{a\le t\le b} g_\varepsilon^i(x,t)\\ &\quad\le \sup_{a\le s\le t\le b} \mathbb{P}_{x,i}|\varphi(X(t)) - \varphi(X(s))| + \sup_{a\le s\le t\le b} \mathbb{P}_{x,i}\{\xi_\varepsilon(t) = l, \xi_\varepsilon(s) \ne l\}\\ &\quad\quad + \mathbb{P}_{x,i}\{a \le \tau \le b\}.\end{aligned} \tag{60}$$

The further reasoning is the same as that used in the proof of Theorem 3, Chapter 2, Section 2 in [2] and this proves condition (56). Since

$$\lim_{\varepsilon\to 0} \mathbb{P}_{x,i}\{\xi_\varepsilon(t) \ne i\} = 0,$$

relation (57) follows from (13) for all $x \in E$, $i \in I$, $t \ge 0$, and for the function

$$g(t) = \delta^{il}\mathbb{P}_{\pi_D}[\varphi(X(t)), t < \tau].$$

Note that

$$\frac{1}{m}\int_0^\infty g(t)\,dt = \delta^{il}\langle \pi, \varphi\rangle,$$

whence (48) follows.

Consider a sequence of functions $p_k^i(t)$, $k \ge 0$, defined by the following recurrence relation:

$$p_0^i(t) = \delta^{il}e^{c_{ii}t},$$
$$p_{k+1}^i(t) = \int_0^t e^{c_{ii}s}\,ds \sum_{j\ne i} c_{ij}p_k^j(t-s).$$

Note that the sum of the series

$$\sum_{k=0}^\infty p_k^i(t) = p^{il}(t)$$

is equal to the entry (i,l) of the matrix $\exp\{tC\}$, where

$$C = \|c_{ij}\|_{i,j=1}^\infty.$$

We prove by induction in $k \ge 0$ that

$$\lim_{\substack{\varepsilon\to 0\\ t\to\infty\\ \varepsilon t\to u}} h_{\varepsilon,k}^i(x,t) = p_k^i(u)\langle \pi, \varphi\rangle \tag{61}$$

for all $u \ge 0$ and $i \in I$ uniformly in $x \in D$. Recall that the functions $h_{\varepsilon,k}^i(x,t)$ are defined in (19).

Equality (61) coincides with (48) for $k = 0$. We show that (48) holds for $k = 1$ uniformly in $0 \le t \le T$ and $x \in D$ for all $T > 0$. We have

$$h^i_{\varepsilon,1}(x,t) = \sum_{j \ne i} \int_D \int_0^1 R^{ij}_\varepsilon(x, dy \times t\,ds) h^j_\varepsilon(y, t(1-s)). \tag{62}$$

Using (48) we get

$$h^j_\varepsilon(y, t(1-s)) = \langle \pi, \varphi \rangle p^j_0(u(1-s)) + r^j_\varepsilon(y(1-s)),$$

where

$$\sup_{y \in D} \sup_{0 \le t \le T} \sup_{0 \le s \le T-\sigma} \left| r^j_\varepsilon(y, 1-s) \right| \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{63}$$

The term corresponding to an index j on the right hand side of (62) can be represented as follows:

$$\begin{aligned} &\int_0^1 R^{ij}_\varepsilon(x, D \times t\,ds) p^j_0(u(1-s)) \langle \pi, \varphi \rangle + \int_D \int_0^{1-\sigma} R^{ij}_\varepsilon(x, dy \times t\,ds) r^j_\varepsilon(y, 1-s) \\ &\quad + \int_D \int_{1-\sigma}^1 R^{ij}_\varepsilon(x, dy \times t\,ds) r^j_\varepsilon(y, 1-s). \end{aligned} \tag{64}$$

According to Lemma 2, the first term of (64) converges as $\varepsilon \to 0$ to $\langle \pi, \varphi \rangle p^j_1(u)$ uniformly in $x \in D$ and $0 \le u \le T$. We apply Lemma 2 to the second term and deduce from (63) that it converges to zero uniformly in $x \in D$ and $0 \le u \le T$. If $\sigma > 0$ is sufficiently small, then the third term also is small uniformly in $x \in D$ and $0 \le u \le T$.

The proof of the induction step from k to $k+1$ for $k \ge 1$ is the same as above, and thus we omit it.

It follows from (20), (21), (61), (17), and (48) that

$$\lim_{\substack{\varepsilon \to 0 \\ t \to \infty \\ \varepsilon t \to u}} f^i_\varepsilon(x,t) = p^{il}(u) \langle \pi, \varphi \rangle$$

for all $x \in E$ and $u > 0$. Thus the theorem is proved if (18) holds.

To complete the proof of the theorem we need to check condition (18). It is easy to see that the functions

$$\widetilde{f}^i_\varepsilon(x,t) = e^{-\varepsilon t} f^i_\varepsilon(x,t)$$

satisfy the system of equations similar to (16); namely,

$$\widetilde{f}^i_\varepsilon(x,t) = \widetilde{g}^i_\varepsilon(x,t) + \widetilde{Q}^i_\varepsilon * \widetilde{f}^i_\varepsilon(x,t) + \sum_{j \ne i} \widetilde{Q}^{ij}_\varepsilon * \widetilde{f}^j_\varepsilon(x,t), \tag{65}$$

where

$$\begin{aligned} \widetilde{g}^i_\varepsilon(x,t) &= e^{-\varepsilon t} g^i_\varepsilon(x,t), \\ \widetilde{Q}^i_\varepsilon(x, dy \times dt) &= e^{-\varepsilon t} Q^i_\varepsilon(x, dy \times dt), \\ \widetilde{Q}^{ij}_\varepsilon(x, dy \times dt) &= e^{-\varepsilon t} Q^{ij}_\varepsilon(x, dy \times dt). \end{aligned}$$

Following the method used in the first part of the paper to derive (17) from (16) we obtain from representation (64) that

$$\widetilde{f}^i_\varepsilon(x,t) = \widetilde{H}^i_\varepsilon * g^i_\varepsilon(x,t) + \sum_{j \ne i} \widetilde{R}^{ij}_\varepsilon * \widetilde{f}^j_\varepsilon(x,t),$$

where $\widetilde{H}^i_\varepsilon$ is the potential of the kernel Q^i_ε,

$$\widetilde{R}^{ij}_\varepsilon = \widetilde{H}^i_\varepsilon * \widetilde{Q}^{ij}_\varepsilon.$$

Now we check that the kernels $\widetilde{R}^{ij}_\varepsilon(x, dy \times dt)$ satisfy condition (18). Recalling the notation for $\widehat{R}^{ij}_{\varepsilon,\alpha}(x, A)$ we obtain

$$\widetilde{R}^{ij}_\varepsilon(x, A \times [0, \infty)) = \widetilde{R}^{ij}_{\varepsilon,1}(x, A), \qquad A \in \mathcal{B}.$$

This together with (41) and Lemma 1 yields

$$\sup_{x \in D} \sum_{j \neq i} \widetilde{R}^{ij}_\varepsilon(x, D \times [0, \infty)) \leq \frac{-c_{ii}}{1 - c_{ii}}$$

for all sufficiently small $\varepsilon > 0$. Putting

$$r = \sup_i \frac{-uc_{ii}}{1 - uc_{ii}}$$

we prove condition (18) with $\widetilde{R}^{ij}_\varepsilon$ instead of R^{ij}_ε.

Assuming that conditions (17)–(19) and (47) hold, we get

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon m} [\mathbb{P}_{\pi_D, i}\{e^{-\varepsilon\tau}, \xi_\varepsilon(\tau) = i\} - 1] = c_{ii} - 1.$$

Applying the part of Theorem 1 already proved to the function $\widetilde{f}^i_\varepsilon(x, t)$, we find that

$$\lim_{\substack{\varepsilon \to 0 \\ t \to \infty \\ \varepsilon t \to u}} \widetilde{f}^i_\varepsilon(x, t) = p^{il}(u)\langle \pi, \varphi \rangle \tag{66}$$

if relations (17)–(19) and (47) hold, where

$$\left\| p^{il}(u) \right\|_{i,l=1}^\infty = \exp\{u(C - I)\} = e^{-u} \exp\{uC\}.$$

On the other hand,

$$\widetilde{f}^i_\varepsilon(x, t) = e^{-\varepsilon t} f^i_\varepsilon(x, t).$$

Comparing this result with equality (66) we complete the proof of the theorem in the general case.

The proof of Theorem 4 follows the same lines; thus we omit it. In the proof one should apply the following result.

Theorem 2. *Assume that conditions* (49)–(51) *hold and*

a) *the step of the kernel $G(x, dy \times dt)$ is equal to one and*

$$G_\varepsilon(x, E \times [0, \infty)) = \sum_{k=0}^\infty G_\varepsilon(x, E \times \{k\})$$

for all $x \in E$ and $n \geq 1$;

b) *$\mathcal{B}$-measurable functions $g_\varepsilon(x, k)$, $\varepsilon > 0$, $k \geq 1$, are such that*

$$\sup_{\varepsilon > 0} \sup_{x \in E} \sum_{k \geq N} g_\varepsilon(x, k) \underset{N \to \infty}{\longrightarrow} 0;$$

c) *there exists a sequence g_k such that*

$$\sum_{k \geq 0} \left| g_k - \int_E \pi(dy) g_\varepsilon(x, k) \right| \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Then

$$\lim_{\substack{\varepsilon \to 0 \\ k \to \infty \\ \gamma_\varepsilon k \to c}} U_\varepsilon * g_\varepsilon(x, k) = e^{-c} \frac{1}{m} \sum_{k \geq 0} g_k.$$

Theorem 3. *Assume conditions* (1), (2), (11), *and* (17)–(19). *Let the parameter t vary in the set* $\{0,1,\dots\}$.

Then

$$\left\{\mathbb{P}_{x,i}[\varphi(X(t)),\xi_\varepsilon(t)=j]-p_{ij}(u)\int_E\pi(dy)\varphi(y)\right\}\underset{\substack{\varepsilon\to0\\ t\to\infty\\ \varepsilon t\to u}}{\longrightarrow}0$$

for all $u\ge0$, $i,j\in I$, $x\in E$, and all $\mathcal{B}$-measurable bounded functions $\varphi(y)$, where $p_{ij}(u)$ is the entry (i,j) of the matrix e^{uC}, $C=\|c_{ij}\|_{i,j=1}^\infty$.

3. Concluding remarks

Remark 1. Condition (16) holds if

$$\sup_{\varepsilon>0}\sup_{x\in E}\mathbb{P}_{x,i}\{\xi_\varepsilon(t)\ne i\}\underset{t\to0}{\longrightarrow}0$$

for all $i\in I$.

Remark 2. We assume condition (16) in order to apply the theorem to the functions $g_\varepsilon^i(x,t)$. Let $\Psi(y,j)$ be a bounded function of two arguments $y\in E$ and $j\in I$ that is continuous in $y\in E$ for all $i\in I$. Then the functions

$$g_\varepsilon^i(x,t)=\mathbb{P}_{x,i}\{\Psi(X(t),\xi_\varepsilon(t)),t<\tau\}=\mathbb{P}_{x,i}\{\Psi(X(t),\xi_\varepsilon(t)),t<(\tau\wedge\zeta_\varepsilon)\}$$

satisfy the assumptions of the theorem if the integer-valued stochastic process

$$\Psi(X(t),\xi_\varepsilon(t)),\qquad t\ge0,$$

is stochastically continuous uniformly in $\varepsilon>0$, that is, if

$$\sup_{\varepsilon>0}\mathbb{P}_{x,i}|\Psi(X(t),\xi_\varepsilon(t))-\Psi(X(s),\xi_\varepsilon(s))|\underset{s\to t}{\longrightarrow}0 \tag{67}$$

for all $x\in E$, $i\in I$, and $t\ge0$.

This remark implies the following result.

Corollary 1. *If conditions* (2) *and* (67) *hold, then*

$$\lim_{\substack{\varepsilon\to0\\ t\to\infty\\ \varepsilon t\to u}}\left\{\mathbb{P}_{x,i}\Psi(X(t),\xi_\varepsilon(t))-\sum_{j=1}^\infty p_{ij}(u)\int_E\Psi(y,j)\,\pi(dy)\right\}=0$$

for all $x\in E$, $i\in I$, and $u\ge0$.

Setting $\Psi(y,j)\equiv1$ we obtain the following result.

Corollary 2. *If condition* (2) *holds, then*

$$\lim_{\substack{\varepsilon\to0\\ t\to\infty\\ \varepsilon t\to u}}\mathbb{P}_{x,i}\{\varepsilon\zeta_\varepsilon\ge u\}-\sum_{j=1}^\infty p_{ij}(u)\}=0$$

for all $x\in E$, $i\in I$, and $u>0$.

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