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CONDITIONS FOR THE UNIFORM CONVERGENCE OF EXPANSIONS OF φ-SUB-GAUSSIAN STOCHASTIC PROCESSES IN FUNCTION SYSTEMS GENERATED BY WAVELETS

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Abstract. The expansions with uncorrelated coefficients in function systems generated by wavelets are constructed in the paper for second order stochastic processes. Conditions for the uniform convergence with probability one on a finite interval are found for expansions whose coefficients are independent. Conditions for the uniform convergence in probability on a finite interval are found for expansions of strictly φ-sub-Gaussian stochastic processes.

1. Introduction

The uniform convergence of random series with probability one or in probability are considered in [2, 7, 9, 17] and in some other papers.

The expansions of φ-sub-Gaussian stochastic processes with uncorrelated coefficients are obtained in this paper for function systems generated by wavelets. We find conditions for the uniform convergence on a finite interval with probability one for expansions with independent coefficients and those for the convergence in probability for expansions of strictly φ-sub-Gaussian stochastic processes.

The paper is organized as follows. Section 2 contains the main notions of the theory of strictly φ-sub-Gaussian random variables and stochastic processes. A theorem on the expansions of centered second order stochastic processes in series with uncorrelated coefficients for function systems generated by wavelet systems is stated in Section 3. In Section 4, conditions for the uniform convergence with probability one on a finite interval are given for the expansions with independent terms. In Section 5, conditions are found for the uniform convergence in probability on a finite interval of expansions of strictly φ-sub-Gaussian stochastic processes.

All the results obtained in the paper are new.

2. Strictly φ-sub-Gaussian stochastic processes

Definition 2.1 ([15]). A continuous even convex function $\varphi = \{\varphi(x), x \in \mathbf{R}\}$ is called an Orlicz N-function if $\varphi(0) = 0$, $\varphi(x) > 0$ for $x \neq 0$, and

$$\lim_{x\to 0}\frac{\varphi(x)}{x} = 0, \qquad \lim_{x\to\infty}\frac{\varphi(x)}{x} = \infty.$$

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Definition 2.2 ([5])**.** We say that condition $\mathbf{Q}$ holds for an N-function φ if

$$\lim_{x\to 0}\inf \frac{\varphi(x)}{x^2} = c > 0$$

(the case of $c = +\infty$ is not excluded).

The following definition of a φ-sub-Gaussian random variable is introduced in [5] and is a slight modification of the corresponding definition in [2].

Definition 2.3. Let $\{\Omega, \mathfrak{F}, \mathsf{P}\}$ be a standard probability space. Let condition $\mathbf{Q}$ hold for an N-function φ. A random variable ξ is called φ-sub-Gaussian if

1) $\mathsf{E}\,\xi = 0$,
2) the exponential moment $\mathsf{E}\exp\{\lambda\xi\}$ exists for all $\lambda \in \mathbf{R}$,
3) there exists a constant $a > 0$ such that the inequality

$$\mathsf{E}\exp\{\lambda\xi\} \le \exp\{\varphi(\lambda a)\}$$

holds for all $\lambda \in \mathbf{R}$.

The collection of φ-sub-Gaussian random variables is denoted by $\mathrm{Sub}_\varphi(\Omega)$. Note that $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\xi) = \sup_{\lambda\neq 0} \frac{\varphi^{(-1)}(\ln \mathsf{E}\exp\{\lambda\xi\})}{|\lambda|}$$

(see [2, 5]). Examples of φ-sub-Gaussian random variables can be found in [5, 2]. Notice that the centered normal random variables belong to the space $\mathrm{Sub}_\varphi(\Omega)$ for $\varphi(x) = x^2/2$. In this case, $\tau^2(\xi) = \sigma^2$.

Definition 2.4 ([8])**.** A family Δ of random variables $\xi \in \mathrm{Sub}_\varphi(\Omega)$ is called strictly φ-sub-Gaussian if there exists a constant $C_\Delta > 0$ such that

$$\tau_\varphi^2\left(\sum_{i\in I}\lambda_i\xi_i\right) \le C_\Delta^2\,\mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2$$

for an arbitrary finite collection of random variables $\xi_i \in \Delta$ and all $\lambda_i \in \mathbf{R}$, $i \in I$. The constant C_Δ is called the determining constant of the family Δ.

Lemma 2.1 ([8])**.** *The linear closure in the space $L_2(\Omega)$ of a strictly φ-sub-Gaussian family Δ is a strictly φ-sub-Gaussian family with the same determining constant.*

Definition 2.5 ([2])**.** A stochastic process $X = \{X(t), t \in T\}$ is called φ-sub-Gaussian if all random variables $X(t)$, $t \in T$, are φ-sub-Gaussian.

Definition 2.6 ([8])**.** A stochastic process $X = \{X(t), t \in T\}$ is called strictly φ-sub-Gaussian if the family of random variables $\{X(t), t \in T\}$ is strictly φ-sub-Gaussian with the determining constant C_T. The constant C_T is called the determining constant of the process X.

Examples of φ-sub-Gaussian stochastic processes can be found in the book [2] and in the paper [5]. Examples of strictly φ-sub-Gaussian random variables and strictly φ-sub-Gaussian stochastic processes are given in [8]. Note that a normal centered stochastic process is φ-sub-Gaussian with $\varphi(x) = x^2/2$ and with the determining constant $C_T = 1$.

3. Expansions of stochastic processes in series with uncorrelated coefficients

Let $\phi = \{\phi(x), x \in \mathbf{R}\}$ be a function of the space $L_2(\mathbf{R})$ and let $\widehat{\phi}(y)$ denote the Fourier transform of ϕ:

$$\widehat{\phi}(y) = \int_{\mathbf{R}} e^{-iyx}\phi(x)\,dx. \tag{1}$$

A function ϕ is called an f-wavelet ([6], [14]) if

1) the function system $\{\phi(x-k), k \in \mathbf{Z}\}$ is orthonormal in the space $L_2(\mathbf{R})$;
2) there exists a periodic function $m_0(y)$ with period 2π such that $m_0 \in L_2(0, 2\pi)$ and almost everywhere

$$\widehat{\phi}(2y) = m_0(y)\widehat{\phi}(y); \tag{2}$$

3) $\widehat{\phi}(0) \neq 0$ and $\widehat{\phi}(y)$ is continuous at zero.

Define the function $\widehat{\psi}(y)$ as follows:

$$\widehat{\psi}(y) = \overline{m_0\left(\frac{y}{2}+\pi\right)}\exp\Big\{-i\frac{y}{2}\Big\}\widehat{\phi}\left(\frac{y}{2}\right).$$

Let $\psi(x)$ be the inverse Fourier transform of the function $\widehat{\psi}(y)$, that is,

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{iyx}\widehat{\psi}(y)\,dy.$$

A function $\psi(x)$ is called the m-wavelet corresponding to the f-wavelet ϕ.

Put

$$\phi_{jk}(x) = 2^{j/2}\phi\left(2^j x - k\right), \qquad \psi_{jk}(x) = 2^{j/2}\psi\left(2^j x - k\right), \qquad j \in \mathbf{Z},\ k \in \mathbf{Z}.$$

It is known that $\{\phi_{0k}, \psi_{jk}, j = 0, \dots, \infty, k \in \mathbf{Z}\}$ is an orthonormal complete system in $L_2(\mathbf{R})$ (see [3, 6]).

Theorem 3.1. *Let $X = \{X(t), t \in \mathbf{R}\}$ be a centered stochastic process such that $\mathsf{E}\,|X(t)|^2 < \infty$ for all $t \in \mathbf{R}$ and let*

$$R(t,s) = \mathsf{E}\,X(t)\overline{X(s)}.$$

Assume that $\phi = \{\phi(\lambda), \lambda \in \mathbf{R}\}$ is an f-wavelet and $\psi = \{\psi(\lambda), \lambda \in \mathbf{R}\}$ is the m-wavelet corresponding to ϕ. Denote

$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), \qquad \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \qquad j \in \mathbf{Z},\ k \in \mathbf{Z}.$$

Assume that $R(t,s)$ is represented in the form

$$R(t,s) = \int_{\mathbf{R}} u(t,\lambda)\overline{u(s,\lambda)}\,d\lambda,$$

where $u(t,\lambda)$, as a function of $\lambda \in \mathbf{R}$, is Borel for all $t \in \mathbf{R}$ and such that

$$\int_{\mathbf{R}} |u(t,\lambda)|^2\,d\lambda < \infty$$

for all $t \in \mathbf{R}$. Then

$$X(t) = \sum_{k\in\mathbf{Z}} \xi_{0k}a_{0k}(t) + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} \eta_{jk}b_{jk}(t), \tag{3}$$

where the series converges for all $t \in \mathbf{R}$ in the mean square sense, that is,

$$\sum_{k\in\mathbf{Z}} |a_{0k}(t)|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} |b_{jk}(t)|^2 < \infty,$$

$$a_{0k}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(t,y)\overline{\widehat{\phi}_{0k}(y)}\,dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(t,y)\overline{\widehat{\phi}(y)}e^{iky}\,dy,$$

$$b_{jk}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(t,y)\overline{\widehat{\psi}_{jk}(y)}\,dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(t,y)2^{-j/2}\exp\left\{i\frac{y}{2^j}k\right\}\overline{\widehat{\psi}\left(\frac{y}{2^j}\right)}\,dy,$$

$\widehat{\phi}_{0k}(y)$ and $\widehat{\psi}_{jk}(y)$ are the Fourier transforms of the functions $\phi_{0k}(x)$ and $\psi_{jk}(x)$, respectively, ξ_{0k} and η_{jk} are centered random variables such that

$$\mathsf{E}\,\xi_{0k}\overline{\xi_{0l}} = \delta_{kl}, \qquad \mathsf{E}\,\eta_{mk}\overline{\eta_{nl}} = \delta_{mn}\delta_{kl}, \qquad \mathsf{E}\,\xi_{0k}\overline{\eta_{nl}} = 0,$$

and δ_{kl} is the Kronecker symbol.

Remark 3.1. Let H_X be the mean square closure of the sums

$$\sum_{k=1}^{m} c_k X(t_k), \qquad t_k \in \mathbf{R},$$

and let $H_{\xi,\eta}$ be the mean square closure of the sums

$$\sum_{j=1}^{m}\sum_{k=1}^{n} c_{kj}\eta_{kj} + \sum_{k=1}^{s} b_k\xi_{0k}.$$

If the system $\{u(t,\lambda)\}$ is complete (in the sense that if $\int_{\mathbf{R}} u(t,\lambda)\overline{f(\lambda)}\,d\lambda = 0$ for some function $f \in L_2(\mathbf{R})$ and all $t \in \mathbf{R}$, then $f(\lambda) = 0$ almost everywhere), then $H_X = H_{\xi,\eta}$. In particular, if a process X is Gaussian, then η_{jk} and ξ_{0k} are independent Gaussian random variables.

A result similar to Theorem 3.1 is proved in [16] under extra restrictions that the process $X(t)$ is stationary, the covariance function R of the process is such that

$$R(\tau) \in L_1(\mathbf{R}),$$

the spectral density is a positive function, and $\widehat{\phi}(y)$ and $\widehat{\psi}(y)$ are functions with finite supports. Theorem 3.1 is proved in [12] without these restrictions. (It also follows from [13].) We prove Theorem 3.1 below together with Remark 3.1.

In what follows we need the following Karhunen theorem.

Theorem 3.2 (Karhunen [4]). *Let $X = \{X(t), t \in T\}$ be a centered second order stochastic process (that is, $\mathsf{E}\,X(t) = 0$ and $\mathsf{E}\,|X(t)|^2 < \infty$ for all t) with the covariance function $R(t,s) = \mathsf{E}\,X(t)\overline{X(s)}$. Let $\{\Lambda, \mathfrak{A}, \mu\}$ be a measurable space with a σ-finite measure and let $\{g(t,\lambda), t \in T, \lambda \in \Lambda\}$ be a system of functions such that $g(t,\cdot)$, as a function of the second argument if t is fixed, belongs to the space $L_2(\Lambda)$. Then the covariance function $R(t,s)$ is represented in the form*

$$R(t,s) = \int_{\Lambda} g(t,\lambda)\overline{g(s,\lambda)}\,d\mu(\lambda)$$

if and only if there exists a random measure $\mathbf{Z}$ with orthogonal values on $\mathfrak{A}$ such that $\mathbf{Z}$ is subordinate to the measure μ and

$$X(t) = \int_{\Lambda} g(t,\lambda)\,d\mathbf{Z}(\lambda).$$

If the system $\{g(t,\lambda)\}$ is complete in $L_2(\Lambda)$, then $H_X = H_{\mathbf{Z}}$, where H_X is the mean square closure of the sums $\sum_{k=1}^{m} c_k X(t_k)$, $t_k \in T$, while $H_{\mathbf{Z}}$ is the mean square closure of the sums $\sum_{k=1}^{n} b_k \mathbf{Z}(S_k)$, $S_k \in \mathfrak{A}$.

Now we turn to the proof of the theorem.

The system $\{\phi_{0k}(x), \psi_{jk}(x), k \in \mathbf{Z}, j = 0, 1, \dots\}$ is an orthonormal basis in $L_2(\mathbf{R})$. The Parseval equality implies that the system

$$\left\{ \frac{1}{\sqrt{2\pi}} \overline{\widehat{\phi}_{0k}(y)}, \frac{1}{\sqrt{2\pi}} \overline{\widehat{\psi}_{jk}(y)}, k \in \mathbf{Z}, j = 0, 1, \dots \right\}$$

also is an orthonormal basis in $L_2(\mathbf{R})$. Thus

$$u(t,y) = \sum_{k\in\mathbf{Z}} a_{0k}(t) \frac{1}{\sqrt{2\pi}} \overline{\widehat{\phi}_{0k}(y)} + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} b_{jk}(t) \frac{1}{\sqrt{2\pi}} \overline{\widehat{\psi}_{jk}(y)},$$

where the series converges in the norm of the space $L_2(\mathbf{R})$. Hence

$$R(t,s) = \int_{\mathbf{R}} u(t,\lambda)\overline{u(s,\lambda)}\, d\lambda = \sum_{k\in\mathbf{Z}} a_{0k}(t)\overline{a_{0k}(s)} + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} b_{jk}(t)\overline{b_{jk}(s)}$$

and the series converges for all $t, s \in \mathbf{R}$.

In what follows we consider the space $\{\Lambda, \mathfrak{A}, \mu\}$, where

$$\Lambda = \{(k,j), k \in \mathbf{Z},\ j = 0, \dots, \infty\},$$

$\mathfrak{A}$ is the family of all subsets of Λ and μ is the measure concentrated at the singletons of the space Λ that assigns the weight 1 to each singleton. Now Theorem 3.1 follows from Karhunen's theorem.

Remark 3.1 also follows, since

$$\{a_{0k}(t), b_{jk}(t), j = 0, \dots, \infty, k \in \mathbf{Z}\}$$

is a complete system in the space $L_2(\mathbf{R})$ if and only if the system $\{u(t,\lambda)\}$ is complete in the space $L_2(\Lambda)$.

Indeed, the system $W = \{a_{0k}(t), b_{jk}(t), j = 0, \dots, \infty, k \in \mathbf{Z}\}$ is complete if the properties

$$\sum_{k\in\mathbf{Z}} |a_{0k}^*|^2 + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} |b_{jk}^*|^2 < \infty$$

and

$$\sum_{k\in\mathbf{Z}} \overline{a_{0k}^*} a_{0k}(t) + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} \overline{b_{jk}^*} b_{jk}(t) = 0 \quad \text{for all } t \in \mathbf{R}$$

imply that $a_{0k}^* = 0$ and $b_{jk}^* = 0$.

We have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} \overline{a_{0k}^*} a_{0k}(t) + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} \overline{b_{jk}^*} b_{jk}(t) \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(t,y) \left(\sum_{k\in\mathbf{Z}} \overline{a_{0k}^* \widehat{\phi}_{0k}(y)} + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} \overline{b_{jk}^* \widehat{\psi}_{jk}(y)} \right) dy \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(t,y) \overline{L(y)}\, dy = 0,
\end{aligned}$$

where

$$L(y) = \sum_{k\in\mathbf{Z}} a_{0k}^* \widehat{\phi}_{0k}(y) + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} b_{jk}^* \widehat{\psi}_{jk}(y)$$

and $L(y) \in L_2(\mathbf{R})$. If the system $\{u(t,y)\}$ is complete, then $L(y)=0$ and thus its Fourier coefficients are zero. If the system W is complete and all a^*_{0k} and b^*_{jk} are equal to zero, then $L(y)=0$, that is, the system $\{u(t,\lambda)\}$ is complete.

In what follows a stationary process means a stationary stochastic process in a wide sense.

Corollary 3.1. *Let $X = \{X(t), t \in \mathbf{R}\}$ be a centered stationary stochastic process, $R(\tau) = \mathsf{E}\, X(t+\tau)\overline{X(t)}$. If*

1) *$R(\tau)$ is a continuous function and the spectral density exists for the process $X(t)$, that is,*

$$R(\tau) = \int_{\mathbf{R}} \exp\{-i\tau\lambda\} f(\lambda)\, d\lambda,$$

2) *f is real-valued, $f(\lambda) \geq 0$, and $\int_{-\infty}^{\infty} f(\lambda)\, d\lambda = R(0) < \infty$,*

then

$$X(t) = \sum_{k\in\mathbf{Z}} \xi_{0k}\alpha_{0k}(t) + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} \eta_{jk}\beta_{jk}(t), \tag{4}$$

where the series converges in the mean square sense for all t, that is,

$$\sum_{k\in\mathbf{Z}} |\alpha_{0k}(t)|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} |\beta_{jk}(t)|^2 < \infty,$$

$$\alpha_{0k}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (f(y))^{1/2} \exp\{-iy(t-k)\}\overline{\widehat{\phi}(y)}\, dy, \tag{5}$$

$$\beta_{jk}(t) = \frac{1}{\sqrt{2\pi}2^{j/2}} \int_{\mathbf{R}} (f(y))^{1/2} \exp\left\{-iy\left(t - \frac{k}{2^j}\right)\right\} \overline{\widehat{\psi}\left(\frac{y}{2^j}\right)}\, dy \tag{6}$$

and where ξ_{0k} and η_{jk} are centered random variables such that

$$\mathsf{E}\, \xi_{0k}\overline{\xi_{0l}} = \delta_{kl}, \qquad \mathsf{E}\, \eta_{mk}\overline{\eta_{nl}} = \delta_{mn}\delta_{kl}, \qquad \mathsf{E}\, \xi_{0k}\overline{\eta_{nl}} = 0.$$

The following results are obtained in [17] for the random variables ξ_{0k} and η_{jk} appearing in the series (4):

$$\xi_{0k} = \int_{\mathbf{R}} X(t)\overline{\Theta_{0k}(t)}\, dt,$$

$$\eta_{jk} = \int_{\mathbf{R}} X(t)\overline{\gamma_{jk}(t)}\, dt,$$

where

$$\widehat{\Theta}_{00}(y) = \left(\widehat{R}(y)\right)^{-1/2}\widehat{\phi}(y), \qquad \Theta_{0k}(t) = \Theta_{00}(t-k),$$

$$\widehat{\gamma}_{j0}(y) = \left(\widehat{R}(y)\right)^{-1/2}\widehat{\psi}_{j0}(y), \qquad \gamma_{jk}(t) = \gamma_{j0}\left(t - 2^{-j}k\right).$$

Moreover,

1) $R(\tau) \in L_1(\mathbf{R})$, $R(\tau)$ is continuous, and $\widehat{R}(\tau) > 0$ for all $\tau \in \mathbf{R}$,
2) $\widehat{\psi}(y)$ and $\widehat{\phi}(y)$ are functions on $\mathbf{R}$ with finite supports.

4. Conditions for the uniform convergence with probability one of series with independent terms

Definition 4.1. We say that a continuous function $\sigma = \{\sigma(h), h > 0\}$ satisfies condition $\mathbf{A}$ if $\sigma(h)$ increases in the domain $h > 0$, $\sigma(+0) = 0$, and

$$\int_0^{\varepsilon} \left|\ln \sigma^{(-1)}(v)\right|^{1/2} dv < \infty$$

for sufficiently small $\varepsilon > 0$, where $\sigma^{(-1)}(v)$ is the inverse function to $\sigma(v)$.

Remark 4.1. Note that condition **A** is satisfied for the function

$$\sigma(h) = \frac{1}{(\ln(e^\alpha + 1/h))^\alpha}, \qquad 0.5 < \alpha, \quad h > 0.$$

The following result is a slight modification of Theorem 3.5.5 of the book [2].

Lemma 4.1 ([1]). *Let $\{\xi_k, k = 1, 2, \dots\}$ be a sequence of independent centered random variables with $\mathsf{E}|\xi_k|^2 = 1$ and let $\{f_k(t), k = 1, 2, \dots\}$ be a sequence of continuous functions defined on a finite interval $T \subset \mathbf{R}$ and such that*

$$\sum_{k=1}^{\infty} f_k^2(t) < \infty, \qquad t \in T. \tag{7}$$

*If a function $\sigma = \{\sigma(h), h > 0\}$ satisfies condition **A** and*

$$\sup_{|t-s| \le h, t, s \in T} |f_k(t) - f_k(s)| \le b_k \sigma(h), \tag{8}$$

where the sequence $\{b_k\}$ is such that

$$\sum_{k=1}^{\infty} b_k^2 < \infty, \tag{9}$$

then the series

$$\sum_{k=1}^{\infty} \xi_k f_k(t)$$

converges uniformly in $t \in T$ with probability one.

Lemma 4.1 implies the following result.

Theorem 4.1. *Let a stochastic process $X = \{X(t), t \in \mathbf{R}\}$ satisfy the assumptions of Theorem 3.1, let the random variables ξ_{0k} and η_{jk}, $j = 0, \dots, \infty$, $k \in \mathbf{Z}$, appearing in expansion (3) be independent, and let condition **A** be satisfied for a function $\sigma = \{\sigma(h), h > 0\}$. Then the series (3) converges uniformly in $t \in T$ with probability one on every bounded interval $T \subset \mathbf{R}$ if*

$$\sup_{|t-s| \le h,\, t, s \in T} |a_{0k}(t) - a_{0k}(s)| \le a_{0k}^* \sigma(h), \tag{10}$$

$$\sup_{|t-s| \le h,\, t, s \in T} |b_{jk}(t) - b_{jk}(s)| \le b_{jk}^* \sigma(h), \tag{11}$$

$$\sum_{k \in \mathbf{Z}} (a_{0k}^*)^2 + \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} (b_{jk}^*)^2 < \infty \tag{12}$$

for sufficiently small h.

Remark 4.2. The random variables ξ_{0k} and η_{jk}, $j = 0, \dots, \infty$, $k \in \mathbf{Z}$, in expansion (3) are independent if, for example, $X(t)$ is a Gaussian stochastic process.

Lemma 4.2. *The following inequality,*

$$v \le \frac{C_{\alpha,T}}{(\ln(e^\alpha + 1/v))^\alpha}, \tag{13}$$

holds for $\alpha > 0$ and $0 < v \le T$, where the constant $C_{\alpha,T}$ depends on α and T only.

Proof. First we prove inequality (13) for $\alpha \ge 1$.

We check the following bound:

$$v^\beta \le F_{\gamma,\beta,T} \frac{\gamma^\gamma}{\beta^\gamma (\ln(e^\gamma + 1/v))^\gamma} \tag{14}$$

for $0 < \beta \le 1$, $\gamma \ge \beta$, and $0 < v \le T$, where $F_{\gamma,\beta,T}$ is a constant depending on γ, β, and T only. Let $u > 0$ and $0 < s \le 1$. Then

$$\ln(1+u) \le \frac{1}{s}\ln(1+u)^s \le \frac{1}{s}\ln(1+u^s) \le \frac{u^s}{s}. \tag{15}$$

Setting $u = 1/v$, $v \ge 0$, in (15) we get

$$\ln(1+1/v) \le 1/(sv^s).$$

Thus

$$v \le \frac{1}{(s\ln(1+1/v))^{1/s}}.$$

Raising both sides to the power $\beta > 0$ we obtain

$$v^\beta \le \frac{1}{(s\ln(1+1/v))^{\beta/s}}.$$

Put $s = \beta/\gamma$. Then

$$v^\beta \le \frac{\gamma^\gamma}{\beta^\gamma(\ln(1+1/v))^\gamma}, \qquad v > 0,\ \beta > 0,\ \gamma \ge \beta. \tag{16}$$

If $0 < v \le T$ and $\gamma > 0$, then

$$\frac{1}{\ln(1+1/v)} \le \frac{\widehat{C}_{\gamma,T}}{\ln(e^\gamma + 1/v)}, \tag{17}$$

where

$$\widehat{C}_{\gamma,T} = \sup_{0<v\le T}\frac{\ln(e^\gamma+1/v)}{\ln(1+1/v)} \le \sup_{0<v\le T}\frac{\ln(e^\gamma(1+1/v))}{\ln(1+1/v)} \le \frac{\gamma}{\ln(1+1/T)} + 1.$$

Thus

$$\frac{1}{\ln(1+1/v)} \le \frac{C_{\gamma,T}}{\ln(e^\gamma + 1/v)} \tag{18}$$

for $0 < v \le T$ and $\gamma > 0$, where $C_{\gamma,T} = \gamma/\ln(1+1/T) + 1$.

We obtain inequality (13) for $\alpha \ge 1$ from (14) if $\beta = 1$ and $\gamma = \alpha$.

Now we turn to the proof of (13) for $\alpha < 1$. If $0 < \alpha < 1$, then

$$v \le \frac{C_{1,T}}{\ln(e+1/v)} \le \frac{C_{1,T}}{(\ln(e+1/v))^\alpha} \le \frac{C_{1,T}}{(\ln(e^\alpha+1/v))^\alpha}$$

for $v \in (0,T]$. This completes the proof of the lemma. □

The following result follows from Lemma 4.1 in the paper [10].

Lemma 4.3. *Let $\alpha > 0$, $u \ge 0$, and $v > 0$. Then*

$$\left|\sin\frac{u}{v}\right| \le \left(\frac{\ln(e^\alpha+u)}{\ln(e^\alpha+v)}\right)^\alpha.$$

Lemma 4.4. *Let $0 < a < b$ and let $X = \{X(t), t \in [a,b]\}$ be a stationary process. Let f be the spectral density of X, $g(y) := \sqrt{f(y)}$, ϕ be some f-wavelet, and let ψ be the corresponding m-wavelet. Put $S_\phi(y) := \overline{\widehat{\phi}(y)}$ and $S(y) := \overline{\widehat{\psi}(y)}$, $t_1, t_2 \in [a,b]$.*

Assume that the derivatives $g'(y)$, $S'(y)$, and $S'_\phi(y)$ exist everywhere and $|S'(y)| \le M$ for all $y \in \mathbf{R}$, $g(y)|S(y/2^j)| \to 0$ as $|y| \to \infty$ for all $j = 0, 1, \ldots;$ $g(y)|S_\phi(y)| \to 0$ as

$|y| \to \infty$, *and*

$$\int_{\mathbf{R}} |y||g'(y)| \left(\ln\left(e^{\alpha} + \frac{|y|}{2}\right)\right)^{\alpha} dy < \infty,$$
$$\int_{\mathbf{R}} |y|g(y) \left(\ln\left(e^{\alpha} + \frac{|y|}{2}\right)\right)^{\alpha} dy < \infty,$$
$$\int_{\mathbf{R}} |g'(y)||S_{\phi}(y)| \left(\ln\left(e^{\alpha} + \frac{|y|}{2}\right)\right)^{\alpha} dy < \infty,$$
$$\int_{\mathbf{R}} g(y)|S_{\phi}(y)| \left(\ln\left(e^{\alpha} + \frac{|y|}{2}\right)\right)^{\alpha} dy < \infty,$$
$$\int_{\mathbf{R}} g(y)|S'_{\phi}(y)| \left(\ln\left(e^{\alpha} + \frac{|y|}{2}\right)\right)^{\alpha} dy < \infty,$$

where $\alpha > 0$. *Then*

$$\sum_{k\in\mathbf{Z}} |\alpha_{0k}(t_1) - \alpha_{0k}(t_2)|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} |\beta_{jk}(t_1) - \beta_{jk}(t_2)|^2 \le \frac{C^*}{\left(\ln\left(e^{\alpha} + \frac{1}{|t_2-t_1|}\right)\right)^{\alpha}}.$$

Here the coefficients $\alpha_{0k}(t)$ *and* $\beta_{jk}(t)$ *are defined by equalities* (5) *and* (6), *respectively, and* C^* *is some constant.*

Proof. In what follows the symbol C with subscripts and/or superscripts stands for different constants.

Let $k \ne 0$. Integrating equality (6) by parts we get

$$\sqrt{2\pi}2^{j/2}\beta_{jk}(t) = \frac{i2^j}{k}\left(\int_{\mathbf{R}} \exp\left\{iy\frac{k}{2^j}\right\} g'(y)e^{-iyt}S\left(y/2^j\right) dy \right.$$
$$- it\int_{\mathbf{R}} \exp\left\{iy\frac{k}{2^j}\right\} g(y)e^{-iyt}S\left(y/2^j\right) dy$$
$$\left. + \frac{1}{2^j}\int_{\mathbf{R}} \exp\left\{iy\frac{k}{2^j}\right\} g(y)e^{-iyt}S'\left(y/2^j\right) dy\right).$$

Thus

$$\sqrt{2\pi}2^{j/2}|\beta_{jk}(t_1) - \beta_{jk}(t_2)| \le \frac{2^j}{|k|}\left(\int_{\mathbf{R}} \left|g'(y)S\left(y/2^j\right)\left(e^{-iyt_1} - e^{-iyt_2}\right)\right| dy \right.$$
$$+ \int_{\mathbf{R}} g(y)\left|S\left(y/2^j\right)\right|\left|t_1\cdot e^{-iyt_1} - t_2\cdot e^{-iyt_2}\right| dy$$
$$\left. + \frac{1}{2^j}\int_{\mathbf{R}} g(y)\left|S'\left(y/2^j\right)\right|\left|e^{-iyt_1} - e^{-iyt_2}\right| dy\right).$$

Let α be an arbitrary positive number. Using the following inequality,

$$\left|t_1\cdot e^{-iyt_1} - t_2\cdot e^{-iyt_2}\right| \le |t_2 - t_1| + 2b\left|\sin\frac{y(t_2-t_1)}{2}\right|,$$

and Lemmas 4.2 and 4.3 we get

$$\sqrt{2\pi}2^{j/2}|\beta_{jk}(t_1) - \beta_{jk}(t_2)| \le \frac{M}{|k|}\sigma(|\Delta t|)C_b^{(1)},$$

where

$$\sigma(|\Delta t|) = \frac{1}{\left(\ln\left(e^{\alpha} + \frac{1}{|t_2-t_1|}\right)\right)^{\alpha}}.$$

Therefore

$$|\beta_{jk}(t_1) - \beta_{jk}(t_2)| \le \frac{C_b^{(1)} M}{\sqrt{2\pi}2^{j/2}|k|}\sigma(|\Delta t|). \tag{19}$$

A similar bound holds for $k \neq 0$:

$$|\alpha_{0k}(t_1) - \alpha_{0k}(t_2)| \le \frac{C_a^{(1)}}{\sqrt{2\pi}|k|}\sigma(|\Delta t|). \tag{20}$$

It is clear that

$$|\alpha_{00}(t_1) - \alpha_{00}(t_2)| \le \frac{2}{\sqrt{2\pi}} \int_{\mathbf{R}} g(y) \left|\sin \frac{y(t_2 - t_1)}{2}\right| |S_\phi(y)|\, dy \le \frac{2C_a^{(2)}}{\sqrt{2\pi}}\sigma(|\Delta t|). \tag{21}$$

Similarly to the proof of a bound for $|\beta_{jk}(t_1) - \beta_{jk}(t_2)|$, one can show that

$$|\beta_{j0}(t_1) - \beta_{j0}(t_2)| \le \frac{2MC_b^{(2)}}{\sqrt{2\pi}2^{3j/2}}\sigma(|\Delta t|). \tag{22}$$

Now the lemma follows from inequalities (19)–(21) and (22). □

Theorem 4.2. *Let a stationary stochastic process $X = \{X(t), t \in [a, b]\}$ satisfy the assumptions of Corollary* 3.1*. Assume that the random variables ξ_{0k} and η_{jk}, $j = 0, \dots, \infty$, $k \in \mathbf{Z}$, in expansion* (4) *are independent. Denote by f the spectral density of the process $X(t)$.*

If an f-wavelet ϕ and the corresponding m-wavelet ψ satisfy the assumptions of Lemma 4.4 *and $\alpha > 0.5$, then the series* (4) *converges uniformly in $t \in [a, b]$ with probability one.*

Proof. The proof follows, since condition **A** holds for the function $\sigma(h) = (\ln(e^\alpha + 1/h))^{-\alpha}$ with $0.5 < \alpha$ and one can apply Theorem 4.1 and Lemma 4.4. □

Remark 4.3. An example of a stochastic process and a wavelet satisfying the assumptions of Theorem 4.1 is given by a centered Gaussian stationary process $X = \{X(t), t \in T\}$ with the covariance function $R(\tau) = e^{-\tau^2}$ and the Haar wavelet system (assumptions of Lemma 4.4 can easily be checked in this case).

5. Conditions for the uniform convergence in probability of expansions of strictly φ-sub-Gaussian stochastic processes

Let $X = \{X(t), t \in \mathbf{R}\}$ be a strictly φ-sub-Gaussian stochastic process such that

$$\mathsf{E}\, X(t)\overline{X(s)} = R(t, s) = \int_{\mathbf{R}} u(t, \lambda)\overline{u(s, \lambda)}\, d\lambda,$$

where $u(t, \lambda)$, $t \in \mathbf{R}$, is a complete function system in the space $L_2(\mathbf{R})$. According to Theorem 3.1 and Remark 3.1 the process $X(t)$ is represented in the form of the series (3), where ξ_{0k} and η_{jk} are uncorrelated φ-sub-Gaussian random variables such that

$$\tau_\varphi(\xi_{0k}) \le C_X,$$
$$\tau_\varphi(\eta_{jk}) \le C_X,$$

and where C_X is the determining constant of the process X.

Definition 5.1. Let $\sigma = \{\sigma(h), h > 0\}$ be a continuous increasing function such that $\sigma(+0) = 0$. We say that condition **B** holds for the function σ if

$$\int_0^\varepsilon \Psi\left(\ln \frac{1}{\sigma^{(-1)}(u)}\right) du < \infty$$

for sufficiently small $\varepsilon > 0$, where $\Psi(u) = u/\varphi^{(-1)}(u)$ and $\sigma^{(-1)}(u)$ is the inverse function to $\sigma(u)$.

Theorem 5.1. *Let $0 < a < b$ and $T = [a, b]$. Assume that*

$$X_n = \{X_n(t), t \in T\} \in \mathrm{Sub}_\varphi(\Omega)$$

*is a separable stochastic process and that condition **B** holds for a function*

$$\sigma = \{\sigma(h), h > 0\}.$$

Let

$$\sup_{|t-s|\le h} \tau_\varphi(X_n(t) - X_n(s)) \le \sigma(h).$$

If, for all $t \in T$, the processes $X_n(t)$ converge in probability to the process $X(t)$, then $X_n(t)$ converge in probability in the space $C(T)$.

Theorem 5.1 is proved in [11] for the space $\mathbf{R}^k$.

Now we can prove the following result.

Theorem 5.2. *Let a strictly φ-sub-Gaussian stochastic process $X = \{X(t), t \in \mathbf{R}\}$ have the determining constant C_X and satisfy the assumptions of Theorem 3.1. Further assume that a function system $u(t, \lambda)$ is complete in $L_2(\mathbf{R})$ and that there exists a function*

$$\sigma_\varphi = \{\sigma_\varphi(h), h > 0\}$$

*such that condition **B** holds. If*

$$\sup_{|t-s|\le h, t,s\in T} |a_{0k}(t) - a_{0k}(s)| \le a^*_{0k}\sigma_\varphi(h), \tag{23}$$

$$\sup_{|t-s|\le h, t,s\in T} |b_{jk}(t) - b_{jk}(s)| \le b^*_{jk}\sigma_\varphi(h), \tag{24}$$

$$\sum_{k\in\mathbf{Z}} (a^*_{0k})^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} (b^*_{jk})^2 < \infty \tag{25}$$

for sufficiently small h, then the series (3) *converges uniformly in probability on a finite interval $T \subset \mathbf{R}$.*

Proof. Theorem 5.2 follows from Theorem 5.1. Indeed put

$$\widehat{X}(t) = \sum_{k=-N_0}^{N_0} a_{0k}(t)\xi_{0k} + \sum_{j=0}^{N-1}\sum_{k=-M_j}^{M_j} b_{jk}(t)\eta_{jk},$$

$$n = n(N_0, N, M_0, M_1, \ldots, M_{N-1}) = \min\{N_0, N, M_0, M_1, \ldots, M_{N-1}\}.$$

Let $|t - s| \le h$. It is clear that

$$\tau_\varphi\big(\widehat{X}(t) - \widehat{X}(s)\big) \le C_X\big(\mathsf{E}\big|\widehat{X}(t) - \widehat{X}(s)\big|^2\big)^{1/2},$$

$$\begin{aligned}\mathsf{E}\big|\widehat{X}(t) - \widehat{X}(s)\big|^2 &= \sum_{k=-N_0}^{N_0} |a_{0k}(t) - a_{0k}(s)|^2 + \sum_{j=0}^{N-1}\sum_{k=-M_j}^{M_j} |b_{jk}(t) - b_{jk}(s)|^2 \\ &\le \sum_{k\in\mathbf{Z}} |a_{0k}(t) - a_{0k}(s)|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} |b_{jk}(t) - b_{jk}(s)|^2 \\ &\le \left(\sum_{k\in\mathbf{Z}} (a^*_{0k})^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} (b^*_{jk})^2\right)\sigma_\varphi(h).\end{aligned}$$

Moreover, for all t, $\widehat{X}(t)$ converges to $X(t)$ in the mean square sense as $n \to \infty$, whence we obtain the convergence in probability. The theorem is proved. □

Corollary 5.1. *Let $p \geq 2$,*

$$\alpha > \frac{p-1}{p},$$

and

$$\varphi(x) = \begin{cases} \frac{|x|^p}{p}, & |x| > 1, \\ \frac{x^2}{p}, & |x| \leq 1. \end{cases}$$

Assume that a stationary strictly φ-sub-Gaussian stochastic process

$$X = \{X(t), t \in [a,b]\}$$

and an f-wavelet ϕ and the corresponding m-wavelet ψ satisfy the assumptions of Lemma 4.4. *Then the series* (3) *converges uniformly in probability to the process $X(t)$ in the interval $[a,b]$.*

Proof. Inequalities (19)–(22) imply that conditions (23)–(25) hold for the functions $\alpha_{0k}(t)$ and $\beta_{jk}(t)$ involved in expansion (4) of the process $X(t)$, where

$$\sigma_\varphi(h) = \frac{C}{(\ln(e^\alpha + 1/h))^\alpha}$$

(here C is some constant). It remains to note that condition **B** holds for the function $\sigma_\varphi(h)$ if $\alpha > (p-1)/p$. □

6. Concluding remarks

We obtained conditions for the uniform convergence of an expansion of a φ-sub-Gaussian process in a function system generated by wavelets. The rate of convergence will be studied elsewhere.

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