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ON THE ABSOLUTE CONTINUITY OF FIXED POINTS OF SMOOTHING TRANSFORMS

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Abstract. Let the distribution of a nonnegative random variable W be such that $W \overset{d}{=} \sum_{i=1}^{J} Y_i W_i$, where $\{Y_i\colon i = 1, \dots, J\}$ are some positive random variables. Under some moment conditions imposed on Y_i we show that the distribution of W is a mixture of the atom at the origin and an absolutely continuous component.

1. Introduction and statement of the result

Consider a family of positive random points $\{Y_i\colon i = 1, \dots, J\}$, where the number J can be deterministic or random as well as finite or infinite with positive probability. Let $\mathcal{L}(\xi)$ denote the distribution of a random variable ξ and let $\mathcal{P}^+$ be the set of probability measures on $[0, \infty)$. The definitions given below can be found in the book [5]. A transform

$$\mathbb{T}\colon \mathcal{P}^+ \to \mathcal{P}^+ \cup \{\delta_\infty\}, \qquad \mathcal{L}(Z) \mapsto \mathcal{L}\left(\sum_{i=1}^{J} Y_i Z_i\right)$$

is called a *homogeneous smoothing transform* if $\{Z_i\colon i \in \mathbf{N}\}$ are independent copies of a random variable Z and if the sequence $\{Z_i\colon i \in \mathbf{N}\}$ is independent of $\{Y_i\colon i = 1, \dots, J\}$.

The distribution of a nonnegative random variable W is called a *fixed point* of a smoothing transform if the following equality holds for the distributions:

$$W \overset{d}{=} \sum_{i=1}^{J} Y_i W_i, \tag{1}$$

where $\{W_i\}$ are independent copies of W and if the sequence $\{W_i\}$ is independent of $\{Y_i\colon i = 1, \dots, J\}$.

Let $\alpha \in (0, 1]$ be given. We say that a distribution μ_α is an α*-elementary fixed point* of a transform $\mathbb{T}$ if its Laplace–Stieltjes transform φ_α is such that

$$\lim_{s \to +0} \frac{1 - \varphi_\alpha(s)}{s^\alpha} = m$$

for some finite number $m > 0$. Note that a fixed point is 1-elementary if and only if it has a finite mean. The set of *elementary fixed points* is the union of all α-elementary fixed points with respect to $\alpha \in (0, 1]$. A fixed point is called *nonelementary* if there is no $\alpha \in (0, 1]$ for which the point is α-elementary.

Corollary 2.4.1 of [5] implies that a 1-fixed point is either a degenerate distribution at the point 1 or it is a mixture of an atom at the origin and a purely continuous component.

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Now we state the main result of the paper, which improves Corollary 2.4.1 of [5] in the sense that it contains conditions under which the purely continuous component is absolutely continuous.

Theorem 1.1. *a) Assume that $J = \infty$ almost surely and that μ is a nondegenerate 1-elementary fixed point. If, for some $j \in \mathbf{N}$ and $a > 0$,*

$$\mathsf{E}\, Y_j^{-a} < \infty,$$

then μ is an absolutely continuous distribution.

b) Let μ be an α-elementary fixed point, $\alpha \in (0,1)$. Then, for some $p \in [0,1)$,

$$\mu(dx) = p\, \delta_0(dx) + (1-p)\, \nu(dx), \tag{2}$$

where ν is an absolutely continuous distribution.

The absolute continuity of fixed points of probability distributions is studied by other authors as well. In particular, the paper [3] contains sufficient conditions for the absolute continuity of the distribution of the limit random variable for the Bellman–Harris branching process satisfying the equality $Z \stackrel{d}{=} A(Z_1 + \cdots + Z_N)$, where $Z_1, Z_2, \dots$ are independent copies of Z such that the sequence $\{Z_i\}$ is independent of (A, N). In the papers [1] and [4], sufficient conditions are found for the existence of an absolutely continuous component of a distribution satisfying equality (1) with $J < \infty$ almost surely. Sufficient conditions are given in [2] for the absolute continuity of a fixed point for the case where the Y_i are points of a Poisson point process.

2. Proof of Theorem 1.1

We study fixed points with a finite mean. Denote by $\phi(t)$ the characteristic function of such a fixed point. According to the assumption of the theorem, $J = \infty$ almost surely. Thus equality (1) is equivalent to the following one:

$$\phi(t) = \mathsf{E} \prod_{i=1}^{\infty} \phi(Y_i t). \tag{3}$$

We need some auxiliary results.

Lemma 2.1. *If $J = \infty$ almost surely, then*

$$\lim_{t\to\infty} |\phi(t)| = 0.$$

Proof. Put $q := \overline{\lim}_{t\to\infty} |\phi(t)|$. Equality (3) implies that

$$q \le \overline{\lim_{t\to\infty}}\, \mathsf{E} \prod_{i=1}^{\infty} |\phi(Y_i t)| = \prod_{i=1}^{\infty} q.$$

It is clear that the latter inequality may hold either for $q = 0$ or for $q = 1$ only. Now we prove that $q < 1$, whence we conclude that $q = 0$. Since a fixed point has a finite mean, we have $\mathsf{E} \sum_{i=1}^{\infty} Y_i = 1$. Thus the expectation of the number of those Y_i that do not exceed 1 is strictly greater than 1. Thus one can choose $\delta \in (0,1)$ such that $\mathsf{E}\, T_\delta > 1$, where $T_\delta := \#\{i \colon Y_i \in [\delta, 1]\}$.

Assume that $q = 1$. Corollary 2.4.1 of [5] implies that the distribution of a fixed point is continuous. Thus $|\phi(t)| < 1$ for all nonzero real t. Fix an arbitrary $\varepsilon > 0$ and choose t_1 and t_2 such that $t_1 < \delta t_2$ and

$$|\phi(t_1)| = |\phi(t_2)| = 1 - \varepsilon \quad \text{and} \quad |\phi(t)| < 1 - \varepsilon \qquad \text{for } t_1 < t < t_2.$$

Further

$$1-\varepsilon = |\phi(t_2)| \le \mathsf{E}\prod_{i=1}^{\infty}|\phi(Y_i t_2)| \le \mathsf{E}\prod_{r\in R}|\phi(Y_r t_2)|,$$

where $R=\{r\colon t_1/t_2 < Y_r \le 1\}$. Thus we derive from the latter inequality that

$$1-\varepsilon \le \mathsf{E}(1-\varepsilon)^{T_\delta}$$

for all sufficiently small ε. This result contradicts the inequality $\mathsf{E}\,T_\delta > 1$. $\square$

Lemma 2.2. *Let*

$$f(t) \le p\,\mathsf{E}\,f(At)$$

for a nonnegative bounded Borel function f, positive random variable A, some $p\in(0,1)$, $t_0\ge 0$ and all $t\ge t_0$. If $p\,\mathsf{E}(A^{-a})<1$ for some $a>0$, then

$$f(t)=O(t^{-a}), \qquad t\to\infty.$$

The proof of Lemma 2.2 can be found in [3].

The following result is Lemma 5.3.2 in the monograph [5].

Lemma 2.3. *Let η_1 and η_2 be independent nonnegative random variables with the distributions χ_1 and χ_2, respectively. If χ_1 is an absolutely continuous distribution with density g_1, then the distribution χ of the random variable $\eta_1\eta_2$ is given by $\chi(dx) = p\,\delta_0(dx)+g(x)\,dx$, where $p:=\chi_2\{0\}\in[0,1)$ and $g(x):=\int_{+0}^{\infty}y^{-1}g_1(xy^{-1})\,\chi_2(dy)$ is the density of the absolutely continuous component of χ. If the n-th derivative of g_1 is bounded and continuous, then the n-th derivative of g exists and is continuous.*

Now we turn to the proof of the theorem. Choose an arbitrary fixed $\varepsilon\in(0,1)$ and $\delta>0$. Put $N_\delta := \sum_{i=1}^{\infty}1_{\{Y_i>\delta\}}$. Since the mean of a fixed point is finite, we have $\mathsf{E}\sum_{i=1}^{\infty}Y_i=1$. Thus

$$1\ge \mathsf{E}\sum_{i=1}^{\infty}Y_i 1_{\{Y_i>\delta\}} \ge \delta\,\mathsf{E}\,N_\delta,$$

whence we conclude that $N_\delta<\infty$ almost surely. By the assumption of the theorem, $N_\delta\uparrow\infty$ as $\delta\downarrow 0$. Lemma 2.1 implies that there exists $t_\varepsilon>0$ such that $|\phi(t)|<\varepsilon$ for all $t>t_\varepsilon$. Thus we obtain from equality (3) that

$$|\phi(t)| \le \mathsf{E}\prod_{i=1}^{\infty}|\phi(Y_i t)| \le \mathsf{E}\,|\phi(Y_j t)|\left[\varepsilon^{N_\delta-1}1_{\{N_\delta\ge1\}}+1_{\{N_\delta=0\}}\right] = p_{\varepsilon,\delta}\,\mathsf{E}\,|\phi(\widetilde{Y}_j t)|$$

for $t>t_\varepsilon/\delta$, where $p_{\varepsilon,\delta}:=\mathsf{E}\,\varepsilon^{N_\delta-1}1_{\{N_\delta\ge1\}}+\mathsf{P}\{N_\delta=0\}$, Y_j is a random variable whose distribution is as in Theorem 1.1, and $\widetilde{Y}_j$ is a positive random variable whose distribution is determined by

$$\mathsf{E}\,g(\widetilde{Y}_j)=\frac{1}{p_{\varepsilon,\delta}}\,\mathsf{E}\,g(Y_j)\left[\varepsilon^{N_\delta-1}1_{\{N_\delta\ge1\}}+1_{\{N_\delta=0\}}\right]$$

considered for an arbitrary nonnegative bounded Borel function g. According to the Lebesgue bounded convergence theorem, $p_{\varepsilon,\delta}\to0$ as $\delta\downarrow0$ and

$$p_{\varepsilon,\delta}\,\mathsf{E}\,\widetilde{Y}_1^{-a}=\mathsf{E}\,Y_1^{-a}\left[\varepsilon^{N_\delta-1}1_{\{N_\delta\ge1\}}+1_{\{N_\delta=0\}}\right]\to0.$$

Thus $p_{\varepsilon,\delta}<1$ for sufficiently small $\delta>0$ and $p_{\varepsilon,\delta}\,\mathsf{E}(\widetilde{Y}_1)^{-a}<1$. Hence

$$|\phi(t)|=O(|t|^{-a}), \qquad |t|\to\infty, \tag{4}$$

in view of Lemma 2.2 and since the function $|\phi(t)|$ is even. If $a>1$, then $|\phi(t)|$ is integrable on the whole axis. Thus a fixed point is absolutely continuous. Assume that $a\in(0,1]$. Choose a natural number n such that $na>1$. For all (nonrandom) nonnegative

$\alpha_1, \dots, \alpha_n$, not all of which are equal to 1, the distribution of $\alpha_1 W_1 + \dots + \alpha_n W_n$ is absolutely continuous, since relation (4) implies that

$$|\mathsf{E}\exp(it(\alpha_1 W_1 + \dots + \alpha_n W_n))| = O(|t|^{-na}), \qquad |t| \to \infty.$$

Therefore

$$\mathsf{P}\left\{\sum_{i=1}^{n} Y_i W_i \in C \;\middle|\; \sigma(Y_j \colon j \in \mathbf{N})\right\} = 0 \quad \text{almost surely}$$

for an arbitrary Borel set $C \subset [0,\infty)$ of zero Lebesgue measure.

Thus

$$\mathsf{P}\left\{\sum_{i=1}^{n} Y_i W_i + \sum_{i=n+1}^{\infty} Y_i W_i \in C \;\middle|\; \sigma(Y_j \colon j \in \mathbf{N})\right\} = 0$$

almost surely. Passing to the mathematical expectation in the latter equality, we get

$$\mathsf{P}\left\{\sum_{i=1}^{\infty} Y_i W_i \in C\right\} = 0,$$

whence we obtain the absolute continuity of fixed points with finite mean.

Now we study α-elementary fixed points, $\alpha \in (0,1)$. Consider the following modified transform:

$$\mathbb{T}_\alpha \colon \mathcal{P}^+ \to \mathcal{P}^+ \cup \{\delta_\infty\}, \qquad \mathcal{L}(Z) \mapsto \mathcal{L}\left(\sum_{i=1}^{J} Y_i^\alpha Z_i\right).$$

By Proposition 3.4.1 of [5], the distribution μ_α of an α-elementary fixed point is such that

$$\mu_\alpha(x,\infty) = \int_0^\infty s_\alpha\left(xt^{-1/\alpha}, \infty\right)\, \mu_1(dx), \qquad x > 0,$$

where s_α is a strictly stable positive distribution with index α and where μ_1 is a fixed point with the finite mean of the modified transform $\mathbb{T}_\alpha$. By Lemma 2.3, the distribution μ_α admits representation (2), since all stable distributions are absolutely continuous.

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