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ON INVESTMENT AND MINIMIZATION OF SHORTFALL RISK FOR A DIFFUSION MODEL WITH JUMPS AND TWO INTEREST RATES VIA MARKET COMPLETION

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Abstract. This paper deals with the problems of investment and shortfall risk minimization in the framework of a two-factor diffusion model with jumps and with different credit and deposit rates. The optimal strategies are derived by means of auxiliary completions of the initial market.

1. Introduction

In well-known financial market models one considers a unique interest rate for both deposit and credit (see Elliott and Kopp [10], Karatzas and Shreve [14], Föllmer and Leukert [12], Nakano [20]). In reality, the credit rate is always higher than the deposit rate. Such a market constraint brings new difficulties in the problems of hedging, investing and shortfall risk minimization (see Bergman [4], Kane and Melnikov [13], Korn [15], Bart [3], and also Cvitanić and Karatzas [8], Cvitanić [6, 7], Föllmer and Kramkov [11], Karatzas and Shreve [14], Cvitanić, Pham and Touzi [9], Soner and Touzi [21] regarding other market constraints).

As in Korn [15], Kane and Melnikov [13], we use a methodology of completions to solve the investment problem with logarithmic utility function and a shortfall risk minimization in a financial market consisting of a diffusion model with jumps and two interest rates.

2. Description of the model and auxiliary results

Let $\{\Omega, \mathcal{F}, F = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\}$ be a standard stochastic basis. Assume there are two risky assets S^i, $i = 1, 2$, whose prices are described by the equations

$$dS^i_t = S^i_{t^-}\left(\mu^i\,dt + \sigma^i\,dW_t - \nu^i\,d\Pi_t\right), \qquad i = 1, 2. \tag{2.1}$$

Here W is a standard Wiener process and Π is a Poisson process with positive intensity λ. The filtration F is generated by the independent processes W and Π, $\mu^i \in \mathbf{R}$, $\sigma^i > 0$, $\nu^i < 1$.

We also assume that there are a deposit account B^1 and a credit account B^2 satisfying to

$$dB^i_t = B^i_t r^i\,dt, \qquad i = 1, 2. \tag{2.2}$$

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Denote by (B^1, B^2, S^1, S^2) the market described by the above assets; any non-negative $\mathcal{F}_T$-measurable random variable f_T is called a *contingent claim* with the maturity time T. In the (B^1, B^2, S^1, S^2)-market, a *portfolio* $\pi = (\beta^1, \beta^2, \gamma^1, \gamma^2)$ is an $\mathcal{F}_t$-predictable process, where we denote respectively by β^i and γ^i the number of units of the ith bond and ith stock in the wealth. The value of the portfolio π is given by

$$V_t = \beta_t^1 B_t^1 + \beta_t^2 B_t^2 + \gamma_t^1 S_t^1 + \gamma_t^2 S_t^2 \quad \text{a.s.} \tag{2.3}$$

A portfolio π is said to be *self-financing* if it satisfies the following property:

$$dV_t = \beta_t^1\, dB_t^1 + \beta_t^2\, dB_t^2 + \gamma_t^1\, dS_t^1 + \gamma_t^2\, dS_t^2 \quad \text{a.s.} \tag{2.4}$$

Such a portfolio will be said to be *admissible* if

$$V_t \geq 0 \quad \text{a.s.} \quad \text{for all } t \geq 0.$$

The set of admissible portfolios with initial capital x is denoted by $\mathcal{A}(x)$.

The investor's position is identified with the *wealth process* $X_t \geq 0$. Here, the process X is the capital of a self-financing and admissible portfolio. Under the above conditions and (2.1)–(2.2), the wealth process X_t has the form

$$dX_t = X_{t^-}\left[\left(1 - \alpha_t^1 - \alpha_t^2\right)^+ r^1\, dt - \left(1 - \alpha_t^1 - \alpha_t^2\right)^- r^2\, dt + \alpha_t^1 \frac{dS_t^1}{S_{t^-}^1} + \alpha_t^2 \frac{dS_t^2}{S_{t^-}^2}\right]. \tag{2.5}$$

Here $\alpha_t^i = \gamma_t^i S_{t^-}^i / X_{t^-}^i$, $i = 1, 2$, is the proportion of cash invested on the ith stock in the wealth process, and $a^+ = \max\{0, a\}$, $a^- = -\min\{0, a\}$.

Note that throughout the paper α will also be called a strategy. Further, the notation X^π or X^α refers to the same wealth process.

Let us consider the special case where the financial market has the same deposit and credit rates: $r^1 = r^2 = r$, and hence, $B^1 = B^2 = B$. In the framework of such a (B, S^1, S^2)-market, the capital generated by an admissible portfolio process

$$\pi := \left(\beta, \gamma^1, \gamma^2\right)$$

is described by

$$X_t = \beta_t B_t + \gamma_t^1 S_t^1 + \gamma_t^2 S_t^2 \quad \text{a.s.}, \tag{2.6}$$

$$\frac{dX_t}{X_{t^-}} = \left[\left(1 - \alpha_t^1 - \alpha_t^2\right) r\, dt + \alpha_t^1 \frac{dS_t^1}{S_{t^-}^1} + \alpha_t^2 \frac{dS_t^2}{S_{t^-}^2}\right]. \tag{2.7}$$

If $\sigma^1 \nu^2 \neq \sigma^2 \nu^1$, then the parameters

$$\begin{aligned} \phi &= -\frac{\left(\mu^1 - r\right)\nu^2 - \left(\mu^2 - r\right)\nu^1}{\sigma^1 \nu^2 - \sigma^2 \nu^1}, \\ \psi &= \frac{\left(\mu^1 - r\right)\sigma^2 - \left(\mu^2 - r\right)\sigma^1}{\sigma^2 \nu^1 - \sigma^1 \nu^2} \lambda^{-1} - 1 \end{aligned} \tag{2.8}$$

define (see Melnikov et al. [17]) a density Z of a unique martingale measure P^* in the (B, S^1, S^2)-market as a stochastic exponent

$$Z_t = \mathcal{E}_t(N) = \exp\left\{\phi W_t - \frac{\phi^2}{2} t + (\lambda - \lambda^*)t + (\ln \lambda^* - \ln \lambda)\Pi_t\right\}, \tag{2.9}$$

where $N_t = \phi W_t + \psi(\Pi_t - \lambda t)$. Under such a measure, the given Poisson process Π has intensity $\lambda^* = \lambda(1 + \psi)$ and $W_t^* = W_t - \phi t$ is a Wiener process.

We consider contingent claims of the form $f_T := f(S_T^1)$.

Let us now turn to the (B^1, B^2, S^1, S^2)-market.

3. Main results and pricing formulas

To study the investment and shortfall risk minimization problems in the framework of a (B^1, B^2, S^1, S^2)-market we define a variety of (B, S^1, S^2) (or (B^d, S^1, S^2))-markets with the interest rates $r = r^d = r^1 + d$, where $d = (d_t)$ is a predictable process such that $d_t \in [0, r^2 - r^1]$.

Proposition 3.1. *Let $d = (d_t)$ be a predictable process with values in the interval*

$$\left[0, r^2 - r^1\right].$$

Assume that $\alpha_d := \alpha = (\alpha^1, \alpha^2)$ generates the wealth processes $X^{\alpha,d}$ and X^α with initial capital x in the (B^d, S^1, S^2) and the (B^1, B^2, S^1, S^2)-markets respectively if further α_d satisfies

$$\left(r^2 - r^1 - d_t\right)\left(1 - \alpha_t^1 - \alpha_t^2\right)^- + d_t\left(1 - \alpha_t^1 - \alpha_t^2\right)^+ = 0. \tag{3.1}$$

Then the above wealth processes coincide.

Proof of Proposition 3.1. Let x be the initial capital associated to α in the (B^d, S^1, S^2)-market. If α satisfies (3.1), then we rewrite the latter equation as follows:

$$\begin{aligned}
&\left(r^2 - r^1 - d_t\right)\left(1 - \alpha_t^1 - \alpha_t^2\right)^- + d_t\left(1 - \alpha_t^1 - \alpha_t^2\right)^+ \\
&\quad = r^2\left(1 - \alpha_t^1 - \alpha_t^2\right)^- - \left(r^1 + d_t\right)\left(1 - \alpha_t^1 - \alpha_t^2\right)^- + d_t\left(1 - \alpha_t^1 - \alpha_t^2\right)^+ \\
&\quad = r^2\left(1 - \alpha_t^1 - \alpha_t^2\right)^- - r^d\left(1 - \alpha_t^1 - \alpha_t^2\right) - r^1\left(1 - \alpha_t^1 - \alpha_t^2\right)^+ = 0.
\end{aligned}$$

Hence,

$$r^d\left(1 - \alpha_t^1 - \alpha_t^2\right) = r^1\left(1 - \alpha_t^1 - \alpha_t^2\right)^+ - r^2\left(1 - \alpha_t^1 - \alpha_t^2\right)^-$$

and

$$\begin{aligned}
\frac{dX_t^{\alpha,d}}{X_{t^-}^{\alpha,d}} &= r^d\left(1 - \alpha_t^1 - \alpha_t^2\right)dt + \alpha_t^1\frac{dS_t^1}{S_{t^-}^1} + \alpha^2\frac{dS_t^2}{S_{t^-}^2} \\
&= \left(r^1\left(1 - \alpha_t^1 - \alpha_t^2\right)^+ - r^2\left(1 - \alpha_t^1 - \alpha_t^2\right)^-\right)dt + \alpha^1\frac{dS_t^1}{S_{t^-}^1} + \alpha^2\frac{dS_t^2}{S_{t^-}^2} \\
&= \frac{dX_t^\alpha}{X_{t^-}^\alpha}.
\end{aligned}$$

Therefore, the wealth processes $X_t^{\alpha,d}(x)$ and $X_t^\alpha(x)$ on respectively (B^d, S^1, S^2) and (B^1, B^2, S^1, S^2)-markets coincide. □

3.1. Investment problem.

In parallel to the hedging agent who wants to find the optimal strategy to hedge his claim f_T (see Kane and Melnikov [13]), an investing agent has to find the optimal strategy that allows him to maximize the expected utility of his terminal wealth. Assume a given utility function $U\colon \mathbf{R}_+ \to \mathbf{R}$ is concave, non-decreasing, continuously differentiable, and such that

$$\lim_{x\to\infty} U'(x) = 0,$$
$$\lim_{x\to 0} U'(x) = \infty.$$

Let $u(x)$ be the cost function in the (B^d, S^1, S^2)-market. The investment problem (see, for instance, Karatzas and Shreve [14], Melnikov et al. [17]) consists in finding

$$\begin{aligned}
u(x) &= \sup_{\pi\in SF} \mathsf{E}\left(U\left(X_T^{\pi,d}(x)\right)\right) = \mathsf{E}\left(U\left(X_T^{\pi^*,d}(x)\right)\right) \\
&= \sup_{Y\in\mathcal{X}} \mathsf{E}\big(U(Y_T(x))\big) = \mathsf{E}\big(U(Y_T^*(x))\big),
\end{aligned} \tag{3.2}$$

where $\chi = \{Y \text{ positive}\colon Y_t(x) = x + \int_0^t \gamma_u \, d\widetilde{X}_s\}$, γ is a predictable process and $\widetilde{X}$ is a P^* local martingale. To solve the problem, we introduce $V(y)$, the conjugate function of $U(x)$, whose relations to the latter are given by

$$V(y) = \sup_{x>0} U(x) - xy, \qquad y > 0, \tag{3.3}$$

$$U(x) = \inf_{y>0} V(y) + xy, \qquad x > 0. \tag{3.4}$$

We denote $I(x) = ((U')^{-1}(x)) = -V'(x)$, $y_0 = \inf\{y\colon v(y) < \infty\}$, and

$$x_0 = \lim_{y \to y_0} (-v'(y)).$$

The functions $v(y)$ and $u(x)$ are such that:

1) the function $u(x) < \infty$ is continuously differentiable for all $x > 0$, strictly concave on $(0, x_0)$, and the function $v(x) < \infty$ is continuously differentiable for all $y > 0$, strictly convex on (y_0, ∞) with

$$v(y) = \sup_{x>0} u(x) - xy, \qquad y > 0, \tag{3.5}$$

$$u(x) = \inf_{y>0} v(y) + xy, \qquad x > 0. \tag{3.6}$$

2) If $y = u'(x)$, where $x < x_0$ and $y < y_0$, then the optimal solution of (3.2) is

$$Y_T^*(x) = I(yZ_T). \tag{3.7}$$

We consider the case $U(x) = \ln(x)$ in the (B^d, S^1, S^2) financial market. Substituting $U(x)$ in (3.3), we derive $V(y) = -\ln(y) - 1$ and

$$\begin{aligned} v(y) = \mathsf{E}[V(yZ_T)] &= -\mathsf{E}[\ln(yZ_T)] - 1 = -\ln(y) - 1 - \mathsf{E}[\ln(Z_T)] \\ &= -\ln(y) - 1 + \frac{\phi^2}{2}T - (\ln(\lambda^*) - \ln(\lambda))\,\lambda T + (\lambda^* - \lambda)\,T. \end{aligned}$$

We substitute the above expression of $v(y)$ into equality (3.6) and find the price function

$$u(x) = \ln(x) + \frac{\phi^2}{2}T - (\ln(\lambda^*) - \ln(\lambda))\,\lambda T + (\lambda^* - \lambda)T. \tag{3.8}$$

Next, we derive the optimal proportions invested in the different assets involved. From equation (3.7), we know that

$$\begin{aligned} Y_T^*(x) = \frac{X_T^{\pi^*,d}}{B_T} &= I(yZ_T) = \frac{1}{yZ_T} = \frac{x}{Z_T} \\ &= x\exp\left\{-\phi W_T + \frac{\phi^2}{2}T - \Pi_T \ln\left(\frac{\lambda^*}{\lambda}\right) + (\lambda^* - \lambda)T\right\} \end{aligned} \tag{3.9}$$

and solving (2.7) for $\alpha_t = \alpha$ yields

$$\begin{aligned} \frac{X_T^{\pi,d}}{B_T^d} = x\exp\Bigg\{&\left(\alpha^1\left(\mu^1 - r_d\right) + \alpha^2\left(\mu^2 - r_d\right) - \frac{\left(\alpha^1\sigma^1 + \alpha^2\sigma^2\right)^2}{2}\right)T \\ &+ \left(\alpha^1\sigma^1 + \alpha^2\sigma^2\right)W_T + \ln\left(1 - \alpha^1\nu^1 - \alpha^2\nu^2\right)\Pi_T\Bigg\}. \end{aligned} \tag{3.10}$$

We identify expressions (3.9) and (3.10) and obtain the values for α^1 and α^2:

$$\alpha^1 = \frac{\phi\nu^2 - \sigma^2\left(\lambda/\lambda^* - 1\right)}{\nu^1\sigma^2 - \nu^2\sigma^1}, \qquad \alpha^2 = \frac{\phi\nu^1 - \sigma^1\left(\lambda/\lambda^* - 1\right)}{\nu^2\sigma^1 - \nu^1\sigma^2}. \tag{3.11}$$

In the (B^d, S^1, S^2)-market, the optimal proportions invested are α^1 on the first stock, α^2 on the second stock and the rest $(1 - \alpha^1 - \alpha^2)$ on the bank account.

Let us turn to the two-interest-rates financial market (B^1, B^2, S^1, S^2). The previous results lead to the following theorem.

Theorem 3.1. *Let the wealth processes $X_t^{\pi,d}(x)$ and $X_t^{\pi}(x)$ in the (B^d, S^1, S^2) and (B^1, B^2, S^1, S^2) financial markets satisfy* (2.5) *and* (2.7)*, respectively, and let relation* (3.1) *hold for the optimal proportions α_t of problem* (3.2) *in the (B^d, S^1, S^2)-market. Then considering a logarithmic utility function, we obtain in the (B^1, B^2, S^1, S^2)-market that*

1) *the price function $u(x)$ is given by* (3.8)*, and*
2) *the optimal proportions invested in the different assets are α^1 on S^1, α^2 on S^2, $(1-\alpha^1-\alpha^2)^+$ on the deposit account (if $(1-\alpha^1-\alpha^2) > 0$) and $(1-\alpha^1-\alpha^2)^-$ on the credit account (if $(1-\alpha^1-\alpha^2) < 0$).*

Proof. Let α^*, the optimal proportions in the (B^d, S^1, S^2)-market, satisfy (3.1); then α^* is optimal for the (B^1, B^2, S^1, S^2)-market.

For any strategy π, we have $X_t^{\pi}(x) \le X_t^{\pi,d}(x)$ and

$$\sup_{\pi\in SF} \mathsf{E}\left[U(X_T^{\pi}(x))\right] \le \sup_{\pi\in SF} \mathsf{E}\left[U\left(X_T^{\pi,d}(x)\right)\right] = \mathsf{E}\left[U\left(X_T^{\pi^*,d}(x)\right)\right] \overset{(3.1)}{=} \mathsf{E}\left[U\left(X_T^{\pi^*}(x)\right)\right].$$

Hence

$$\sup_{\pi\in SF} \mathsf{E}\left[U(X_T^{\pi}(x))\right] = \mathsf{E}\left[U\left(X_T^{\pi^*}(x)\right)\right] = \mathsf{E}\left[U\left(X_T^{\pi^*,d}(x)\right)\right] = u(x). \tag{3.12}$$

Regarding the optimal proportions in the (B^1, B^2, S^1, S^2)-market, we derive from (3.1) and (3.12) that

$$X_T^{\pi^*}(x) = X_T^{\pi^*,d}(x) = Y_T^{*,d}(x)e^{r^d T}. \tag{3.13}$$

Solving (2.5) for $X_T^{\pi^*}(x)$ and identifying the latter with $Y_T^{*,d}e^{r^d T}$, the optimal proportions invested in the (B^1, B^2, S^1, S^2)-market are α^1 on S^1 and α^2 on S^2. Since α^1 and α^2 are constants, we invest $(1-\alpha^1-\alpha^2)^+$ on B^1 if $(1-\alpha^1-\alpha^2) > 0$, and we invest $-(1-\alpha^1-\alpha^2)^-$ on B^2 if $(1-\alpha^1-\alpha^2) < 0$. □

Consider a particular case $\sigma^1 = 0$, $\nu^2 = 0$ and assume $(1-\alpha^1-\alpha^2) \ge 0$ (only the lending rate r^1 is applicable). Then, the market (B^d, S^1, S^2) is defined by

$$\begin{cases} dB_t = r^d B_t\, dt, & B_0 > 0, \\ dS_t^1 = S_t^1\left(\mu^1\, dt - \nu^1\, d\Pi_t\right), & S_0^1 > 0, \\ dS_t^2 = S_t^2\left(\mu^2\, dt + \sigma_2\, dW_t\right), & S_0^2 > 0. \end{cases}$$

From relation (2.8) we obtain that

$$\phi = -\frac{\mu^2 - r^1}{\sigma_2}, \qquad \lambda^* = \frac{\mu^1 - r^1}{\nu^1}. \tag{3.14}$$

Exploiting (3.14) and (3.11), we derive

$$\alpha^1 = \frac{\mu^1 - r^1 - \lambda\nu^1}{\nu^1(\mu^1 - r)}, \qquad \alpha^2 = \frac{\mu^2 - r^1}{(\sigma_2)^2}. \tag{3.15}$$

Note that α_1 and α_2 are the Merton points in a pure jump and pure diffusion model, respectively. In the setting of a two-interest-rates financial market with the above assumptions, the optimal proportions invested are defined by (3.15) for S^1 and S^2 correspondingly, and $(1-\alpha^1-\alpha^2)$ in B^1 and 0 in B^2.

3.2. **Minimization of a shortfall risk.** In the (B^d, S^1, S^2)-market, we found in Kane and Melnikov [13] the optimal strategy and initial capital required to hedge perfectly a claim $f(S_T^1)$. We also derived in the previous section the optimal investment strategy and terminal wealth of an expected utility maximization problem.

Now we consider an investor whose initial capital x is less than the required

$$E^{d,*}\left[e^{-r^d T} f_T\right].$$

In such a case, a perfect hedge is no longer possible, but one can minimize the risk of shortfall given the initial constraint on the cost:

$$u(x) = \inf_{\substack{\pi\in\mathcal{A},\\ x<E^{d,*}\left[f(S_T^1)e^{-r^d T}\right]}} \mathsf{E}\left[l_p\left(\left(f_T - X_T^{\pi,d}(x)\right)^+\right)\right]. \tag{3.16}$$

Note that $u(x)$ is different from that in Section 3.1. The loss function is $l_p(x) = x^p/p$ with $p > 1$, $\mathcal{A} = \left\{\pi\colon \mathsf{E}\left[\sup_{0\le t\le T}|X_t^\pi(0)|\right] < \infty\right\}$ and $f_T \in L^{p+\varepsilon}(\Omega, \mathcal{F}_T, \mathsf{P})$ for some $\varepsilon > 0$. From Nakano [20], the solution of the problem (3.16) in the setting of a (B^d, S^1, S^2)-market consists in finding the perfect hedge of the claim f_T and the optimal strategy for an expected utility maximization problem (Section 3.2 helps in solving this part; see also Karatzas–Shreve [14], Nakano [20]). Such a solution is characterized by what follows.

1) The optimal solution for (3.16) is $\pi^* = \pi_{f_T} - \pi_0$, where π_{f_T} is the perfect hedge for f_T and π_0 is the optimal strategy for

$$J(z) := \inf_{\pi\in\mathcal{A}_0(z)} \mathsf{E}\left[l_p\left(X_T^{\pi,d}(z)\right)\right] \tag{3.17}$$

with $z = x_{f_T} - x$. We denote $\mathcal{A}_0(z) = \{\pi \in \mathcal{A} \text{ and } X_t^\pi \ge 0,\ t \in [0, T] \text{ a.s.}\}$.

2) Let $\alpha_0(t)$ be the optimal portfolio proportions associated to the solution π_0 of (3.17). Then $\alpha_0(t) := \alpha_0 = (\alpha_0^1, \alpha_0^2)$ of $J(z)$ is given by

$$\alpha_0^1 = \frac{\frac{\phi\nu^2}{p-1} + \sigma^2\left(\frac{\lambda^*}{\lambda}\right)^{q-1}}{\nu^2\sigma^1 - \nu^1\sigma^2}, \qquad \alpha_0^2 = \frac{\frac{\phi\nu^1}{p-1} + \sigma^1\left(\frac{\lambda^*}{\lambda}\right)^{q-1}}{\sigma^2\nu^1 - \sigma^1\nu^2}, \tag{3.18}$$

where q is such that $1/p + 1/q = 1$.

3) The price function is

$$u(x) = l_p(x_{f_T} - x)e^{(-(p-1)aT)}, \tag{3.19}$$

with

$$a = -qr^d + \frac{1}{2}q(q-1)\phi^2 - \lambda\left((q-1) - q\left(\frac{\lambda^*}{\lambda}\right) + \left(\frac{\lambda^*}{\lambda}\right)^q\right).$$

4) The optimal terminal wealth is given by

$$X_T^{\pi_{f_T}-\pi_0,d}(x) = f_T - (x_{f_T} - x)(Z_T)^{q-1}\exp\left\{\left(-\left(a + \frac{r^d}{p-1}\right)T\right)\right\}.$$

Consider now the identical problem in a two-interest-rates financial market:

$$u(x) = \inf_{\substack{\pi\in\mathcal{A},\\ x<C_-}} \mathsf{E}\left[l_p\left((f_T - X_T^\pi(x))^+\right)\right]. \tag{3.20}$$

We give the solution of the problem (3.20).

Theorem 3.2. *Let $X_t^{\pi,d}(x)$ and $X_t^\pi(x)$ be the wealth processes in the (B^d, S^1, S^2) and (B^1, B^2, S^1, S^2)-markets, and let the initial capital x satisfy respectively relations (2.5) and (2.7). Further, assume that the optimal proportion α_t in the (B^d, S^1, S^2)-market* *satisfies* (3.1). *Assume also that the optimal strategy* α_{f_T} *hedging* f_T *in the* (B^d, S^1, S^2)*-market satisfies the conditions provided in Proposition 3.1.*

Then, in the (B^1, B^2, S^1, S^2)*-market,*

1) *the price function* (3.20) *is given by equation* (3.19)*,*
2) *the optimal proportions invested are*

$$\alpha_t^1 = \frac{\alpha_f^1 X_{t^-}^{\pi_{f_T}}(x_{f_T}) - \alpha_0^1 X_{t^-}^{\pi_0}(x_{f_T} - x)}{X_{t^-}^{\pi_{f_T}-\pi_0}(x)} \quad \text{on } S^1,$$
$$\alpha_t^2 = \frac{\alpha_f^2 X_{t^-}^{\pi_{f_T}}(x_{f_T}) - \alpha_0^2 X_{t^-}^{\pi_0}(x_{f_T} - x)}{X_{t^-}^{\pi_{f_T}-\pi_0}(x)} \quad \text{on } S^2,$$

and $(1 - \alpha_t^1 - \alpha_t^2)^+$ *on the deposit account and* $(1 - \alpha_t^1 - \alpha_t^2)^-$ *on the credit account.*

Proof. Let the optimal proportions α_t^* invested for the problem (3.16) in the (B^d, S^1, S^2)-market satisfy (3.1). Then,

$$X_t^{\pi^*,d}(x) = X_t^{\pi^*}(x) \quad \text{for all } 0 \le t \le T$$

and α^* is optimal for the same problem in the (B^1, B^2, S^1, S^2) financial market.

The (B^1, B^2, S^1, S^2)-market admits a higher deposit rate and a lower lending rate than the (B^d, S^1, S^2)-market. Therefore, for any strategy π, $X_t^\pi(x) \le X(x)_t^{\pi,d}$ and from the monotonicity of l_p it follows that

$$\begin{aligned} &\inf_{\pi\in\mathcal{A},\,\{x<E^{d,*}[e^{-r^dT}f_T]\}} \mathsf{E}\left[l_p\left(\left(f_T - X_T^{\pi,d}(x)\right)^+\right)\right] \\ &\qquad \le \inf_{\pi\in\mathcal{A},\,\{x<E^{d,*}[e^{-r^dT}f_T]\}} \mathsf{E}\left[l_p\left((f_T - X_T^\pi(x))^+\right)\right]. \end{aligned} \tag{3.21}$$

Now if α^* is optimal for the left hand side of inequality (3.21) and satisfies equality (3.1), then it is also optimal for the right hand side.

The price function $u(x)$ is given by

$$\begin{aligned} u(x) &= \inf_{\pi\in\mathcal{A},\,\{x<E^{d,*}[e^{-r^dT}f_T]\}} \mathsf{E}\left[l_p\left(\left(f_T - X_T^{\pi,d}(x)\right)^+\right)\right] \\ &= \mathsf{E}\left[l_p\left(\left(f_T - X_T^{\pi^*,d}(x)\right)^+\right)\right] \\ &= \mathsf{E}\left[l_p\left((f_T - X_T^{\pi^*}(x))^+\right)\right] \\ &= \inf_{\pi\in\mathcal{A},\,\{x<E^{d,*}[e^{-r^dT}f_T]\}} \mathsf{E}\left[l_p\left((f_T - X_T^\pi(x))^+\right)\right]. \end{aligned}$$

Since the optimal proportion α_{f_T} hedging f_T satisfies the conditions of Proposition 3.1 (i.e. $C_- = C_{r^d}$), we obtain

$$u(x) = \inf_{\{\pi\in\mathcal{A},\,\{x<C_-\}\}} \mathsf{E}\left[l_p\left((f_T - X_T^\pi(x))^+\right)\right].$$

The optimal terminal wealth is $X_T^\pi(x) = X_T^{\pi,d}(x) = X_T^{\pi_f}(x_f) - X_T^{\pi_0}(x - x_f)$.

The optimal proportions invested are given by

$$\begin{aligned} \pi &= \pi_f - \pi_0, \\ \gamma_t S &= \gamma_{t,f} S - \gamma_{t,0} S, \\ \alpha_t X_{t^-}^\pi &= \alpha_{t,f} X_{t^-}^{\pi_f} - \alpha_{t,0} X_{t^-}^{\pi_0}. \end{aligned} \tag{3.22}$$

□

Bibliography

1. K. K. Aase, *Contingent claim valuation when the security price is a combination of an Ito process and a random point process*, Stochastic Process. Appl. **28** (1988), 185–220. MR952829 (89k:90015)
2. J. Bardhan and X. Chao, *Pricing options on securities with discontinuous returns*, Stochastic Process. Appl. **48** (1993), 123–137. MR1237171 (94g:90011)
3. Y. Bart, *Option hedging in the binomial model with differing interest rates*, Uspekhi Math. Nauk **53** (1998), no. 5, 227–228; English transl. in Russian Math. Surveys **53** (1998), 1084–1085. MR1691190
4. Y. Bergman, *Option pricing with different interest rates for borrowing and for lending*, Working Paper, University of California, Berkeley **109** (1981).
5. D. Colwell and R. Elliott, *Discontinuous asset prices and non-attainable contingent claims and corporate policy*, Math. Finance **3** (1993), 295–318.
6. J. Cvitanić, *Optimal trading under constraints*, Lectures Notes in Mathematics, vol. 1656, Springer-Verlag, Berlin, 1997, pp. 123–190. MR1478201
7. J. Cvitanić, *Theory of Portfolio Optimization in Markets with Frictions*, Handbooks in Math. Finance: Option Pricing, Interest Rates and Risk Management (E. Jouini and M. Musiela, eds.), Cambridge University Press, 2001. MR1848563
8. J. Cvitanić and I. Karatzas, *Hedging contingent claims with constrained portfolio*, Annals Appl. Probab. **3(3)** (1993), 652–681. MR1233619 (95c:90022)
9. J. Cvitanić, H. Pham, and N. Touzi, *Super-replication in stochastic volatility models under portfolio constraints*, J. Appl. Probab. **36** (1999), 523–545. MR1724796 (2001a:91048)
10. R. Elliott and P. E. Kopp, *Mathematics of Financial Markets*, Springer-Verlag, Berlin, 1998. MR2098795 (2005g:91001)
11. H. Föllmer and D. O. Kramkov, *Optional decompositions under constraints*, Probab. Theory Related Fields **109** (1997), 1–25. MR1469917 (98j:60065)
12. H. Föllmer and P. Leukert, *Efficient hedging: Cost versus shortfall risk*, Finance Stoch. **4** (2000), 117–146. MR1780323 (2001f:91054)
13. S. Kane and A. Melnikov, *On pricing contingent claims in a two interest rates jump diffusion model via market completions*, Theory Probab. Math. Statist. **77** (2007), 57–69. MR2432772 (2009f:91062)
14. I. Karatzas and S. Shreve, *Methods of Mathematical Finance*, Springer-Verlag, New York, 1998. MR1640352 (2000e:91076)
15. R. Korn, *Contingent claim valuation in a market with different interest rates*, Math. Methods Oper. Res. **42** (1995), 255–274. MR1358829
16. R. Krutchenko and A. V. Melnikov, *Quantile hedging for a jump-diffusion financial market*, Trends in Mathematics (M. Kohlmann, ed.), Birkhäuser Verlag, Basel/Switzerland, 2001, pp. 215–229. MR1882833
17. A. V. Melnikov, M. Nechaev, and S. Volkov, *Mathematics of Financial Obligations*, American Mathematical Society, Providence, R.I., 2002. MR1918716 (2003f:91055)
18. F. Mercurio and W. Runggaldier, *Option pricing for jump-diffusions: Approximations and their interpretation*, Math. Finance **3** (1993), 191–200.
19. R. C. Merton, *Continuous-time finance*, Basil–Blackwell, Oxford, 1990.
20. Y. Nakano, *Minimization of shortfall risk in a jump-diffusion model*, Statist. Probab. Lett. **67** (2004), 87–95. MR2039936 (2005b:62137)
21. H. Soner and N. Touzi, *Superreplication under gamma constraints*, J. Control Optim. **39** (2000), 73–96. MR1780909 (2002h:91068)

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