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EXISTENCE OF A LIMIT DISTRIBUTION OF A SOLUTION OF A LINEAR INHOMOGENEOUS STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. We find conditions for the existence of a limit distribution (as $t \to \infty$) of a vector process ξ defined in $\mathbb{R}_+$ and determined by an inhomogeneous stochastic differential equation $\xi(t) = \xi(0) - \xi \circ \alpha + f * \nu + g * \mu$, where α is a nonrandom continuous increasing function, ν and μ are independent Poisson and centered Poisson measures, respectively.

1. Introduction

Let $\mathcal{E}$ be a σ-ring in a certain set Θ, α a continuous increasing real-valued function defined in $\mathbb{R}_+$, f and g measurable $\mathbb{R}^d$-valued functions defined in $\mathbb{R}_+ \times \Theta$, and $\xi(0)$ an $\mathbb{R}^d$-valued random variable. Further let ν and υ be independent Poisson measures in $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$ with compensators $\widetilde{\nu}$ and $\widetilde{\upsilon}$, respectively. We also assume that ν and υ are independent of $\xi(0)$. Put $\mu = \upsilon - \widetilde{\upsilon}$ and define a stochastic process ξ by the following equation:

$$\xi(t) = \xi(0) - \int_0^t \xi(s)\, d\alpha(s) + \int_0^t \int_\Theta f(s,\theta)\, \nu(ds, d\theta) + \int_0^t \int_\Theta g(s,\theta)\, \mu(ds, d\theta)$$

or, using the notation of [1, 2],

$$\xi = \xi(0) - \xi \circ \alpha + f * \nu + g * \mu. \tag{1}$$

We study conditions for the existence of the limit distribution of $\xi(t)$ are $t \to \infty$.

Necessary and sufficient conditions for the existence of the limit distribution of ξ are found in [3]–[5] for the case of $f(s,\theta) = \theta$, $g = 0$, $\alpha(s) = cs$, and $\widetilde{\nu}(ds, d\theta) = \lambda\, ds\, \Pi(d\theta)$. Ergodic properties of a vector Ornstein–Uhlenbeck process governed by a generalized Lévy process (that is, by a process with independent increments and continuous trajectories) are studied in Proposition 2.2 of [6] (also see Theorems 4.1 and 4.2 of [8]) and [7, Theorem 2.8]. It is assumed in all the papers mentioned above that the governing process is homogeneous in time.

The model studied in this paper includes the integrals with respect to two Poisson measures (one of them is centered) and both measures are inhomogeneous in time.

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2. Main results

In what follows $z \in \mathbb{R}^d$ denotes a row vector, while all others vectors are column vectors; $\int$ means $\int_\Theta$. We also put $\mathsf{E}^0 = \mathsf{E}[\cdots|\xi(0)]$, $\varphi(s,\theta,z) = e^{izf(s,\theta)} - 1$,

$$\gamma(s,\theta,z) = e^{izg(s,\theta)} - izg(s,\theta) - 1,$$

$K = \varphi * \widetilde{\nu} + \gamma * \widetilde{\upsilon}$, $\eta(t,z) = e^{iz\xi(t)}$, $X(t,z) = \mathsf{E}^0\,\eta(t,z)$, $\eta(t) = \eta(t,\cdot)$, and $X(t) = X(t,\cdot)$.

Assume that

$$\int_0^t\!\!\int |f(s,\theta)|\,\nu(ds,d\theta) < \infty \quad \text{a.s.,} \tag{2}$$

$$\int_0^t\!\!\int |f(s,\theta)|\,\mathrm{I}\{|f|<\varepsilon\}\,\widetilde{\nu}(ds,d\theta) < \infty, \tag{3}$$

$$\int_0^t\!\!\int |g(s,\theta)|^2\,\mathrm{I}\{|g|<\varepsilon\}\,\widetilde{\upsilon}(ds,d\theta) < \infty, \tag{4}$$

$$\int_0^t\!\!\int |g(s,\theta)|\,\mathrm{I}\{|g|\ge\varepsilon\}\,\widetilde{\upsilon}(ds,d\theta) < \infty \tag{5}$$

for all $t > 0$ and $\varepsilon > 0$.

According to the definition of a Poisson measure [1], the measures $\widetilde{\nu}$ and $\widetilde{\upsilon}$ are σ-finite, and thus (3) and (4) are not trivial restrictions, indeed.

Theorem 1. *Let conditions* (2)–(5) *hold. Assume that, for all z, the function $K(\cdot,z)$ has a continuous derivative $h(\cdot,z)$ and that, for all t, the function $h(t,\cdot)$ is continuously differentiable. Then*

$$X(t,z) = X\left(0, ze^{-\alpha(t)}\right)\exp\left\{\int_0^t h\left(s, ze^{\alpha(s)-\alpha(t)}\right)\,ds\right\}. \tag{6}$$

Proof. First we prove the theorem under certain auxiliary assumptions (see steps 1° and 2° below) and then consider the general case (see steps 3° and 4°).

1°. Assume that the function α is continuously differentiable and that f and g are bounded. Let $U = f * \nu + g * \mu$. Since α is nonrandom and U does not depend on $\xi(0)$, we get from equality (1),

$$\mathsf{E}^0\,|\xi(t)| \le |\xi(0)| + \mathsf{E}\,|U(t)| + \int_0^t \mathsf{E}^0\,|\xi(s)|\,d\alpha(s). \tag{7}$$

The boundedness of the function f together with condition (3) where ε is an arbitrary number implies that

$$|f| * \widetilde{\nu}(t) < \infty \tag{8}$$

for all t. Similarly, the boundedness of the function g together with condition (4) yields for all t,

$$|g|^2 * \widetilde{\upsilon}(t) < \infty. \tag{9}$$

Since the functions f, g, $\widetilde{\nu}$, and $\widetilde{\upsilon}$ are nonrandom, $\mathsf{E}\,|f| * \nu = |f| * \widetilde{\nu}$ by Fubini's theorem and $\mathsf{E}\,|g * \mu|^2 \le |g|^2 * \widetilde{\upsilon}$ by Theorem 3.5.1 of [2]. This together with (8) and (9) implies $\mathsf{E}\,|U(t)| < \infty$, whence

$$\mathsf{E}^0\,|\xi(t)| < \infty \tag{10}$$

by (7) and the Gronwall–Bellman lemma.

2°. Let $r = |z|$ and $a = z/r$. Given a, let $Y(t,r) = X(t,ra)$. It is clear that

$$\frac{\partial\eta}{\partial r} = ia\xi e^{ira\xi}, \tag{11}$$

whence $|\partial\eta/\partial r| \le |\xi|$. This together with (10) yields

$$\mathsf{E}^0\left(\frac{\partial\eta}{\partial r}\circ\alpha\right) = \left(\mathsf{E}^0\,\frac{\partial\eta}{\partial r}\right)\circ\alpha$$

by Fubini's theorem and

$$\mathsf{E}^0\,\frac{\partial\eta}{\partial r} = \frac{\partial\,\mathsf{E}^0\,\eta}{\partial r} \equiv \frac{\partial Y}{\partial r}$$

by the theorem on the differentiation of integrals [9]. Comparing these equalities with (11) we conclude that

$$r\frac{\partial Y}{\partial r} = iz\,\mathsf{E}^0\,\xi\eta. \tag{12}$$

By Itô's formula [1, Theorem 3.4.6] we obtain from (1),

$$\eta = \eta(0) - i(z\xi\eta)\circ\alpha + (\varphi\eta^-)*\nu + \left(\left(e^{izg}-1\right)\eta^-\right)*\mu + (\gamma\eta^-)*\widetilde{\upsilon}.$$

Considering the conditional expectation with respect to $\xi(0)$ of both sides of the latter equality and taking into account that the functions f and g are nonrandom and that $K(\cdot,z)$ is continuous, we get

$$Y = \eta(0) - iz\left(\mathsf{E}^0\,\xi\eta\right)\circ\alpha + Y\circ K.$$

In view of (12), this equality is equivalent to the following Cauchy problem:

$$\frac{\partial Y}{\partial t} = -r\dot{\alpha}\frac{\partial Y}{\partial r} + hY, \qquad Y(0,r) = e^{iz\xi(0)}, \tag{13}$$

since α and $K(\cdot,z)$ are continuously differentiable. For continuous $h(\cdot,z)$ and continuously differentiable $h(t,\cdot)$, the solution of the latter Cauchy problem is given by

$$Y(t,r) = Y\left(0, re^{-\alpha(t)}\right)\exp\left\{\int_0^t h\left(s, rae^{\alpha(s)-\alpha(t)}\right)ds\right\},$$

which is equivalent to (6).

3°. Consider a general nonnegative function $q\in C^1(\mathbb{R}_+)$ such that $\int_0^\infty q(t)\,dt = 1$ and $q(t)=0$ for $|t|>1$. Let $q_n(t) = nq(nt)$ and

$$\alpha_n(t) = \int_0^\infty q_n(t-s)\alpha(s)\,ds \equiv \int_0^\infty \alpha(t-s)q_n(s)\,ds.$$

Note that $q_n\ge 0$, $q_n(t)=0$ for $|t|>\frac{1}{n}$, and $\int_0^\infty q_n(s)\,ds = 1$. It is clear that

$$\lim_{n\to\infty}\sup_{s\le t}|\alpha_n(s)-\alpha(s)| = 0 \tag{14}$$

for all t. Moreover $\dot{\alpha}_n(t) = \int_0^t \dot{q}_n(t-s)\alpha(s)\,ds$, and thus α_n is a continuously differentiable function.

Now let ξ_n be a solution of the equation $\xi_n = \xi(0) - \xi_n\circ\alpha_n + f*\nu + g*\mu$. The case of the theorem proved above shows that the corresponding function X_n is given by (6). We prove that

$$\sup_{s\le t}|\xi_n(s)-\xi(s)| \xrightarrow{\mathsf{P}} 0. \tag{15}$$

This relation implies the convergence of η_n to η and, as a consequence, the convergence of X_n to X.

Denote $\zeta = \xi e^\alpha$ and $\zeta_n = \xi_n e^{\alpha_n}$. The integration by parts formula ([1, §3.4], [2, §2.3]) implies

$$\zeta = \xi(0) + e^\alpha\circ\xi + \xi\circ e^\alpha + [\xi, e^\alpha].$$

Since the function e^α is continuous and has a bounded variation on every interval, we have $[e^\alpha]=0$. Moreover, the processes ξ and ξ_n, both being semimartingales, have a finite square variation on every interval ([2, Theorem 2.1.3]). Thus $[\xi, e^{\alpha}] = 0 = [\xi_n, e^{\alpha}]$. Therefore

$$\zeta = \xi(0) + (e^{\alpha} f) * \nu + (e^{\alpha} g) * \mu. \tag{16}$$

A similar formula holds for ζ_n, too. This implies that

$$\begin{aligned} \mathsf{P}\left\{\sup_{s\le t} |\zeta_n(s) - \zeta(s)| \ge \varepsilon\right\} \le \mathsf{P}\left\{\sup_{s\le t} |((e^{\alpha} - e^{\alpha_n}) f) * \nu(s)| \ge \frac{\varepsilon}{2}\right\} \\ + \mathsf{P}\left\{\sup_{s\le t} |((e^{\alpha} - e^{\alpha_n}) g) * \mu(s)| \ge \frac{\varepsilon}{2}\right\} \end{aligned} \tag{17}$$

for all $\varepsilon > 0$.

By Lenglart–Rebolledo's inequality ([2, §1.9], [1, Theorem 3.3.7])

$$\mathsf{P}\left\{\sup_{s\le t} |((e^{\alpha} - e^{\alpha_n}) g) * \mu| \ge \frac{\varepsilon}{2}\right\} \le \frac{4\delta}{\varepsilon^2} + \mathsf{P}\left\{\left((e^{\alpha} - e^{\alpha_n})^2 |g|^2\right) * \widetilde{\upsilon} \ge \delta\right\}$$

for all $\varepsilon > 0$ and $\delta > 0$. Now (15) follows from (17), (14), (2), and (9) by the Lebesgue dominated convergence theorem.

4°. Now we consider general functions f and g. Let

$$f^N = f\,\mathrm{I}\{|f| \le N\}, \qquad g^N = f\,\mathrm{I}\{|g| \le N\},$$

and $\zeta^N = \xi^N e^{\alpha}$, where ξ^N is a solution of the equation

$$\xi^N = \xi(0) - \xi^N \circ \alpha + f^N * \nu + g^N * \mu.$$

It follows from (16) by using a similar notation for ζ^N that

$$\left|\zeta - \zeta^N\right| \le \left|\left(e^{\alpha}\left(f - f^N\right)\right) * \nu\right| + \left|\left(e^{\alpha}\left(g - g^N\right)\right) * \mu\right|.$$

It is obvious that $|f - f^N| \le |f|$; thus condition (2) and the pointwise convergence of (f^N) to f proves

$$\lim_{N\to\infty} \left|e^{\alpha}\left(f - f^N\right)\right| * \nu = 0$$

by the Lebesgue dominated convergence theorem. By condition (5),

$$\left|g - g^N\right| * \widetilde{\upsilon}(t) < \infty,$$

whence $|g - g^N| * \upsilon(t) < \infty$. Thus $(g - g^N) * \mu = (g - g^N) * \upsilon - (g - g^N) * \widetilde{\upsilon}$. Then

$$\lim_{N\to\infty} \left|e^{\alpha}\left(g - g^N\right)\right| * \upsilon = 0,$$
$$\lim_{N\to\infty} \left|e^{\alpha}\left(g - g^N\right)\right| * \widetilde{\upsilon} = 0$$

by the Lebesgue dominated convergence theorem, since the functions (g^N) pointwise converge to g. □

Let $\alpha^{-1}(t) = \sup\{s\colon \alpha(s) \le t\}$ (we apply the same notation for other increasing functions),

$$R(t, z) = \int_0^t h\left(s, z e^{\alpha(s) - \alpha(t)}\right) ds,$$

and $Q(t, z) = R(\alpha^{-1}(t), z)$. Assume that

$$\lim_{t\to\infty} \alpha(t) = \infty. \tag{18}$$

This, in particular, means that $\alpha^{-1}(t) < \infty$ for all t. Lévy's continuity theorem claims that the existence of the limit distribution of $\xi(t)$ as $t \to \infty$ is equivalent to the pointwise convergence of $X(t, \cdot)$ to some function that is continuous at zero. Equality (6) shows that, under the assumptions of Theorem 1, the latter property is equivalent to the convergence of $R(t, \cdot)$ to some function $R(\cdot)$ such that $\lim_{z\to 0} R(z) = 0$.

Consider the following condition.

Condition L. The limit $R(z) = \lim_{t\to\infty} Q(t,z)$ exists.

Assume also that

$$\widetilde{\nu}(ds, d\theta) = \Pi(s, d\theta)\, d\Lambda_1(s), \tag{19}$$

$$\widetilde{\upsilon}(ds, d\theta) = \Upsilon(s, d\theta)\, d\Lambda_2(s), \tag{20}$$

where Λ_1 and Λ_2 are nonrandom continuous increasing functions.

Remark 1. It is clear that expansions (19) and (20) are not unique, but any choice of the expansions is suitable for our goals. In what follows we fix arbitrary expansions among (19)–(20).

Denote

$$Q_1(t,z) = \int_0^t d\Lambda_1\left(\alpha^{-1}(s)\right) \int \varphi\left(\alpha^{-1}(s), \theta, ze^{s-t}\right) \Pi\left(\alpha^{-1}(s), d\theta\right),$$

$$Q_2(t,z) = \int_0^t d\Lambda_2\left(\alpha^{-1}(s)\right) \int \gamma\left(\alpha^{-1}(s), \theta, ze^{s-t}\right) \Upsilon\left(\alpha^{-1}(s), d\theta\right),$$

$$Q_j^*(p,z) = \int_0^\infty e^{-pt} Q_j(t,z)\, dt, \qquad \beta_j(s) = \exp\left\{-\alpha\left(\Lambda_j^{-1}(s)\right)\right\}.$$

Then $Q = Q_1 + Q_2$.

Assume for a moment that, for some $p \ge 0$ and z,

$$p\int_0^\infty e^{-p\tau}\, d\tau \int_0^\infty d\Lambda_1\left(\alpha^{-1}(s)\right) \left| \int \varphi\left(\alpha^{-1}(s), \theta, ze^{-\tau}\right) \Pi\left(\alpha^{-1}(s), d\theta\right) \right| < \infty, \tag{21}$$

$$p\int_0^\infty e^{-p\tau}\, d\tau \int_0^\infty d\Lambda_2\left(\alpha^{-1}(s)\right) \left| \int \gamma\left(\alpha^{-1}(s), \theta, ze^{-\tau}\right) \Upsilon\left(\alpha^{-1}(s), d\theta\right) \right| < \infty. \tag{22}$$

Lemma 1. *Let conditions* (2)–(5) *and* (19)–(22) *hold. Assume that the functions* Λ_j *are absolutely continuous. Then*

$$pQ_1^*(p,z) = p\int_0^\infty \beta_1(v)^p \left(\int \left(\int_0^r \left(\frac{u}{r}\right)^p \frac{\varphi\left(\Lambda_1^{-1}(v), \theta, ua\right)}{u}\, du \right) \Pi\left(\Lambda_1^{-1}(v), d\theta\right) \right) dv, \tag{23}$$

$$pQ_2^*(p,z) = p\int_0^\infty \beta_2(v)^p \left(\int \left(\int_0^r \left(\frac{u}{r}\right)^p \frac{\gamma\left(\Lambda_2^{-1}(v), \theta, ua\right)}{u}\, du \right) \Upsilon\left(\Lambda_2^{-1}(v), d\theta\right) \right) dv. \tag{24}$$

As will be shown in Corollary 1, the factor $(u/r)^p$ does not influence the asymptotic behavior of the whole expression as $p \to 0$.

Proof. First we prove the result under the assumption that the function α is strictly increasing. Note that the function α^{-1} is continuous in this case. According to conditions (21) and (22) and by Fubini's theorem, one can change the order of the integrals with respect to t and s in the corresponding expressions for Q_j^*. We have

$$pQ_1^*(p,z) = p\int_0^\infty d\Lambda_1\left(\alpha^{-1}(s)\right) \int \Pi\left(\alpha^{-1}(s), d\theta\right) \int_s^\infty e^{-pt} \varphi\left(\alpha^{-1}(s), \theta, ze^{s-t}\right) dt,$$

$$pQ_2^*(p,z) = p\int_0^\infty d\Lambda_2\left(\alpha^{-1}(s)\right) \int \Upsilon\left(\alpha^{-1}(s), d\theta\right) \int_s^\infty e^{-pt} \gamma\left(\alpha^{-1}(s), \theta, ze^{s-t}\right) dt.$$

Changing the variables $t = s + \ln(r/u)$ in the inner integral yields

$$pQ_1^*(p,z) = p\int_0^\infty e^{-ps}\left(\int\left(\int_0^r \left(\frac{u}{r}\right)^p \frac{\varphi\left(\alpha^{-1}(s),\theta,ua\right)}{u}\,du\right)\Pi\left(\alpha^{-1}(s),d\theta\right)\right)d\Lambda_1\left(\alpha^{-1}(s)\right),$$

$$pQ_2^*(p,z) = p\int_0^\infty e^{-ps}\left(\int\left(\int_0^r \left(\frac{u}{r}\right)^p \frac{\gamma\left(\alpha^{-1}(s),\theta,ua\right)}{u}\,du\right)\Upsilon\left(\alpha^{-1}(s),d\theta\right)\right)d\Lambda_2\left(\alpha^{-1}(s)\right).$$

Finally, changing the variables $\Lambda_1(\alpha^{-1}(s)) = v$ in the first integral and $\Lambda_2(\alpha^{-1}(s)) = v$ in the second one we prove (23) and (24).

Consider an arbitrary sequence of strictly increasing functions α_n approximating α. The functions β_n and φ_n are chosen in the same way with respect to β and φ.

It follows from (23) and a similar expression for Q_{1n}^* that

$$Q_{1n}^* - Q_1^* = \int_0^\infty dv\int_0^r \left(\frac{u}{r}\right)^p \frac{1}{u}\left(\int \beta_{1n}(v)^p\varphi_n\left(\Lambda_1^{-1}(v),\theta,ua\right)\Pi\left(\alpha_n^{-1}(s),d\theta\right) - \int \beta_1(v)^p\varphi\left(\Lambda_1^{-1}(v),\theta,ua\right)\Pi\left(\alpha^{-1}(s),d\theta\right)\right)du.$$

The sequence (α_n) converges pointwise to α, β_1 is continuous, and φ is continuous in z. Thus the Lebesgue dominated convergence theorem implies the pointwise convergence of the sequence

$$\left(\int \beta_{1n}(v)^p\varphi_n\left(\Lambda_1^{-1}(v),\theta,ua\right)\,\Pi\left(\alpha_n^{-1}(s),d\theta\right)\right)$$

to

$$\int \beta_1(v)^p\varphi\left(\Lambda_1^{-1}(v),\theta,ua\right)\Pi\left(\alpha^{-1}(s),d\theta\right).$$

By condition (21) and by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}|p(Q_1^* - Q_{1n}^*)| = 0.$$

The equality $\lim_{n\to\infty}|p(Q_2^* - Q_{2n}^*)| = 0$ is proved in the same fashion. □

Corollary 1. *Let all the assumptions of Lemma 1 hold. Then*

$$|Q_1^*(p,z)| \le \left|\int_0^\infty \beta_1(v)^p\,dv\int \Pi\left(\Lambda_1^{-1}(v),d\theta\right)\int_0^r \frac{\varphi\left(\Lambda_1^{-1}(v),\theta,ua\right)}{u}\,du\right|,$$

$$|Q_2^*(p,z)| \le \left|\int_0^\infty \beta_2(v)^p\,dv\int \Upsilon\left(\Lambda_2^{-1}(v),d\theta\right)\int_0^r \frac{\gamma\left(\Lambda_2^{-1}(v),\theta,ua\right)}{u}\,du\right|.$$

Consider the following condition: for some $c > 0$,

$$\sup_{s\ge 0}\int \ln\left|f(\Lambda_1^{-1}(s),\theta)\right|\mathrm{I}\{|f| > c\}\Pi\left(\Lambda_1^{-1}(s),d\theta\right) < \infty, \tag{25}$$

$$\sup_{s\ge 0}\int \left|f(\Lambda_1^{-1}(s),\theta)\right|\mathrm{I}\{|f| \le c\}\Pi\left(\Lambda_1^{-1}(s),d\theta\right) < \infty, \tag{26}$$

$$\sup_{s\ge 0}\int \left|g(\Lambda_2^{-1}(s),\theta)\right|\mathrm{I}\{|g| > c\}\Upsilon\left(\Lambda_2^{-1}(s),d\theta\right) < \infty, \tag{27}$$

$$\sup_{s\ge 0}\int \left|g(\Lambda_2^{-1}(s),\theta)\right|^2\mathrm{I}\{|g| \le c\}\Upsilon\left(\Lambda_2^{-1}(s),d\theta\right) < \infty. \tag{28}$$

Remark 2. Conditions (25)–(28) are more restrictive than conditions (3)–(5).

The following result follows from Theorem 1 and Lemma 1.

Theorem 2. *Let conditions* (2), (18), *and* **L** *hold. Assume that expansions* (19) *and* (20) *exist such that* (25)–(28) *hold. Let the limits*

$$\varkappa_j \equiv \lim_{p\to 0} p \int_0^\infty \beta_j(s)^p \, ds$$

exist and the functions Λ_j *be absolutely continuous. Then* $\xi(t)$ *has a limit distribution as* $t \to \infty$.

Remark 3. Condition **L** is not easy to check. Lemma 2 below reduces this condition to simpler ones.

Proof. 1°. First we assume that conditions (21) and (22) hold. Then Lemma 1 is satisfied and, according to a Tauberian theorem, one needs to show that

$$\lim_{r\to 0} \varlimsup_{p\to 0} pQ_j^*(p,z) = 0, \qquad j = 1,2. \tag{29}$$

Put

$$G_1(s,r) = \int \Pi\left(\Lambda_1^{-1}(s), d\theta\right) \int_0^r \frac{\varphi\left(\Lambda_1^{-1}(s), \theta, ua\right)}{u} \, du,$$

$$G_2(s,r) = \int \Upsilon\left(\Lambda_2^{-1}(s), d\theta\right) \int_0^r \frac{\gamma\left(\Lambda_2^{-1}(s), \theta, ua\right)}{u} \, du.$$

We check that (29) follows from

$$\lim_{r\to 0} \varlimsup_{t\to\infty} |G_j(t,r)| = 0, \qquad j = 1,2. \tag{30}$$

Fix an arbitrary $\varepsilon > 0$ and choose T_ε such that

$$|G_j(s,r)| \le \varlimsup_{t\to\infty} |G_j(t,r)| + \varepsilon$$

for all $s > T_\varepsilon$. Then

$$\int_0^\infty p\beta_j^p(s)|G_j(s,r)| \, ds \le \int_0^T p\beta_j^p(s)|G_j(s,r)| \, ds + \left(\varlimsup_{t\to\infty} |G_j(t,r)| + \varepsilon\right) p \int_0^\infty \beta_j^p(s) \, ds$$

for $T > T_\varepsilon$. Conditions (21) and (22) and the Lebesgue dominated convergence theorem prove the convergence to zero of the first term on the right hand side of the latter inequality. Since ε is arbitrary,

$$\varlimsup_{p\to 0} p \int_0^\infty p\beta_j^p(s)|G_j(s,r)| \, ds \le \varkappa_j \varlimsup_{t\to\infty} |G_j(t,r)|. \tag{31}$$

2°. We prove (30) for $j = 1$. For arbitrary $A \in \mathcal{B}(\mathbb{R})$ and $s \in \mathbb{R}_+$, put

$$\overline{\Pi}(s,A) = \Pi\left(\Lambda_1^{-1}(s), \left\{\theta\colon af\left(\Lambda_1^{-1}(s), \theta\right) \in A\right\}\right).$$

Then $G_1(s,r) = J_1(s,r) + J_2(s,r)$, where

$$J_1(s,r) = \int_{|x|\le c} \overline{\Pi}(s,dx) \int_0^r \frac{e^{iux} - 1}{u} \, du$$

and

$$J_2(s,r) = \int_{|x|> c} \overline{\Pi}(s,dx) \int_0^r \frac{e^{iux} - 1}{u} \, du.$$

We show that $\lim_{r\to 0} J_k(s,r) = 0$, $k = 1,2$.

The inequality

$$\left| \int_0^r \frac{e^{iux} - 1}{u} \, du \right| \le 2|x|r$$

yields the bound

$$|J_1(s,r)| \le 2r \int_{|x|\le c} |x|\,\overline{\Pi}(s,dx). \tag{32}$$

Changing the variables $u = t/x$ in the integral J_2 we get

$$|J_2(s,r)| \le \left|\int_{|x|>c} \overline{\Pi}(s,dx) \int_0^{rx} \frac{1-\cos t}{t}\,dt\right| + \left|\int_{|x|>c} \overline{\Pi}(s,dx) \int_0^{rx} \frac{\sin t}{t}\,dt\right|. \tag{33}$$

The uniform boundedness of the integral $\int_0^{rx} (\sin t)/t\,dt$ implies the convergence to zero of the second term on the right hand side of (33) as $r \to 0$.

We apply the equality

$$\int_0^{rx} \frac{1-\cos t}{t}\,dt = \int_0^{1\wedge rx} \frac{1-\cos t}{t}\,dt + \int_{1\wedge rx}^{rx} \frac{1-\cos t}{t}\,dt$$

to the first term on the right hand side of (33) and note that

$$\left|\int_{|x|>c} \overline{\Pi}(s,dx) \int_0^{1\wedge rx} \frac{1-\cos t}{t}\,dt\right| \le \left(\int_{|x|>c} (rx\wedge 1)\,\overline{\Pi}(s,dx)\right) \to 0$$

as $r \to 0$. Further we use the relations

$$\int_{1\wedge rx}^{rx} \frac{1-\cos t}{t}\,dt = \int_1^{1\vee rx} \frac{1-\cos t}{t}\,dt = \ln(1\vee rx) - \int_1^{1\vee rx} \frac{\cos t}{t}\,dt.$$

Now (30) follows from (32), (33), (25), (26), and the uniform boundedness of the integral $\int_1^{1\vee rx} (\cos t)/t\,dt$.

3°. Now let $j = 2$. Similarly to 2° we put

$$\overline{\Upsilon}(s,A) = \Upsilon\left(\Lambda_2^{-1}(s), \left\{\theta\colon ag\left(\Lambda_2^{-1}(s),\theta\right) \in A\right\}\right).$$

Then $G_2(s,r) = \Phi_1(s,r) + \Phi_2(s,r)$, where

$$\Phi_1(s,r) = \int_{|x|\le c} \overline{\Upsilon}(s,dx) \int_0^r \frac{e^{iux} - iux - 1}{u}\,du,$$

$$\Phi_2(s,r) = \int_{|x|>c} \overline{\Upsilon}(s,dx) \int_0^r \frac{e^{iux} - iux - 1}{u}\,du.$$

We show that

$$\lim_{r\to 0} \Phi_k(s,r) = 0, \qquad k = 1,2. \tag{34}$$

We start the proof of (34) with the case of $k = 1$. The inequality

$$\left|\int_0^r \frac{e^{iux} - iux - 1}{u}\,du\right| \le x^2 r^2$$

yields the bound

$$|\Phi_1(s,r)| \le r^2 \int_{|x|\le c} x^2\,\overline{\Upsilon}(s,dx). \tag{35}$$

For the case of $k = 2$, we use the inequality

$$\left|\int_0^r \frac{e^{iux} - iux - 1}{u}\,du\right| \le |x|\left(r^2 + 2r\right)$$

to obtain

$$|\Phi_2(s,r)| \le \left(r^2 + 2r\right) \int_{|x|>c} |x|\,\overline{\Upsilon}(s,dx).$$

This together with (35), (27), and (28) proves (34).

It follows from (30) and (31) that the expressions for $pQ_j^*(p,r)$ obtained after the change of order of the integrals (Lemma 1) are finite for sufficiently small p and r, whence we conclude that (21) and (22) hold. $\square$

Let $\Lambda_j(t) = t$ and $\alpha(t) = \lambda t$. Consider the following conditions.

Condition B1. There exist measurable nonrandom functions a and b defined on $\mathbb{R}_+ \times \Theta$ and σ-finite measures π and ρ defined on $\mathcal{E}$ such that $\Pi(t, d\theta) = a(t,\theta)\,\pi(d\theta)$ and $\Upsilon(t, d\theta) = b(t,\theta)\,\rho(d\theta)$.

Condition B2. For all θ,

$$\begin{aligned}
&\overline{\lim_{T\to\infty}} \sup_{t_1,t_2>2T} \int_0^T |f(t_1,\theta) - f(t_2,\theta)|\,du = 0,\\
&\overline{\lim_{T\to\infty}} \sup_{t_1,t_2>2T} \int_0^T |a(t_1,\theta) - a(t_2,\theta)|\,du = 0,\\
&\overline{\lim_{T\to\infty}} \sup_{t_1,t_2>2T} \int_0^T |g(t_1,\theta) - g(t_2,\theta)|\,du = 0,\\
&\overline{\lim_{T\to\infty}} \sup_{t_1,t_2>2T} \int_0^T |b(t_1,\theta) - b(t_2,\theta)|\,du = 0.
\end{aligned}$$

Condition B3. There exist finite functions Q_j and m_j such that

$$|f(t,\theta)| \le Q_1(\theta), \qquad |g(t,\theta)| \le Q_2(\theta), \qquad a(t,\theta) \le m_1(\theta), \qquad b(t,\theta) \le m_2(\theta)$$

for all $t \ge 0$.

Condition B4. The following integrals

$$\begin{aligned}
&\int_{Q_1\le c} Q_1(\theta) m_1(\theta)\,\pi(d\theta), \qquad \int_{Q_1>c} \ln Q_1(\theta) m_1(\theta)\,\pi(d\theta),\\
&\int_{Q_2\le c} Q_2(\theta)^2 m_2(\theta)\,\rho(d\theta), \qquad \int_{Q_2>c} Q_2(\theta) m_2(\theta)\,\rho(d\theta)
\end{aligned}$$

are finite.

Lemma 2. *Conditions* **B1**–**B4**, (2), (19), *and* (20) *imply Condition* **L**.

Proof. Write

$$R(t,z) = \int_0^t ds \left(\int \varphi\left(s,\theta, ze^{\lambda(s-t)}\right) \Pi(s, d\theta) + \int \gamma\left(s,\theta, ze^{\lambda(s-t)}\right) \Upsilon(s, d\theta) \right). \tag{36}$$

Put

$$R_1(t,z) = \int_0^t du \int \varphi\left(t-u,\theta, ze^{-\lambda u}\right) a(t-u,\theta)\,\pi(d\theta), \tag{37}$$

$$R_2(t,z) = \int_0^t du \int \gamma\left(t-u,\theta, ze^{-\lambda u}\right) b(t-u,\theta)\,\rho(d\theta). \tag{38}$$

Changing the variables $s = t - u$ in the integral on the right hand side of (36) we get $R = R_1 + R_2$.

First we prove that the limit of

$$V(t,\theta,z) = \int_0^t \varphi\left(t-u,\theta, ze^{-\lambda u}\right) a(t-u,\theta)\,du$$

exists as $t \to \infty$.

Consider $\operatorname{Im} V(t,\theta,z) = \int_0^t \sin\left(e^{-\lambda u} z f(t-u,\theta)\right) a(t-u,\theta)\, du$. For arbitrary T and $t_1, t_2 > 2T$, we have

$$\begin{aligned} \left|\operatorname{Im} V(t_1,\theta,z) - \operatorname{Im} V(t_2,\theta,z)\right| \le \sum_{j=1}^{2} &\left| \int_T^{t_j} a(t_j-u,\theta) \sin\left(e^{-\lambda u} z f(t_j-u,\theta)\right) du \right| \\ &+ \left| \int_0^T \Big[a(t_1-u,\theta)\sin\left(e^{-\lambda u} z f(t_1-u,\theta)\right) \right. \\ &\qquad \left. - a(t_2-u,\theta)\sin\left(e^{-\lambda u} z f(t_2-u,\theta)\right)\Big] du \right|. \end{aligned} \tag{39}$$

The first two terms on the right hand side of (39) obviously are less than

$$2|z| m_1(\theta) Q_1(\theta) \int_T^\infty e^{-\lambda u}\, du = \frac{2}{\lambda} |z| m_1(\theta) Q_1(\theta) e^{-\lambda T}.$$

The third term on the right hand side of (39) is bounded by

$$m_1(\theta) \int_0^T |f(t_1-u,\theta) - f(t_2-u,\theta)|\, du + Q_1(\theta) \int_0^T |a(t_1-u,\theta) - a(t_2-u,\theta)|\, du.$$

Taking into account Condition **B2** we get

$$\lim_{t_1,t_2\to\infty} |\operatorname{Im} V(t_1,\theta,z) - \operatorname{Im} V(t_2,\theta,z)| = 0. \tag{40}$$

Now we find a bound for $\operatorname{Im} R_1(t,z)$. Put

$$\Theta_1 = \{\theta \colon Q_1(\theta) \le c\}, \qquad \Theta_2 = \Theta \setminus \Theta_1, \qquad H_k(t,z) = \int_{\Theta_k} \operatorname{Im} V(t,\theta,z)\, \pi(d\theta).$$

Then $\operatorname{Im} R_1(t,z) = H_1(t,z) + H_2(t,z)$.

We have $|H_1(t,z)| \le \lambda^{-1} |z| \int_{\Theta_1} Q_1(\theta)\, \pi(d\theta)$. For the case of $H_2(t,z)$, we change the variable $v = \lambda u$ and represent the inner integral as a sum of two integrals, namely

$$\begin{aligned} H_2(t,z) \le \frac{1}{\lambda} \int_{\Theta_2} & m_1(\theta)\, \pi(d\theta) \\ &\times \left(\int_0^{\ln Q_1} \sin\left(e^{-v} z f(t-v/\lambda,\theta)\right)\, dv + \int_{\ln Q_1}^{\lambda t} \sin\left(z f(t-v/\lambda,\theta) e^{-v}\right)\, dv \right). \end{aligned}$$

This implies the bound

$$\begin{aligned} |H_2(t,z)| &\le \frac{1}{\lambda} \int_{\Theta_2} m_1(\theta)\, \pi(d\theta) \left(\int_0^{\ln Q_1} dv + \int_{\ln Q_1}^\infty |z| Q_1(\theta) e^{-v}\, dv \right) \\ &\le \frac{|z|+1}{\lambda} \int_{\Theta_2} (\ln Q_1(\theta)) m_1(\theta)\, \pi(d\theta). \end{aligned}$$

Now equality (40), Condition **B4**, and the Lebesgue dominated convergence theorem imply that the limit of $\operatorname{Im} R_1(t,z)$ exists as $t \to \infty$.

Note that

$$\operatorname{Re} V(t,\theta,z) = 2 \int_0^t \sin^2\left(e^{-\lambda u} z f(t-u,\theta)/2\right) a(t-u,\theta)\, du.$$

Reasoning in the same way as above we derive from the latter equality that

$$\lim_{t_1,t_2\to\infty} |\operatorname{Re} V(t_1,\theta,z) - \operatorname{Re} V(t_2,\theta,z)| = 0.$$

The existence of the limit as $t \to \infty$ for the function

$$\int_0^t \gamma\left(t-u, \theta, z e^{-\lambda u}\right) b(t-u, \theta)\, du$$

is proved in a similar manner. The bound for $R_2(t,z)$ is given by

$$\frac{1}{\lambda}\left(|z|^2 \int_{Q_2 \le c} Q_2(\theta)^2\, \rho(d\theta) + 2|z| \int_{Q_2 > c} Q_2(\theta)\, \rho(d\theta)\right). \qquad \square$$

The assumptions of Theorem 2 become simpler if f, g, Π, and Υ do not depend on s. The following assertion generalizes a result due to Zakusilo [5].

Corollary 2. *Let conditions* (2), (19), *and* (20) *hold. Assume that there exist monotone bounded derivatives* $\varkappa_j(t) = \left(\Lambda_j(\alpha^{-1}(t))\right)'$. *If the integrals*

$$\int_{|f|>c} \ln|f(\theta)|\, \Pi(d\theta), \qquad \int_{|f|\le c} |f(\theta)|\, \Pi(d\theta),$$
$$\int_{|g|>c} |g(\theta)|\, \Upsilon(d\theta), \qquad \int_{|g|\le c} |g(\theta)|^2\, \Upsilon(d\theta)$$

are finite for all positive c, then ξ has a limit distribution as $t \to \infty$. The logarithm of the characteristic function of the limit distribution is equal to

$$\psi(z) = \kappa_1 \int \Pi(d\theta) \int_0^{|z|} \frac{\varphi(\theta, uz/|z|)}{u}\, du + \kappa_2 \int \Upsilon(d\theta) \int_0^{|z|} \frac{\gamma(\theta, uz/|z|)}{u}\, du,$$

where $\kappa_j = \lim_{t\to\infty} \varkappa_j(t)$.

Proof. The assumptions of the corollary imply

$$Q_1(t,z) = \int_0^t \varkappa_1(s)\, ds \int \varphi\left(\theta, e^{s-t} z\right)\, \Pi(d\theta), \tag{41}$$

$$Q_2(t,z) = \int_0^t \varkappa_2(s)\, ds \int \gamma\left(\theta, e^{s-t} z\right)\, \Upsilon(d\theta). \tag{42}$$

Changing the variables $s = t + \ln(v/|z|)$ on the right hand sides of both equalities (41) and (42) we obtain

$$Q_1(t,z) = \int_{|z|e^{-t}}^{|z|} \varkappa_1\left(t + \ln\frac{v}{|z|}\right) dv \int \frac{\varphi(\theta, vz/|z|)}{v}\, \Pi(d\theta),$$
$$Q_2(t,z) = \int_{|z|e^{-t}}^{|z|} \varkappa_2\left(t + \ln\frac{v}{|z|}\right) dv \int \frac{\gamma(\theta, vz/|z|)}{v}\, \Upsilon(d\theta).$$

The two latter relations imply that the functions Q_j, $j=1,2$, are monotone in t, since so are the functions $\varkappa_j$, $j=1,2$. Therefore the classical Tauberian theorem gives

$$\lim_{t\to\infty} Q_j(t,z) = \lim_{p\to 0} p Q_j^*(p,z).$$

Note that conditions (25)–(28) hold.

Corollary 1 and Theorem 2 imply that

$$\psi(z) = \int \Pi(d\theta) \int_0^{|z|} \frac{\varphi(\theta, uz/|z|)}{u}\, du \lim_{p\to 0} p \int_0^\infty \beta_1(v)^p\, dv$$
$$+ \int \Upsilon(d\theta) \int_0^{|z|} \frac{\gamma(\theta, uz/|z|)}{u}\, du \lim_{p\to 0} p \int_0^\infty \beta_2(v)^p\, dv.$$

Since $\varkappa_j$ is monotone, we deduce by the classical Tauberian theorem that

$$\lim_{p\to 0} p\int_0^\infty \beta_j(t)^p\,dt = \lim_{p\to 0} p\int_0^\infty e^{-pt}\varkappa_j(t)\,dt = \kappa_j. \qquad \square$$

Example 1. The functions for which nontrivial κ_j exist can be found from the relation $\Lambda_j(t) = \lambda_j\alpha(t)$, which is a consequence of the equation $\Lambda_j(\alpha^{-1}(t)) = \lambda_j t$. For example, $\alpha(t) = \ln(t+1)$ and $\Lambda_j(t) = \lambda_j \ln(t+1)$ or $\alpha(t) = t^\delta$ and $\Lambda_j(t) = \lambda_j t^\delta$, $\delta > 0$.

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