

## SOME APPLICATIONS OF THE GNEDENKO–KOROLYUK METHOD TO EMPIRICAL DISTRIBUTIONS

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ABSTRACT. A new proof of the Kolmogorov theorem on the asymptotic behavior of the deviation between a theoretical and an empirical distribution function is presented. We use the Gnedenko–Korolyuk approach based on some combinatorial properties of the merged sample constructed from two other independent samples. Some statistical applications of the Gnedenko–Korolyuk theorem are discussed.

### 1. INTRODUCTION

For independent identically distributed random variables

$$\xi_1, \xi_2, \dots, \xi_n$$

assuming values in  $\mathbb{R}^1$  and having the distribution function  $F(t)$ , consider the empirical distribution function

$$F_n^*(t) = \frac{1}{n} \sum_{i=1}^n I_{(-\infty, t)}(\xi_i), \quad -\infty < t < \infty.$$

Kolmogorov in his well-known paper [1] introduced the following statistic:

$$D_n = \sup_{-\infty < t < \infty} |F_n^*(t) - F(t)|$$

and proved the following classical result.

**Theorem 1.1** (Kolmogorov). *If  $F(t)$  is a continuous distribution function, then*

$$(1) \quad \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}D_n < z) = K(z) = 1 - 2 \sum_{n=1}^{\infty} (-1)^{n-1} \exp(-2n^2 z^2).$$

Kolmogorov used a different notation in [1], namely  $D$  and  $\Phi(z)$  instead of  $D_n$  and  $K(z)$ , respectively.

This fundamental result determined in fact the main stream of further research related to the empirical functions. The next steps are due to Smirnov [2] who found the distribution of the so-called  $\omega^2$  statistic and showed that

$$(2) \quad \begin{aligned} \lim_{m \rightarrow \infty} \mathbb{P}(\sqrt{\hat{n}}D_{m,n}^+ < z) &= 1 - \exp(-2z^2), \\ \lim_{m \rightarrow \infty} \mathbb{P}(\sqrt{\hat{n}}D_{m,n} < z) &= K(z), \end{aligned}$$

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where

$$D_{m,n}^+ = \sup_{-\infty < t < \infty} (F_n^*(t) - G_m^*(t)), \quad D_{m,n} = \sup_{-\infty < t < \infty} |F_n^*(t) - G_m^*(t)|,$$

$\hat{n} = mn/(m+n)$ , and  $F_n^*(t)$  and  $G_m^*(t)$  are the empirical distribution functions constructed from two independent identically distributed samples of sizes  $n$  and  $m$ , respectively,  $m \leq n$ .

Kolmogorov and Smirnov used lengthy calculations when proving relations (1) and (2), respectively. Gnedenko and Korolyuk [3] found in 1951 an ingenious combinatorial proof of equality (2) for  $m = n$ . Moreover, the explicit distribution of the statistic  $D_{n,n}$  is obtained in [3] for finite  $n$ . Below is their main result.

**Theorem 1.2** (Gnedenko–Korolyuk). *If  $F(t)$  is a continuous distribution function of two independent samples and  $c = \lceil z\sqrt{2n} \rceil$  is the minimal integer that does not exceed  $z\sqrt{2n}$ , then, for  $(2n)^{-1/2} < z \leq \sqrt{n/2}$ ,<sup>1</sup>*

$$(3) \quad \Phi_n^+(z) = \mathbb{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n}^+ < z \right\} = 1 - \frac{C_{2n}^{n-c}}{C_{2n}^n},$$

$$(4) \quad \Phi_n(z) = \mathbb{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n} < z \right\} = 1 - \frac{2}{C_{2n}^n} \sum_{k=1}^{\lfloor n/c \rfloor} (-1)^{k-1} C_{2n}^{n-kc},$$

and

$$\Phi_n^+(z) = \Phi_n(z) = \begin{cases} 0, & \text{for } z \leq (2n)^{-1/2}, \\ 1, & \text{for } z > \sqrt{n/2}. \end{cases}$$

The explicit expressions for the distributions of the statistics  $D_{m,n}^+$  and  $D_{m,n}$  are found by Korolyuk [4] for all  $m$  and  $n$ . However these expressions are much more complicated as compared to relations (3) and (4).

Further studies of the asymptotic behavior of the function  $F_n(t)$  lead to the investigation of the weak convergence of the empirical process

$$\beta_n(t) = \sqrt{n}(F_n^*(t) - F(t)), \quad -\infty < t < \infty,$$

to the Brownian bridge [5]. Surveys of results in this direction can be found in [6]–[8].

Following the Gnedenko–Korolyuk approach, we provide below a direct proof of the Kolmogorov theorem. From our standpoint, the proof presented here is simpler than other known proofs of this result (see, for example, [9, 10]) and does not use general results on the weak convergence of measures such as Prokhorov’s theorem. We also discuss some statistical applications of the Gnedenko–Korolyuk theorem.

## 2. GNEDENKO–KOROLYUK METHOD

Apart from the sample  $\xi_1, \xi_2, \dots, \xi_n$ , we consider another sample of the same size

$$\eta_1, \eta_2, \dots, \eta_n.$$

Denote by  $F_n^*(x)$  and  $G_n^*(x)$  the empirical distribution functions for these samples. Equalities (2)–(4) can be used to test the homogeneity of the samples  $(\xi_i)$  and  $(\eta_i)$ .

Following Gnedenko and Korolyuk [3], we prove Theorem 1.2 and the asymptotic relation (2) for  $m = n$ . In doing so, we discuss the combinatorial part of the proof in more detail as compared to the proof in [3].

First we present two auxiliary combinatorial results.

<sup>1</sup>EDITORIAL NOTE: Here and in what follows the authors use the  $C$ -notation for binomial coefficients,  $C_n^m = \binom{n}{m}$ .

**Lemma 2.1.** *Let  $L$  be the class of polygonal lines that have  $2n$  segments each and possess the following properties:*

- (1) *the lines start at the point  $(0; 0)$  and end at the point  $(2n; 0)$ ,*
- (2) *the slope of any segment is either 1 or  $-1$ ,*
- (3) *the slopes can change only at integer points.*

*Then the number of polygonal lines of class  $L$  that have no common points with the line  $y = c$  is equal to  $C_{2n}^n - C_{2n}^{n-c}$ , where  $c \in \mathbb{N}$ ,  $c \leq n$ .*

*Proof.* The cardinality of the class  $L$  equals  $C_{2n}^n$  (this can easily be proved by the one-to-one correspondence between the polygonal lines and sequences of length  $2n$  containing equal amounts of the symbols 1 and  $-1$ ). Now we find the number of polygonal lines of the class  $L$  that have common points with the line  $y = c$ .

Consider a polygonal line. Reflecting, with respect to the line  $y = c$ , the part of the polygonal line to the right of the first intersection point with  $y = c$  we obtain a new polygonal line starting at the point  $(0; 0)$  and ending at the point  $(2n; 2c)$ . Conversely, every polygonal line ending at the point  $(2n; 2c)$  corresponds to a polygonal line ending at the point  $(2n; 0)$ ; the latter line is constructed as described above. The number of polygonal lines starting at  $(0; 0)$  and ending at  $(2n; 2c)$  equals  $C_{2n}^{n-c}$  (because of the one-to-one correspondence between those lines and sequences consisting of  $n + c$  symbols 1 and  $n - c$  symbols  $-1$ ).

Therefore the number of polygonal lines of the class  $L$  that do not intersect the line  $y = c$  is equal to  $C_{2n}^n - C_{2n}^{n-c}$ .  $\square$

The following result is a generalization of Lemma 2.1.

**Corollary 2.1.** *Let  $L(i; j)$  be the class of polygonal lines corresponding to sequences consisting of  $i$  symbols 1 and  $j$  symbols  $-1$ . Thus those lines consist of  $i + j$  segments, start at the point  $(0; 0)$ , and end at the point  $(i + j; i - j)$ .*

*Then the number of lines of the class  $L(i, j)$  that have no common points with the line  $y = c$  is equal to  $C_{i+j}^i - C_{i+j}^{c+j}$  if  $i - j < c$  or 0 if  $i - j \geq c$ .*

*Proof.* The cardinality of the class  $L(i, j)$  equals  $C_{i+j}^i$ . If  $i - j \geq c$ , then any polygonal line of  $L(i, j)$  intersects  $y = c$ . Thus the total number of those polygonal lines that have no common points with the line  $y = c$  equals zero.

If  $i - j < c$ , one can follow the same reasoning as in the proof of Lemma 2.1. Note that the end point of the segment intersecting the line  $y = c$  corresponds to the point  $(i + j; i - j)$  under the reflection. The total number of polygonal lines with this property is  $C_{i+j}^{c+j}$ . Thus the number of lines that do not intersect the line  $y = c$  is equal to  $C_{i+j}^i - C_{i+j}^{c+j}$ .  $\square$

**Lemma 2.2.** *The total number of polygonal lines of the class  $L$  that do not intersect both lines  $y = c$  and  $y = -c$  is equal to  $C_{2n}^n - 2 \sum_{i=1}^{\lfloor n/c \rfloor} (-1)^{i-1} C_{2n}^{n-ic}$ ,  $c \in \mathbb{N}$ ,  $c \leq n$ .*

*Proof.* Since  $|L| = C_{2n}^n$ , we need to find the number of polygonal lines that have common points with at least one line,  $y = c$  or  $y = -c$ .

Consider such a line. Let, for example, it intersect  $y = c$ . It follows from the proof of Lemma 2.1 that the total number of such lines equals  $C_{2n}^{n-c}$ . Similar reasoning is applied to the lines that intersect  $y = -c$ . The difference between these two cases is that, in the latter case, the correspondence is constructed by reflection with respect to  $y = -c$  and then the polygonal line starts at the point  $(0; 0)$  and ends at the point  $(2n; -2c)$ . The number of those polygonal lines is  $C_{2n}^{n-c}$ .

There are, however, polygonal lines that intersect both  $y = c$  and  $y = -c$ . Let  $L_1$  be the class of polygonal lines that intersect at least one of the lines  $y = c$  or  $y = -c$  and

let  $L_2$  be the class of polygonal lines that intersect both  $y = c$  and  $y = -c$ . The class of polygonal lines that start at the point  $(0; 0)$  and end at the point  $(2n; d)$  is denoted by  $L(d)$ .

Then

$$|L_1| = |L(2c)| + |L(-2c)| - |L_2| = C_{2n}^{n-c} + C_{2n}^{n-c} - |L_2| = 2C_{2n}^{n-c} - |L_2|.$$

Consider an arbitrary polygonal line of the class  $L_2$ . Let it intersect the line  $y = c$  for the first time at the point  $x_1$ . Let  $x_2$  be the first point after  $x_1$  where the polygonal line intersects  $y = -c$ . Reflect the part of the polygonal line after  $x_1$  with respect to  $y = c$ . Then the point  $(x_2; -c)$  corresponds to  $(x_2; 3c)$ , while the point  $(2n; 0)$  corresponds to  $(2n; 2c)$ . Now reflect the part of the polygonal line after  $x_2$  with respect to  $y = 3c$ . Then the end point  $(2n; 2c)$  of the polygonal line corresponds to the point  $(2n; 4c)$ .

As a result we obtain a polygonal line of the class  $L(4c)$ . Starting with a line of  $L(4c)$  one can uniquely obtain the initial line of the class  $L_2$  by applying the backward rule. The cardinality of the class  $L(4c)$  equals  $C_{2n}^{n-2c}$ .

The bijection between those lines of  $L_2$  that first intersect  $y = -c$  at some point  $x_1$  and then intersect  $y = c$  at some point  $x_2$  and lines of  $L(-4c)$  is constructed similarly.

There are some polygonal lines that belong to both classes  $L(4c)$  and  $L(-4c)$ . These lines have the property that there are  $a < b < g$  such that  $a$  and  $g$  are some points of intersection with  $y = c$ , while  $b$  is a point of intersection with  $y = -c$ , or otherwise. Denote this class of polygonal lines by  $L_3$ . Then

$$|L_2| = |L(4c)| + |L(-4c)| - |L_3| = C_{2n}^{n-2c} + C_{2n}^{n-2c} - |L_3| = 2C_{2n}^{n-2c} - |L_3|.$$

Consider a polygonal line of  $L_3$ . Let there exist three points  $x_1 < x_2 < x_3$  such that  $x_1$  and  $x_3$  are points of intersection with  $y = c$ , while  $x_2$  is a point of intersection with  $y = -c$  and the line has no common points with  $y = c$  before  $x_1$ , it has no common points with  $y = -c$  in the interval  $(x_1, x_2)$ , and it does not intersect  $y = c$  in the interval  $(x_2, x_3)$ . Reflecting the line step by step after  $x_1$  with respect to  $y = c$ , then after  $x_2$  with respect to  $y = 3c$ , and after  $x_3$  with respect to  $y = 5c$  we obtain a polygonal line of the class  $L(6c)$ . Starting with a line of  $L(6c)$  one can uniquely obtain the initial line of the class  $L_3$  by applying the backward rule.

The same reasoning applies if  $x_1$  and  $x_3$  are points of intersection with  $y = -c$ , while  $x_2$  is a point of intersection with  $y = c$ ; the only difference between these two cases is that, in the latter case, the resulting line belongs to  $L(-6c)$ .

Again, some of the polygonal lines belong to both classes  $L(6c)$  and  $L(-6c)$ . Those lines are characterized by the property that they first intersect  $y = c$ , then  $y = -c$ , then again  $y = c$ , and finally they intersect  $y = -c$ , or they do the same for  $y = c$  and  $y = -c$  interchanged. The class of those lines is denoted by  $L_4$ . Then

$$|L_3| = |L(6c)| + |L(-6c)| - |L_4| = C_{2n}^{n-3c} + C_{2n}^{n-3c} - |L_4| = 2C_{2n}^{n-3c} - |L_4|.$$

As before we derive the equality

$$|L_i| = 2C_{2n}^{n-ic} - |L_{i+1}|,$$

where  $L_i$  is the class of polygonal lines that sequentially intersect  $y = c$  and  $y = -c$  at least  $i$  times.

This procedure will end after a finite number of steps, since any polygonal line can intersect  $y = c$  and  $y = -c$  sequentially no more than  $[n/c]$  times. Thus

$$|L_{[n/c]}| = 2C_{2n}^{n-[n/c]c},$$

since any polygonal line of this class is not counted twice (there are no polygonal lines having  $[n/c] + 1$  or more points of intersection).

Now we combine the above results. Denote the number of polygonal lines of interest by  $N$ . Then

$$N = C_{2n}^n - |L_1|,$$

$$|L_1| = 2C_{2n}^{n-c} - |L_2|, \quad |L_2| = 2C_{2n}^{n-2c} - |L_3|, \quad \dots, \quad |L_{[n/c]}| = 2C_{2n}^{n-[n/c]c}.$$

Combining all the above expressions,

$$\begin{aligned} N &= C_{2n}^n - 2C_{2n}^{n-c} + 2C_{2n}^{n-2c} - 2C_{2n}^{n-3c} + \dots + (-1)^{[n/c]} 2C_{2n}^{n-[n/c]c} \\ &= C_{2n}^n - 2 \sum_{i=1}^{[n/c]} (-1)^{i-1} C_{2n}^{n-ic}. \end{aligned} \quad \square$$

*Proof of Theorem 1.2.* Denote by  $K_1(x) = nF_n^*(x)$  the number of those elements of the first sample that are not less than  $x$ . We use the similar notation  $K_2(x) = nG_n^*(x)$  for the second sample. Then

$$\mathbb{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n}^+ < z \right\} = \mathbb{P} \left\{ nD_{n,n}^+ < z\sqrt{2n} \right\} = \mathbb{P} \left\{ \sup_{-\infty < x < \infty} (K_1(x) - K_2(x)) < z\sqrt{2n} \right\}.$$

Now we merge two samples  $\xi_1, \xi_2, \dots, \xi_n$  and  $\eta_1, \eta_2, \dots, \eta_n$  and obtain the sample  $z_1, z_2, \dots, z_{2n}$ ; we write its elements in the ascending order

$$z_{(1)} \leq z_{(2)} \leq \dots \leq z_{(2n)}.$$

Consider the following random variables:

$$\varepsilon_k = \begin{cases} 1, & z_{(k)} \in (\xi_i), \\ -1, & z_{(k)} \in (\eta_i), \end{cases}$$

$$S_n = \sum_{k=1}^n \varepsilon_k, \quad S_0 = 0, \quad S_{2n} = 0.$$

Thus  $S_k = K_1(z_{(k)} + 0) - K_2(z_{(k)} + 0)$  and

$$\begin{aligned} \mathbb{P} \left\{ \sup_{-\infty < x < \infty} (K_1(x) - K_2(x)) < z\sqrt{2n} \right\} &= \mathbb{P} \left\{ \sup_{k \in \{1, \dots, 2n\}} S_k < z\sqrt{2n} \right\} \\ &= \mathbb{P} \left\{ \sup_{k \in \{1, \dots, 2n\}} S_k < c \right\}. \end{aligned}$$

According to the symmetry (recall that the random variables  $\xi_i$  and  $\eta_k$  have the same distribution function), all allocations of the symbols 1 and  $-1$  in the sequence  $(\varepsilon_i)$  have the same probability.

Notice that an arbitrary sequence  $(S_n)$  can be treated as a certain polygonal line of the class  $L$  introduced above. The inequality  $\sup_{k \in \{1, \dots, 2n\}} S_k < c$  means that the polygonal line does not intersect  $y = c$ . The probability  $\mathbb{P} \{ \sup_{k \in \{1, \dots, 2n\}} S_k < c \}$  equals the fraction of those lines in  $L$  for which the latter inequality holds, whence

$$\mathbb{P} \left\{ \sup_{k \in \{1, \dots, 2n\}} S_k < c \right\} = \frac{C_{2n}^n - C_{2n}^{n-c}}{C_{2n}^n} = 1 - \frac{C_{2n}^{n-c}}{C_{2n}^n}$$

by Lemma 2.1. Thus

$$\mathbb{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n}^+ < z \right\} = 1 - \frac{C_{2n}^{n-c}}{C_{2n}^n}.$$

Similarly we prove another equality:

$$\begin{aligned} \mathbb{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n} < z \right\} &= \mathbb{P} \left\{ n D_{(n,n)} < z \sqrt{2n} \right\} = \mathbb{P} \left\{ \sup_{-\infty < x < \infty} |K_1(x) - K_2(x)| < z \sqrt{2n} \right\} \\ &= \mathbb{P} \left\{ \sup_{-\infty < x < \infty} |S_k| < z \sqrt{2n} \right\} = \mathbb{P} \left\{ \sup_{k \in \{1, \dots, 2n\}} |S_k| < c \right\}. \end{aligned}$$

The inequality

$$\sup_{k \in \{1, \dots, 2n\}} |S_k| < c$$

means that the polygonal line does not intersect both lines  $y = c$  and  $y = -c$ . Thus

$$\begin{aligned} \mathbb{P} \left\{ \sup_{k \in \{1, \dots, 2n\}} |S_k| < c \right\} &= \frac{C_{2n}^n - 2 \sum_{k=1}^{\lfloor n/c \rfloor} (-1)^{k-1} C_{2n}^{n-kc}}{C_{2n}^n} \\ &= 1 - \frac{2}{C_{2n}^n} \sum_{k=1}^{\lfloor n/c \rfloor} (-1)^{k-1} C_{2n}^{n-kc} \end{aligned}$$

by Lemma 2.2. □

**Corollary 2.2** (Smirnov).

$$(5) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n}^+ < z \right\} = \begin{cases} 1 - e^{-2z^2}, & z > 0, \\ 0, & z \leq 0, \end{cases}$$

$$(6) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n} < z \right\} = \begin{cases} K(z), & z > 0, \\ 0, & z \leq 0. \end{cases}$$

*Proof.* This result can be obtained by passing to the limit in equalities (3)–(4) in the Gnedenko–Korolyuk theorem.

For this, we determine the asymptotic behavior of the ratio

$$C_{2n}^{n-kc} / C_{2n}^n, \quad \text{where } c = \lceil z \sqrt{2n} \rceil, \quad k \in \mathbb{Z}.$$

By Stirling's formula,

$$n! = n^n e^{-n} \sqrt{2\pi n} (1 + o(1)), \quad n \rightarrow \infty.$$

Then

$$\begin{aligned} \frac{C_{2n}^{n-kc}}{C_{2n}^n} &= \frac{n! n!}{(n-kc)! (n+kc)!} \\ &= \frac{n^{2n} e^{-2n} 2\pi n (1 + o(1))}{(n-kc)^{n-kc} (n+kc)^{n+kc} e^{-2n} \sqrt{2\pi(n-kc)} \sqrt{2\pi(n+kc)}} \\ &= \frac{n^{2n} n (1 + o(1))}{(n-kc)^{n-kc} (n+kc)^{n+kc} \sqrt{n^2 - k^2 c^2}} = \frac{(1 + o(1))}{(1 - kc/n)^{n-kc} (1 + kc/n)^{n+kc}}. \end{aligned}$$

Passing to the logarithms, we get

$$\begin{aligned}
\ln \left( \frac{C_{2n}^{n-kc}}{C_{2n}^n} \right) &= \ln(1 + o(1)) - \ln \left( 1 - \frac{kc}{n} \right)^{n-kc} - \ln \left( 1 + \frac{kc}{n} \right)^{n+kc} \\
&= -(n - kc) \ln \left( 1 - \frac{kc}{n} \right) - (n + kc) \ln \left( 1 + \frac{kc}{n} \right) + o(1) \\
&= -(n - kc) \left( -\frac{kc}{n} - \frac{k^2 c^2}{2n^2} + o \left( \frac{k^2 c^2}{n^2} \right) \right) \\
&\quad - (n + kc) \left( \frac{kc}{n} - \frac{k^2 c^2}{2n^2} + o \left( \frac{k^2 c^2}{n^2} \right) \right) + o(1) \\
&= -\frac{k^2 c^2}{n} + (n - kc) o \left( \frac{k^2 c^2}{n^2} \right) + (n + kc) o \left( \frac{k^2 c^2}{n^2} \right) + o(1).
\end{aligned}$$

Since  $c = \lceil z\sqrt{2n} \rceil = z\sqrt{2n} + O(1)$ , we have

$$\ln \left( \frac{C_{2n}^{n-kc}}{C_{2n}^n} \right) = -2k^2 z^2 + o(1).$$

Therefore

$$\frac{C_{2n}^{n-kc}}{C_{2n}^n} = e^{-2k^2 z^2 + o(1)}.$$

Both statistics  $D_{n,n}^+$  and  $D_{n,n}$  are nonnegative, whence the corollary follows for  $z \leq 0$ . For  $z > 0$ , we apply the Gnedenko-Korolyuk theorem and the latter equality:

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n}^+ < z \right\} = \lim_{n \rightarrow \infty} \left( 1 - \frac{C_{2n}^{n-c}}{C_{2n}^n} \right) = \lim_{n \rightarrow \infty} \left( 1 - e^{-2z^2 + o(1)} \right) = 1 - e^{-2z^2},$$

whence equality (5) follows.

To prove (6) for some integer  $N$  we write

$$\begin{aligned}
(7) \quad \Delta(n, z) &= \left| \mathbf{P} \left\{ \sqrt{\frac{n}{2}} D_{n,n} < z \right\} - K(z) \right| \\
&= \left| \sum_{k=-\lceil n/c \rceil}^{\lfloor n/c \rfloor} (-1)^k \frac{C_{2n}^{n-kc}}{C_{2n}^n} - \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 z^2} \right| \\
&\leq \left| \sum_{|k| \geq N} (-1)^k e^{-2k^2 z^2} \right| + \left| \sum_{N \leq |k| < \lceil n/c \rceil} (-1)^k \frac{C_{2n}^{n-kc}}{C_{2n}^n} \right| \\
&\quad + \left| \sum_{|k| < N} (-1)^k \frac{C_{2n}^{n-kc}}{C_{2n}^n} - \sum_{|k| < N} (-1)^k e^{-2k^2 z^2} \right|.
\end{aligned}$$

Fix some  $\varepsilon > 0$  and  $z > 0$  and choose  $N$  such that

$$\exp(-2N^2 z^2) < \varepsilon/16.$$

The first term on the right hand side of inequality (7) is a double sum of a monotone sign-alternating series. The sum of this series is estimated by the absolute value of its first term. Thus this term is estimated from above by  $\varepsilon/8$ .

Since

$$C_{2n}^{n-kc} > C_{2n}^{n-(k+1)c},$$

a similar method applies to the second term on the right hand side of inequality (7). Therefore this term is estimated by  $4C_{2n}^{n-Nc}/C_{2n}^n$ . For large  $n$ , the bound is

$$4e^{-2N^2z^2+o(1)} < \varepsilon/3.$$

Further, consider the third term on the right hand side of (7) for a fixed  $N$ :

$$\left| \sum_{|k|<N} (-1)^k \left( \frac{C_{2n}^{n-kc}}{C_{2n}^n} - e^{-2k^2z^2} \right) \right|.$$

Since the sum is finite and every difference approaches zero as  $n \rightarrow \infty$ , there exists  $n_0$  for which the term of interest does not exceed  $\varepsilon/2$  for all  $n > n_0$ . Therefore

$$\Delta(n, z) \leq \frac{\varepsilon}{8} + \frac{\varepsilon}{3} + \frac{\varepsilon}{2} < \varepsilon$$

for all  $n > n_0$ . This implies equality (6).  $\square$

### 3. PROOF OF THE KOLMOGOROV THEOREM

Let  $W(t)$  be the standard Brownian motion and let  $W^0(t) = W(t) - tW(1)$  be the Brownian bridge, that is, a Gaussian stochastic process such that

$$\mathbb{E} W^0(t) = 0, \quad \mathbb{E} W^0(t)W^0(s) = \min(t, s) - ts.$$

If a distribution function  $F(t)$  is continuous, then Kolmogorov [1] noticed that the limit distribution of the statistic  $D_n$  does not depend on  $F(t)$ . Thus we assume without loss of generality that the sample  $\xi_1, \dots, \xi_n$  is uniformly distributed in the interval  $[0, 1]$  so that  $F(t) = t$  for  $t \in [0, 1]$ , and  $\beta_n(t) = \sqrt{n}(F_n^*(t) - t)$  is a uniform empirical process.

Put  $X_i(t) = I(\xi_i < t) - t$ . Then

$$\begin{aligned} \beta_n(t) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i(t) = \frac{S_n(t)}{\sqrt{n}}, \\ \mathbb{E} X_i(t) &= 0, \quad \mathbb{E} X_i(t)X_i(s) = \min(t, s) - ts, \\ \mathbb{E} \beta_n(t) &= 0, \quad \mathbb{E}(\beta_n(t)\beta_n(s)) = \frac{1}{n} \sum_{i=1}^n \mathbb{E} X_i(t)X_i(s) = \min(t, s) - ts. \end{aligned}$$

Thus  $\beta_n(t)$  has the same mean value and covariance function as the Brownian bridge. By the central limit theorem in  $\mathbb{R}^k$ ,

$$(8) \quad (\beta_n(t_1), \dots, \beta_n(t_k)) = \left( \frac{S_n(t_1)}{\sqrt{n}}, \dots, \frac{S_n(t_k)}{\sqrt{n}} \right) \xrightarrow{D} (W^0(t_1), \dots, W^0(t_k))$$

for arbitrary points

$$0 \leq t_1 < \dots < t_k \leq 1;$$

that is, the finite-dimensional distributions of  $\beta_n(t)$  weakly converge to those of  $W^0(t)$  as  $n \rightarrow \infty$ .

Note that

$$\sqrt{n}D_n^+ = \sup_{t \in [0,1]} \beta_n(t), \quad \sqrt{n}D_n = \sup_{t \in [0,1]} |\beta_n(t)|.$$

Thus the Kolmogorov theorem follows from

$$(9) \quad \max_{0 \leq t \leq 1} |\beta_n(t)| \xrightarrow{D} \max_{0 \leq t \leq 1} |W^0(t)|$$

as  $n \rightarrow \infty$  and

$$(10) \quad \mathbb{P} \left\{ \max_{0 \leq t \leq 1} |W^0(t)| < z \right\} = K(z).$$



In what follows we need the following auxiliary result.

**Lemma 3.1.** *For all  $\varepsilon > 0$ ,*

$$(11) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{|t-s| < \delta} |\beta_n(t) - \beta_n(s)| > \varepsilon \right\} = 0.$$

*Proof.* The reasoning below is a modification of a similar reasoning in the book [11, Chapter 6, §5]. Let  $S = \{k2^{-N} : N > 1, 0 \leq k \leq 2^N\}$  be the family of binary numbers in  $[0, 1]$  and let

$$\alpha_N = \sup_{1 \leq k \leq 2^N} \left| \beta_n \left( \frac{k}{2^N} \right) - \beta_n \left( \frac{k-1}{2^N} \right) \right|.$$

It is clear that

$$(12) \quad |\beta_n(s_1) - \beta_n(s_2)| \leq 2 \sum_{r=m(\delta)}^m \alpha_r$$

for all  $s_1, s_2 \in S$  such that  $s_1 = k_1 2^{-m}$ ,  $s_2 = k_2 2^{-m}$ , and  $|s_1 - s_2| < \delta$ , where

$$m(\delta) = \min \{k : \delta 2^k \geq 1\}.$$

Fix  $\delta > 0$  and let  $m$  be sufficiently large in order that  $\delta 2^{m-1} \geq 1$ . Let  $0 \leq s < t \leq 1$  and  $|s-t| < \delta$ . Then there are integer numbers  $k_1$  and  $k_2$  such that  $k_2 2^{-m} \leq s \leq (k_2+1)2^{-m}$  and  $k_1 2^{-m} \leq t \leq (k_1+1)2^{-m}$ . It is clear that the function  $\beta_n(t) + \sqrt{nt}$  increases in  $t$ , whence

$$\beta_n \left( \frac{k_1}{2^m} \right) + \sqrt{n} \left( \frac{k_1}{2^m} \right) \leq \beta_n(t) + \sqrt{nt} \leq \beta_n \left( \frac{k_1+1}{2^m} \right) + \sqrt{n} \left( \frac{k_1+1}{2^m} \right).$$

Similar bounds hold for  $\beta_n(s) + \sqrt{ns}$ , too. Then

$$|\beta_n(t) - \beta_n(s)| \leq \frac{2\sqrt{n}}{2^m} + \left| \beta_n \left( \frac{k_1+1}{2^m} \right) - \beta_n \left( \frac{k_2}{2^m} \right) \right| + \left| \beta_n \left( \frac{k_2+1}{2^m} \right) - \beta_n \left( \frac{k_1}{2^m} \right) \right|.$$

Since  $m$  is such that

$$|(k_1+1)2^{-m} - k_2 2^{-m}| \leq \delta + 2 \cdot 2^{-m} \leq 2\delta,$$

we have

$$(13) \quad \sup_{|t-s| < \delta} |\beta_n(t) - \beta_n(s)| \leq \frac{2\sqrt{n}}{2^m} + 2 \sup_{|k_1-k_2| \leq \delta 2^{m+1}} \left| \beta_n \left( \frac{k_1}{2^m} \right) - \beta_n \left( \frac{k_2}{2^m} \right) \right|.$$

Now we choose  $m = m_n$  such that

$$(14) \quad \frac{\sqrt{n}}{2^{m_n}} \rightarrow 0, \quad \frac{n}{2^{m_n}} \geq 1.$$

Put

$$\Delta_n = \sup_{|k_1-k_2| \leq \delta 2^{m_n}} |\beta_n(k_1 2^{-m_n}) - \beta_n(k_2 2^{-m_n})|.$$

In view of (13), it remains to prove that

$$(15) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \mathbf{P}\{\Delta_n > \varepsilon\} = 0.$$

We use bound (12) and the inequality

$$\alpha_N^p \leq \sum_{k=1}^{2^N} \left| \beta_n \left( \frac{k}{2^N} \right) - \beta_n \left( \frac{k-1}{2^N} \right) \right|^p.$$

Then

$$\begin{aligned}
\mathbb{E} \Delta_n &\leq 2 \sum_{r=m(\delta)}^{m_n} \mathbb{E} \alpha_r \leq 2 \sum_{r=m(\delta)}^{m_n} \mathbb{E} (\alpha_r^4)^{1/4} \\
(16) \quad &\leq 2 \sum_{r=m(\delta)}^{m_n} \left( \sum_{k=1}^{2^r} \mathbb{E} \left| \beta_n \left( \frac{k}{2^r} \right) - \beta_n \left( \frac{k-1}{2^r} \right) \right|^4 \right)^{1/4}.
\end{aligned}$$

Since

$$\beta_n(t+h) - \beta_n(t) = \sqrt{n} \left( \frac{\nu_n}{n} - h \right),$$

where  $\nu_n$  is a binomial random variable with parameters  $n$  and  $p = h$ , we obtain

$$\mathbb{E} |\beta_n(t+h) - \beta_n(t)|^4 = n^2 \mathbb{E} \left| \frac{\nu_n}{n} - h \right|^4 \leq 3 \left( h^2 + \frac{h}{n} \right) \leq 3h^2 + \frac{h}{2m_n} \leq 4h^2$$

(see [12, Chapter 6, §34]). Here we used inequality (14). Substituting the latter bound in (16) we get

$$\mathbb{E} \Delta_n \leq 2 \sum_{r=m(\delta)}^{m_n} \left( 2^r \frac{4}{2^{2r}} \right)^{1/4} \leq 2\sqrt{2} \sum_{r \geq m(\delta)} 2^{-r/4} \rightarrow 0, \quad \delta \rightarrow 0.$$

This implies equality (15), whence (11) follows.  $\square$

**Proposition 3.1.** *Let  $\beta_n(t)$  be a uniform empirical process and let  $W^0(t)$  be the Brownian bridge. Then (9) holds as  $n \rightarrow \infty$ .*

*Proof.* Put

$$\bar{\beta}_n = \max_{0 \leq t \leq 1} |\beta_n(t)|, \quad \bar{\beta}_n^{(k)} = \max_{0 \leq i \leq k} |\beta_n(i/k)|.$$

The convergence of finite-dimensional distributions (8) implies

$$\bar{\beta}_n^{(k)} \xrightarrow{D} \max_{0 \leq i \leq k} \left| W^0 \left( \frac{i}{k} \right) \right|, \quad n \rightarrow \infty.$$

It is clear that  $\mathbb{P}\{\bar{\beta}_n < x\} \leq \mathbb{P}\{\bar{\beta}_n^{(k)} < x\}$ . Thus

$$\overline{\lim}_{n \rightarrow \infty} \mathbb{P}\{\bar{\beta}_n < x\} \leq \mathbb{P} \left\{ \max_{0 \leq i \leq k} \left| W^0 \left( \frac{i}{k} \right) \right| < x \right\}.$$

Further

$$\begin{aligned}
\mathbb{P}\{\bar{\beta}_n \geq x\} &= \mathbb{P} \left\{ \bar{\beta}_n \geq x, \bar{\beta}_n^{(k)} > x - \varepsilon \right\} + \mathbb{P} \left\{ \bar{\beta}_n \geq x, \bar{\beta}_n^{(k)} \leq x - \varepsilon \right\} \\
&\leq \mathbb{P} \left\{ \bar{\beta}_n^{(k)} > x - \varepsilon \right\} + \mathbb{P} \left\{ \bar{\beta}_n \geq x, \bar{\beta}_n^{(k)} \leq x - \varepsilon \right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}\{\bar{\beta}_n < x\} &\geq \mathbb{P} \left\{ \bar{\beta}_n^{(k)} \leq x - \varepsilon \right\} - \mathbb{P} \left\{ \bar{\beta}_n \geq x, \bar{\beta}_n^{(k)} \leq x - \varepsilon \right\} \\
&\geq \mathbb{P} \left\{ \bar{\beta}_n^{(k)} \leq x - \varepsilon \right\} - P_n \left( \frac{1}{k} \right),
\end{aligned}$$

where

$$\begin{aligned}
P_n(\delta) &= \mathbb{P} \left\{ \sup_{|t-s| \leq \delta} |\beta_n(t) - \beta_n(s)| > \varepsilon \right\} \\
&\geq \mathbb{P} \left\{ \max_{0 \leq t \leq 1} |\beta_n(t)| \geq x, \max_{0 \leq i \leq k} \left| \beta_n \left( \frac{i}{k} \right) \right| \leq x - \varepsilon \right\}.
\end{aligned}$$

Combining together all the above estimates for probabilities and passing to the limit as  $n \rightarrow \infty$  we obtain

$$\begin{aligned} \mathbb{P} \left\{ \max_{0 \leq i \leq k} \left| W^0 \left( \frac{i}{k} \right) \right| < x \right\} &\geq \overline{\lim}_{n \rightarrow \infty} \mathbb{P} \{ \bar{\beta}_n < x \} \geq \underline{\lim}_{n \rightarrow \infty} \mathbb{P} \{ \bar{\beta}_n < x \} \\ &\geq \mathbb{P} \left\{ \max_{0 \leq i \leq k} \left| W^0 \left( \frac{i}{k} \right) \right| < x - \varepsilon \right\} - \overline{\lim} P_n \left( \frac{1}{k} \right). \end{aligned}$$

The Brownian bridge  $W^0(t)$  as well as the Brownian process  $W(t)$  is continuous. Therefore

$$\mathbb{P} \left\{ \max_{0 \leq i \leq k} \left| W^0 \left( \frac{i}{k} \right) \right| < x \right\} \rightarrow \mathbb{P} \left\{ \sup_{0 \leq t \leq 1} |W^0(t)| < x \right\}$$

as  $k \rightarrow \infty$ . Moreover Lemma 3.1 implies that  $\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P_n(\delta) = 0$ . Passing to the limit in the above inequalities as  $k \rightarrow \infty$  and then as  $\varepsilon \rightarrow 0$  we get

$$\lim_{n \rightarrow \infty} \mathbb{P} \{ \bar{\beta}_n < x \} = \mathbb{P} \left\{ \sup_{0 \leq t \leq 1} |W^0(t)| < x \right\}$$

for all  $x$ , where the distribution function is right continuous. This completes the proof of Proposition 3.1.  $\square$

It remains to show that (10) holds in order to complete the proof of Kolmogorov's theorem. Of course, this is a well-known result. However its direct proof is not trivial at all (see, for example, [5, 9]). Below we present a new proof of equality (10) based on the approach proposed by Gnedenko and Korolyuk.

Consider two empirical distribution functions  $F_n^*(t)$  and  $G_n^*(t)$  constructed from independent samples uniformly distributed in the interval  $[0, 1]$ ; both samples are of size  $n$ . Put

$$\beta_n^{(s)}(t) = \sqrt{\frac{n}{2}} (F_n^*(t) - G_n^*(t)).$$

It is clear that the process  $\beta_n^{(s)}(t)$  can be represented as follows:

$$(17) \quad \beta_n^{(s)}(t) = \frac{\beta_n'(t) - \beta_n''(t)}{\sqrt{2}},$$

where  $\beta_n'(t)$  and  $\beta_n''(t)$  are two independent uniform empirical processes. Thus

$$\mathbb{E} \beta_n^{(s)}(t) = 0, \quad \mathbb{E} \beta_n^{(s)}(t) \beta_n^{(s)}(s) = \min(t, s) - ts.$$

As in the case of the process  $\beta_n(t)$ , the latter equality implies the convergence of finite dimensional distributions of  $\beta_n^{(s)}(t)$  to those of the Brownian bridge  $W^0(t)$ .

Lemma 3.1 and representation (17) imply the following result.

**Lemma 3.2.** *For all  $\varepsilon > 0$ ,*

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{|t-s| < \delta} |\beta_n^{(s)}(t) - \beta_n^{(s)}(s)| > \varepsilon \right\} = 0.$$

Using Lemma 3.2 and following the lines of the proof of Proposition 3.1 we get the following result.

**Proposition 3.2.**

$$\sup_{0 \leq t \leq 1} |\beta_n^{(s)}(t)| \xrightarrow{D} \sup_{0 \leq t \leq 1} |W^0(t)|.$$

Equality (10) follows from Proposition 3.2, since

$$\sqrt{\frac{n}{2}}D_{n,n} = \sup_{0 \leq t \leq 1} |\beta_n^{(s)}(t)|$$

and in view of Corollary 2.2. The Kolmogorov theorem is proved.

#### 4. REMARKS

1. The Smirnov limit theorem

$$\lim_{n \rightarrow \infty} \mathbf{P} \{ \sqrt{n}D_n^+ < z \} = 1 - e^{-2z^2}$$

can also be proved by the Gnedenko–Korolyuk method.

**2. Discrete distributions.** Recall that equalities (1)–(4) are proved for the case of a continuous distribution function  $F(t)$ . It is known in the general case that the empirical process weakly converges to the process  $W^0(F(t))$ , where  $W^0(t)$  is a Brownian bridge ([9, Chapter 3, §16, Theorem 16.4]). If the distribution function  $F(t)$  is discontinuous, then the distribution of the random variable  $\sup_t |W^0(F(t))|$  is not the Kolmogorov distribution and depends on the distribution function  $F(t)$  (see [13]). Naturally, Theorem 1.2 does not hold for discrete random variables. Based on the Gnedenko–Korolyuk approach, we propose to use some randomized statistics  $DR_{n,n}^+$  and  $DR_{n,n}$  (instead of Smirnov's statistic) whose distribution does not depend on the distribution function  $F(t)$ .

Let the samples  $(\xi_1, \xi_2, \dots, \xi_n)$  and  $(\eta_1, \eta_2, \dots, \eta_n)$  be the observations after the random variables  $\xi$  and  $\eta$ , respectively. Without loss of generality, we may assume that their values are  $(1, 2, \dots, m, \dots)$ . To construct statistics  $DR_{n,n}^+$  and  $DR_{n,n}$  we put, similarly to the continuous case,

$$z_1 = \xi_1, \quad \dots, \quad z_n = \xi_n, \quad z_{n+1} = \eta_1, \quad \dots, \quad z_{2n} = \eta_n.$$

Then we write the sample  $(z_i)$  in the ascending order

$$(18) \quad z_{(1)} \leq z_{(2)} \leq \dots \leq z_{(2n)}.$$

Fix some  $k \geq 1$  and let

$$(19) \quad z_{(i)} = k \quad \text{for } i = i_{k-1} + 1, \dots, i_k, \quad i_k - i_{k-1} = m_k$$

(for example, we may assume that the first  $i$  members are taken from the sample  $(\xi_k)$  and the last  $i$  members are taken from the sample  $(\eta_k)$ ). We introduce a partial order in sequence (19); namely, if  $\nu_k = (\nu_{k1}, \dots, \nu_{km_k})$  is a uniform random variable defined on all permutations of the sequence  $(i_{k-1} + 1, \dots, i_k)$  and if  $\nu_k$  does not depend on samples  $(\xi_i)$  and  $(\eta_i)$ , then we make the following transformation:

$$(i_{k-1} + 1, \dots, i_k) \rightarrow (\nu_{k1}, \dots, \nu_{km_k}).$$

After this randomization is done for all  $k \geq 1$ , we obtain the sequence

$$(20) \quad z_{(1)}^* \leq z_{(2)}^* \leq \dots \leq z_{(2n)}^*$$

instead of the sequence (18). The further procedure is analogous to that considered in the continuous case. Put

$$\varepsilon_k^* = \begin{cases} 1, & z_{(k)}^* \in (\xi_i), \\ -1, & z_{(k)}^* \in (\eta_i), \end{cases}$$

$$S_n^* = \sum_{k=1}^n \varepsilon_k^*, \quad S_0^* = 0, \quad S_{2n}^* = 0,$$

$$DR_{n,n}^+ = \sup_{0 \leq k \leq 2n} S_k^*, \quad DR_{n,n} = \sup_{0 \leq k \leq 2n} |S_k^*|.$$

Then the Gnedenko-Korolyuk theorem implies the following corollary.

**Corollary 4.1.** *Let the random variables  $\xi$  and  $\eta$  have the same discrete distribution. If  $D_{n,n}^+$  and  $D_{n,n}$  are changed in Theorem 1.2 for  $DR_{n,n}^+$  and  $DR_{n,n}$ , respectively, then equalities (3) and (4) remain true.*

**3. Testing compound hypotheses.** It is an important problem to decide whether a sample is drawn from a given parametric family of distributions. The theory of this problem is quite complicated, and its results are not easy to use in practice (see [8]).

Let  $\mathbf{F} = \{F((t-a)/\sigma), (a, \sigma) \in Q\}$  be a family of distribution functions depending on the shift and scale parameters,  $a$  and  $\sigma$ , respectively, where  $F(t)$  is a continuous distribution function. Consider a sample  $(\xi_i)$  of size  $n$  drawn from an unknown distribution function  $F_0(t)$ . We want to test the hypothesis  $H_0 = \{F_0 \in \mathbf{F}\}$ . A modified Kolmogorov statistic

$$\hat{D}_n = \sup_{-\infty < t < \infty} \left| F_n^*(t) - F\left(\frac{t-\hat{a}}{\hat{\sigma}}\right) \right|$$

is often used for this purpose where  $\hat{a}$  and  $\hat{\sigma}$  are maximum likelihood estimates of the parameters  $a$  and  $\sigma$ . It is known that the limit distribution of the statistic  $\hat{D}_n$  depends on the distribution function  $F(t)$  (see Theorems 1 and 2 in [14, Chapter 3, §17]). Thus one needs a separate table of critical points of the distribution of  $\hat{D}_n$  for every distribution function  $F(t)$  in order to test the above hypothesis.

Using the Gnedenko-Korolyuk theorem one can construct the following simple procedure for testing the hypothesis  $H_0$ .

- a) Given a sample  $(\xi_i)$ , simulate a sample  $(\eta_i)$  of the same size  $n$  and with the distribution function  $F(t)$  (we assume that  $a = 0$  and  $\sigma = 1$ ).
- b) Starting from the samples  $(\xi_i)$  and  $(\eta_i)$ , construct new samples  $(\bar{\xi}_j)$  and  $(\bar{\eta}_j)$ , where

$$\bar{\xi}_j = \frac{\xi_{4j-2} - \xi_{4j-3}}{\xi_{4j} - \xi_{4j-1}}, \quad \bar{\eta}_j = \frac{\eta_{4j-2} - \eta_{4j-3}}{\eta_{4j} - \eta_{4j-1}}, \quad j = 1, 2, \dots, m, \quad m = \left\lfloor \frac{n}{4} \right\rfloor.$$

- c) Evaluate the empirical distribution functions  $\bar{F}_m^*(t)$  and  $\bar{G}_m^*(t)$  from the samples  $(\bar{\xi}_j)$  and  $(\bar{\eta}_j)$ , respectively, and the corresponding statistic  $\bar{D}_{m,m}$ . If the hypothesis  $H_0$  is true, then the random variable  $\sqrt{\frac{m}{2}} \bar{D}_{m,m}$  has the distribution  $\Phi_m(z)$  defined by equality (4).
- d) Now the hypothesis  $H_0$  can be tested in a standard way.

It is clear that a large portion of information is lost when passing from the sample  $(\xi_j)$  to the samples  $(\bar{\xi}_j)$  and  $(\bar{\eta}_j)$ . Nevertheless the losses are negligible if the size of the samples is large.

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