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CONVERGENCE OF OPTION REWARDS FOR MARKOV TYPE PRICE PROCESSES MODULATED BY STOCHASTIC INDICES. I

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D. S. SILVESTROV, H. JÖNSSON, AND F. STENBERG

Abstract. A general price process represented by a two-component Markov process is considered. Its first component is interpreted as a price process and the second component as an index process modulating the price component. American type options with pay-off functions admitting power type upper bounds are studied. Both the transition characteristics of the price processes and the pay-off functions are assumed to depend on a perturbation parameter $\delta \geq 0$ and to converge to the corresponding limit characteristics as $\delta \to 0$. In the first part of the paper, asymptotically uniform skeleton approximations connecting reward functionals for continuous and discrete time models are given. In the second part of the paper, these skeleton approximations are used for obtaining results about the convergence of reward functionals of American type options with perturbed price processes for both cases of discrete and continuous time. Examples related to modulated exponential price processes with independent increments are given.

1. Introduction

This paper is devoted to studies of conditions for the convergence of reward functionals of American type options with Markov type price processes modulated by stochastic indices.

The idea behind these models is that the randomness of these models depends on the global market environment, in particular on some indicators or indices. One example would be a model where the price process depends on the level of market index reflecting a bullish, bearish, or stable market behavior. Another example is a model where the overall market volatility is indicating a high, moderate, or low volatility environment.

The main objective of the present paper is to study the continuous time optimal stopping problem originating from the American option pricing under these processes, to derive approximations of the reward functionals for the continuous time models by imbedded discrete time models, and to prove the convergence of these reward functionals.

Markov type price processes modulated by stochastic indices and option pricing for such processes were studied in Hull and White (1987) [23], Kijima and Yoshida (1993) [29], Naik (1993) [38], Di Masi, Kabanov, and Runggaldier (1995) [10], Svishchuk (1995) [53], Bollen (1998) [4], Shiryaev (1999) [44], Guo (2001a) [20], (2001b) [21], Kukush and Silvestrov (2000) [31], (2001) [32], (2004) [33], Buffington and Elliot (2002) [5], Guo and Zhang (2004) [22], Silvestrov and Stenberg (2004) [51], Elliot, Chan, and Su (2005) [15], Aingworth, Das, and Motwani (2006) [1], Di Graziano and Rogers (2006) [9], Jobert and

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Rogers (2006) [24], Peskir and Shiryaev (2006) [40], Yao, Zhang, and Zhou (2006) [54], and Stenberg (2006) [52].

We would also like to mention the books by Rolski, Schmidli, Schmidt, and Teugels (1999) [42], Shiryaev (1999) [44], Shreve (2004) [45], (2005) [46], and Peskir and Shiryaev (2006) [40] for an account of various models of stochastic price processes and optimal stopping problems for options. The books by Silvestrov (1980) [48] and Koroliuk and Limnios (2005) [30] contain a description of a variety of models of stochastic processes with semi-Markov modulation (switchings).

We consider a variant of price processes modulated by stochastic indices as was introduced in Kukush and Silvestrov (2000) [31], (2001) [32], (2004) [33]. The object of our studies is a two-component process $Z^{(\delta)}(t) = \big(Y^{(\delta)}(t), X^{(\delta)}(t)\big)$, where the first component $Y^{(\delta)}(t)$ is a real-valued càdlàg process and the second component $X^{(\delta)}(t)$ is a measurable process with a general metric phase space. The first component is interpreted as a log-price process while the second component is interpreted as a stochastic index modulating the price process.

As mentioned above, one can treat the process $X^{(\delta)}(t)$ as a global price index "modulating" market prices, or a jump process representing some market regime index. The stochastic index can indicate, for example, a growing, declining, or stable market situation, or a high, moderate, or low level of volatility, or describe a credit rating dynamics modulating the price process $Y^{(\delta)}(t)$.

The log-price process $Y^{(\delta)}(t)$ as well as the corresponding price process

$$S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$$

are not assumed to be Markov processes, but the two-component process $Z^{(\delta)}(t)$ is assumed to be a continuous time inhomogeneous two-component Markov process. Thus, the component $X^{(\delta)}(t)$ reflects the information which together with the information represented by the log-price process $Y^{(\delta)}(t)$ allows one to assume that the two-component vector $\big(Y^{(\delta)}(t), X^{(\delta)}(t)\big)$ is a Markov stochastic process.

In the literature, the values of options in discrete time markets have been used to approximate the value of the corresponding option in continuous time. The paper by Cox, Ross and Rubinstein (1979) [7] is a seminal paper, where the convergence of European option values for the binomial tree model to the Black–Scholes value for geometrical Brownian motion was shown.

Further results on the convergence of the values of European and American options can be found in Barone-Adesi and Whaley (1987) [3], Lamberton (1993) [34], Amin and Khanna (1994) [2], Cutland, Kopp, Willinger, and Wyman (1997) [8], Mulinacci and Pratelli (1998) [37], Prigent (2003) [41], Neiuwenhuis and Vellekoop (2004) [39], Silvestrov and Stenberg (2004) [51], Dupuis and Wang (2005) [14], Jönsson (2005) [26], Coquet and Toldo (2007) [6], and Stenberg (2006) [52]. In particular, Amin and Khanna (1994) [2] gave conditions for the convergence of the values of American options in a discrete-time model to the value of the option in a continuous-time model, provided that the sequence of processes describing the value of the underlying asset converges weakly to a diffusion. They also show results when the limiting process is a diffusion with discrete jumps at fixed moments. Martingale based methods are used in the book by Prigent (2003) [41]; the topic of this book covers the recent results on the weak convergence in financial markets, both for European and American type options. We would also like to mention the papers by Mackevičius (1973) [35], (1975) [36], Fährmann (1978) [16], (1979) [17], (1982) [18], Dochviri and Shashiashvili (1992) [13], and Dochviri (1988) [11], (1993) [12], where the convergence in optimal stopping problems is studied for general Markov processes.

It is well known that explicit formulas for optimal rewards for American type options do not exist even for standard pay-off functions and simple price processes. The methods used in this case are based on approximations of price processes by simpler ones, for example by binomial tree price processes. Models with complex nonstandard pay-off functions may also require an approximation of these pay-offs by simpler ones, for example by piecewise linear pay-off functions. Results concerning the convergence of rewards for perturbed price processes play a crucial role and serve as a justification for the corresponding approximation algorithms.

Our results differ from those in the papers mentioned above by a generality of models for price processes and nonstandard pay-off functions as well as conditions for the convergence.

We consider very general models of càdlàg Markov type price processes modulated by stochastic indices. So far, conditions for the convergence of rewards have not been investigated for such general models.

We consider the so-called triangular array models, in which the processes under consideration depend on a small perturbation parameter $\delta \geq 0$. It is assumed that transition probabilities of the perturbed processes $Z^{(\delta)}(t)$ converge in some sense to the corresponding transition probabilities of the limiting process $Z^{(0)}(t)$ as $\delta \to 0$. Thus the processes $Z^{(\delta)}(t)$ can be considered to be a perturbed modification of the corresponding limit process $Z^{(0)}(t)$. An example is the binomial tree model converging to the corresponding geometrical Brownian motion.

We do not assume explicitly the condition of the finite-dimensional weak convergence for the corresponding processes, which is usual in general limit theorems for Markov type processes. Our conditions also do not use any assumptions on the convergence of auxiliary processes in probability which are common in martingale based methods. The latter type of conditions usually do involve some special imbedding constructions replacing perturbed and limiting processes on a common probability space that may be difficult to realize for complex models of price processes.

Instead of the conditions mentioned above, we introduce new general conditions for the local uniform convergence of the corresponding transition probabilities. These conditions do imply the finite-dimensional weak convergence for the price processes and can be effectively used in applications. We also use effective conditions of the exponential moment compactness for the increments of the log-price processes, which are natural for applications to Markov type processes.

We also consider American type options with non-standard pay-off functions $g^{(\delta)}(t, s)$, which are non-negative and whose growth is not faster than polynomial. The pay-off functions are also assumed to be perturbed and to converge as $\delta \to 0$ to the corresponding limit pay-off functions $g^{(0)}(t, s)$. This is a useful assumption. For example, Kukush and Silvestrov (2000) [31] showed how one can approximate reward functions for options with general convex pay-off functions by reward functions for options with more simple piece-wise linear pay-off functions.

As is well known, the optimal stopping moment for the exercise of an American option has the form of the first hitting time into the optimal price-time stopping domain. It is worth noting that, under the general assumptions on the pay-off functions listed above, the structure of the reward functions and the corresponding optimal stopping domain can be rather complicated. For example, as shown in Kukush and Silvestrov (2000) [31], (2001) [32], (2004) [33], Jönsson (2001) [25], and Jönsson, Kukush, and Silvestrov (2004) [27], (2005) [28], the optimal stopping domains can possess a multi-threshold structure.

Despite the complex structure of the optimal stopping domains, we can prove the convergence of the reward functionals which represent the optimal expected rewards in the class of all Markov stopping moments.

Our approach is based on the skeleton approximations for price processes given in Kukush and Silvestrov (2001) [32], where continuous time reward functionals have been approximated by their analogs for imbedded skeleton type discrete time models. In that paper, the skeleton approximations are given in a form suitable for applications to continuous price processes. We improve these approximations to be useful for càdlàg price processes and, moreover, prove that the latter approximations are asymptotically uniform as the perturbation parameter δ tends to 0. Another important element of our approach is a recursive method for an asymptotic analysis of reward functionals for discrete time models developed in Jönsson (2005) [26]. Key examples of price processes modulated by semi-Markov indices and the corresponding convergence results are also given in Silvestrov and Stenberg (2004) [51] and Stenberg (2006) [52].

The paper contains two parts. The first part includes an introduction and two sections. In Section 2, we introduce Markov type price processes modulated by stochastic indices and American type options with general pay-off functions. Section 3 contains results about asymptotically uniform skeleton approximations. These results have their own value and allow one to approximate reward functionals for continuous time price processes by similar functionals for simpler imbedded discrete time models.

The second part of the paper contains four sections. In Section 4, results concerning conditions for the convergence of reward functionals in discrete time models are given. Section 5 presents general results on the convergence of reward functionals for American type options. In Sections 6 and 7, we illustrate our general convergence results by applying them to exponential price processes with independent increments and exponential Lévy price processes modulated by semi-Markov stochastic indices, and some other models.

This paper is an improved and extended version of the report by Silvestrov, Jönsson, and Stenberg (2006) [49]. The main results are also presented in a short paper by Silvestrov, Jönsson, and Stenberg (2007) [50].

2. American type options under price processes modulated by stochastic indices

Let, for every $\delta \geq 0$, $Z^{(\delta)}(t) = \big(Y^{(\delta)}(t), X^{(\delta)}(t)\big)$, $t \geq 0$, be a Markov process with the phase space $\mathbf{Z} = \mathbb{R}_1 \times \mathbb{X}$, where $\mathbb{R}_1$ is the real line and $\mathbb{X}$ is a Polish space (a separable, complete metric space). The transition probabilities and initial distribution of $Z^{(\delta)}(t)$ are denoted by $P^{(\delta)}(t, z, t+u, A)$ and $P^{(\delta)}(A)$, respectively.

Note that $\mathbf{Z}$ is a Polish space for the metrics

$$d_{\mathbf{Z}}(z', z'') = \big(|y' - y''|^2 + d_{\mathbb{X}}(x', x'')^2\big)^{1/2},$$

where $z' = (y', x')$, $z'' = (y'', x'')$, and $d_{\mathbb{X}}(x', x'')$ is the metric in the space $\mathbb{X}$.

Let $\mathcal{B}_{\mathbf{Z}} = \sigma(\mathcal{B}_1 \times \mathcal{B}_{\mathbb{X}})$ be the Borel σ-field, where $\mathcal{B}_1$ and $\mathcal{B}_{\mathbb{X}}$ are the Borel σ-fields in $\mathbb{R}_1$ and $\mathbb{X}$, respectively, and let the transition probabilities and the initial distribution be probability measures on $\mathcal{B}_{\mathbf{Z}}$.

The process $Z^{(\delta)}(t)$, $t \geq 0$, is defined on a probability space $\big(\Omega^{(\delta)}, \mathcal{F}^{(\delta)}, \mathsf{P}^{(\delta)}\big)$. Note that these spaces can be different for different δ; i.e., we consider a triangular array model.

We assume that $Z^{(\delta)}(t)$, $t \geq 0$, is a measurable process; i.e., $Z^{(\delta)}(t, \omega)$ is a measurable function in $(t, \omega) \in [0, \infty) \times \Omega^{(\delta)}$. Also, we assume that the first component $Y^{(\delta)}(t)$,

$t \geq 0$, is a càdlàg process; i.e., $Y^{(\delta)}(t)$ is almost surely continuous from the right and has limits from the left at all points $t \geq 0$.

The component $Y^{(\delta)}(t)$ is interpreted as a log-price process and the component $X^{(\delta)}(t)$ as a stochastic index modulating the log-price process $Y^{(\delta)}(t)$.

The price process is defined by

$$S^{(\delta)}(t) = \exp\{Y^{(\delta)}(t)\}, \qquad t \geq 0. \tag{1}$$

Consider the two-component process $V^{(\delta)}(t) = \big(S^{(\delta)}(t), X^{(\delta)}(t)\big)$, $t \geq 0$. Due to the one-to-one mapping and continuity properties of the exponential function, $V^{(\delta)}(t)$ is also a measurable Markov process with the phase space $\mathbb{V} = (0, \infty) \times \mathbb{X}$. The first component $S^{(\delta)}(t)$, $t \geq 0$, is a càdlàg process. The process $V^{(\delta)}(t)$ has the transition probabilities $Q^{(\delta)}(t, v, t+u, A) = P^{(\delta)}(t, z, t+u, \ln A)$ and the initial distribution $Q^{(\delta)}(A) = P^{(\delta)}(\ln A)$, where $v = (s, x) \in \mathbb{V}$, $z = (\ln s, x) \in \mathbf{Z}$, and

$$\ln A = \{z = (y, x) \colon y = \ln s, (s, x) \in A\}, \qquad A \in \mathcal{B}_{\mathbb{V}} = \sigma(\mathcal{B}_+ \times \mathcal{B}_{\mathbb{X}}).$$

Here $\mathcal{B}_+$ is the Borel σ-algebra of subsets of $(0, \infty)$.

Let, for every $\delta \geq 0$, $g^{(\delta)}(t, s)$, $(t, s) \in [0, \infty) \times (0, \infty)$, be a pay-off function. We assume that $g^{(\delta)}(t, s)$ is a nonnegative measurable (Borel) function.

The typical example of a pay-off function is

$$g^{(\delta)}(t, s) = e^{-R_t^{(\delta)}} a_t^{(\delta)} \left[s - K_t^{(\delta)}\right]_+, \tag{2}$$

where $a_t^{(\delta)}$, $t \geq 0$, and $K_t^{(\delta)}$, $t \geq 0$, are two nonnegative measurable functions and $R_t^{(\delta)}$, $t \geq 0$, is a nondecreasing function such that $R_0^{(\delta)} = 0$.

Here, $R_t^{(\delta)}$ is the accumulated continuously compounded riskless interest rate. Typically, $R_t^{(\delta)} = \int_0^t r^{(\delta)}(s)\, ds$, where $r^{(\delta)}(s) \geq 0$ is a nonnegative measurable function representing an instant riskless interest rate at the moment s.

The functions $a_t^{(\delta)}$, $t \geq 0$, and $K_t^{(\delta)}$, $t \geq 0$, are parameters of an option contract. The case where $a_t^{(\delta)} = a^{(\delta)}$ and $K_t^{(\delta)} = K^{(\delta)}$ do not depend on t corresponds to the standard American call option.

Let $\mathcal{F}_t^{(\delta)}$, $t \geq 0$, be a natural filtration of σ-fields generated by the process $Z^{(\delta)}(t)$, $t \geq 0$. We consider Markov moments $\tau^{(\delta)}$ with respect to the filtration $\mathcal{F}_t^{(\delta)}$, $t \geq 0$. This means that $\tau^{(\delta)}$ is a random variable which takes values in $[0, \infty]$ and is such that $\{\omega \colon \tau^{(\delta)}(\omega) \leq t\} \in \mathcal{F}_t^{(\delta)}$, $t \geq 0$.

Note that $\mathcal{F}_t^{(\delta)}$, $t \geq 0$, is a natural filtration of the σ-field generated by the process $V^{(\delta)}(t)$, $t \geq 0$.

Denote by $\mathcal{M}_{\max,T}^{(\delta)}$ the class of all Markov moments $\tau^{(\delta)} \leq T$, where $T > 0$, and consider a class of Markov moments $\mathcal{M}_T^{(\delta)} \subseteq \mathcal{M}_{\max,T}^{(\delta)}$.

Our aim is to maximize the expected pay-off for a given stopping moment over a class $\mathcal{M}_T^{(\delta)}$, that is, to find

$$\Phi\left(\mathcal{M}_T^{(\delta)}\right) = \sup_{\tau^{(\delta)} \in \mathcal{M}_T^{(\delta)}} \mathsf{E} g^{(\delta)}\left(\tau^{(\delta)}, S^{(\delta)}\left(\tau^{(\delta)}\right)\right). \tag{3}$$

The reward functional $\Phi\big(\mathcal{M}_T^{(\delta)}\big)$ can take the value $+\infty$. However, the conditions imposed below on price processes and pay-off functions guarantee that $\Phi\big(\mathcal{M}_{\max,T}^{(\delta)}\big) < \infty$ for all sufficiently small δ.

Note that we do not impose any monotonicity conditions on the pay-off functions $g^{(\delta)}(t, s)$. Note, however, that the cases where the pay-off function $g^{(\delta)}(t, s)$ is nondecreasing or nonincreasing in the argument s correspond to call and put American type options, respectively.

The first condition assumes the absolute continuity of pay-off functions and imposes power type upper bounds on their partial derivatives:

$\mathbf{A_1}$: There exists $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$:

- **(a)** the function $g^{(\delta)}(t, s)$ is absolutely continuous in t with respect to the Lebesgue measure for every fixed $s \in (0, \infty)$, and also the function $g^{(\delta)}(t, s)$ is absolutely continuous in s with respect to the Lebesgue measure for every fixed $t \in [0, T]$;
- **(b)** for every $s \in (0, \infty)$, the partial derivative in t is such that
$$\left| \frac{\partial g^{(\delta)}(t, s)}{\partial t} \right| \leq K_1 + K_2 s^{\gamma_1}$$
for almost all $t \in [0, T]$ with respect to the Lebesgue measure, where $0 \leq K_1, K_2 < \infty$ and $\gamma_1 \geq 0$;
- **(c)** for every $t \in [0, T]$, the partial derivative in s is such that
$$\left| \frac{\partial g^{(\delta)}(t, s)}{\partial s} \right| \leq K_3 + K_4 s^{\gamma_2}$$
for almost all $s \in (0, \infty)$ with respect to the Lebesgue measure, where $0 \leq K_3, K_4 < \infty$ and $\gamma_2 \geq 0$;
- **(d)** for every $t \in [0, T]$, $g^{(\delta)}(t, 0) \leq K_5 < \infty$ for some $0 \leq K_5 < \infty$, where
$$g^{(\delta)}(t, 0) = \overline{\lim_{s \to 0}} \, g^{(\delta)}(t, s).$$

Note that condition $\mathbf{A_1(a)}$ includes the case where the corresponding partial derivatives exist at all points of the interval $[0, T]$ or $(0, \infty)$, respectively, except for some subsets of zero Lebesgue measures. Note also that conditions $\mathbf{A_1(b)}$ and $\mathbf{A_1(c)}$ include the case where the corresponding upper bounds hold at points of the sets where the corresponding derivatives exist except for some subsets (of these sets) of zero Lebesgue measures.

Note that condition $\mathbf{A_1}$ implies that the function $g^{(\delta)}(t, s)$ is continuous with respect to the pair of arguments $t \in [0, T]$ and $s \in (0, \infty)$.

For example, condition $\mathbf{A_1}$ holds for the pay-off function given in (2) if the functions $R_t^{(\delta)}$, $a_t^{(\delta)}$, and $K_t^{(\delta)}$ have bounded first derivatives in the interval $[0, T]$. In this case, $\gamma_1 = 1$ and $\gamma_2 = 0$.

Taking into account the formula $S^{(\delta)}(t) = e^{Y^{(\delta)}(t)}$ connecting the price process $S^{(\delta)}(t)$ and the log-price process $Y^{(\delta)}(t)$, condition $\mathbf{A_1}$ can be rewritten in an equivalent form in terms of the function $g^{(\delta)}(t, e^y)$, $(t, y) \in [0, T] \times \mathbb{R}_1$.

Put
$$g_1^{(\delta)}(t, s) = \frac{\partial g^{(\delta)}(t, s)}{\partial t} \quad \text{and} \quad g_2^{(\delta)}(t, s) = \frac{\partial g^{(\delta)}(t, s)}{\partial s}.$$
Then $\partial g^{(\delta)}(t, e^y)/\partial t = g_1^{(\delta)}(t, e^y)$ and $\partial g^{(\delta)}(t, e^y)/\partial y = g_2^{(\delta)}(t, e^y)e^y$. An equivalent variant of condition $\mathbf{A_1}$ takes the following form:

$\mathbf{A_1'}$: There exists $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$:

- **(a)** the function $g^{(\delta)}(t, e^y)$ is absolutely continuous in t with respect to the Lebesgue measure for every fixed $y \in \mathbb{R}_1$, and also the function $g^{(\delta)}(t, e^y)$ is absolutely continuous in y with respect to the Lebesgue measure for every fixed $t \in [0, T]$;

(b) for every $y \in \mathbb{R}_1$, the partial derivative in t is such that

$$\left|\frac{\partial g^{(\delta)}(t, e^y)}{\partial t}\right| \leq K_1 + K_2 e^{\gamma_1 y}$$

for almost all $t \in [0, T]$ with respect to the Lebesgue measure, where $0 \leq K_1, K_2 < \infty$ and $\gamma_1 \geq 0$;

(c) for almost all $y \in \mathbb{R}_1$ with respect to the Lebesgue measure, the partial derivative in y is such that

$$\left|\frac{\partial g^{(\delta)}(t, e^y)}{\partial y}\right| \leq (K_3 + K_4 e^{\gamma_2 y}) e^y$$

for every $t \in [0, T]$ where $0 \leq K_3, K_4 < \infty$ and $\gamma_2 \geq 0$;

(d) for every $t \in [0, T]$, $g^{(\delta)}(t, -\infty) \leq K_5$ for some $0 \leq K_5 < \infty$, where

$$g^{(\delta)}(t, -\infty) = \overline{\lim_{y \to -\infty}} g^{(\delta)}(t, e^y).$$

As usual, we use the notation $\mathsf{E}_{z,t}$ and $\mathsf{P}_{z,t}$ for the expectations and probabilities evaluated under the condition $Z^{(\delta)}(t) = z$.

For $\beta, c, T > 0$, define the exponential moment modulus of compactness for the càdlàg process $Y^{(\delta)}(t)$, $t \geq 0$:

$$\Delta_\beta\big(Y^{(\delta)}(\cdot), c, T\big) = \sup_{0 \leq t \leq t+u \leq t+c \leq T} \sup_{z \in \mathbf{Z}} \mathsf{E}_{z,t} \left(\exp\left\{\beta \left|Y^{(\delta)}(t+u) - Y^{(\delta)}(t)\right|\right\} - 1\right).$$

We also need the following conditions of the exponential moment compactness for the log-price processes:

$\mathbf{C_1}$: $\lim_{c \to 0} \overline{\lim}_{\delta \to 0} \Delta_\beta\big(Y^{(\delta)}(\cdot), c, T\big) = 0$ for some $\beta > \gamma = \max(\gamma_1, \gamma_2 + 1)$, where γ_1 and γ_2 are the parameters introduced in condition $\mathbf{A_1}$,

and

$\mathbf{C_2}$: $\overline{\lim}_{\delta \to 0} \mathsf{E} \exp\big\{\beta |Y^{(\delta)}(0)|\big\} < \infty$, where β is the parameter introduced in condition $\mathbf{C_1}$.

Now we get asymptotically uniform upper bounds for moments of the maximums of log-price and price processes. Explicit expressions for the constants are given in the proofs of the corresponding lemmas.

Lemma 1. *Let conditions $\mathbf{C_1}$ and $\mathbf{C_2}$ hold. Then there exist $0 < \delta_1 \leq \delta_0$ and a constant $L_1 < \infty$ such that*

$$\mathsf{E} \exp\left\{\beta \sup_{0 \leq u \leq T} \left|Y^{(\delta)}(u)\right|\right\} \leq L_1 \tag{4}$$

for all $\delta \leq \delta_1$.

Lemma 2. *Let conditions $\mathbf{A_1}$, $\mathbf{C_1}$, and $\mathbf{C_2}$ hold. Then there exists a constant $L_2 < \infty$ such that*

$$\mathsf{E}\left(\sup_{0 \leq u \leq T} g^{(\delta)}\big(u, S^{(\delta)}(u)\big)\right)^{\beta/\gamma} \leq L_2 \tag{5}$$

for all $\delta \leq \delta_1$.

Proof of Lemma 1. Define the random variables

$$S_\beta^{(\delta)}(t) = \exp\left\{\beta \sup_{0 \leq u \leq t} \left|Y^{(\delta)}(u)\right|\right\}.$$

Note that

$$S_\beta^{(\delta)}(t) = \begin{cases} \exp\{\beta|Y^{(\delta)}(0)|\} & \text{if } t = 0, \\ \sup_{0\le u\le t}\exp\{\beta|Y^{(\delta)}(u)|\} & \text{if } 0 < t \le T. \end{cases} \tag{6}$$

Now we introduce the random variables

$$W_\beta^{(\delta)}[t', t''] = \sup_{t'\le t\le t''} \exp\left\{\beta\left|Y^{(\delta)}(t) - Y^{(\delta)}(t')\right|\right\}, \qquad 0 \le t' \le t'' \le T.$$

Consider a partition $\tilde{\Pi}_m = \{0 = v_{0,m} < \cdots < v_{m,m} = T\}$ of the interval $[0, T]$ by the points $v_{n,m} = nT/m$, $n = 0, \dots, m$. Using equality (6) we get the following inequalities for $n = 1, \dots, m$:

$$\begin{aligned} S_\beta^{(\delta)}(v_{n,m}) &\le S_\beta^{(\delta)}(v_{n-1,m}) + \sup_{v_{n-1,m}\le u\le v_{n,m}} \exp\left\{\beta\left|Y^{(\delta)}(u)\right|\right\} \\ &\le S_\beta^{(\delta)}(v_{n-1,m}) + \exp\left\{\beta\left|Y^{(\delta)}(v_{n-1,m})\right|\right\} W_\beta^{(\delta)}[v_{n-1,m}, v_{n,m}] \\ &\le S_\beta^{(\delta)}(v_{n-1,m})\left(W_\beta^{(\delta)}[v_{n-1,m}, v_{n,m}] + 1\right). \end{aligned} \tag{7}$$

Condition $\mathbf{C_1}$ implies that, for any constant $e^{-\beta} < L_5 < 1$, one can choose a number $c = c(L_5) > 0$ and then $\delta_1 = \delta_1(c) \le \delta_0$ such that

$$\frac{\Delta_\beta\big(Y^{(\delta)}(\cdot), c, T\big) + 1}{e^\beta} \le L_5 \tag{8}$$

for all $\delta \le \delta_1$.

Also condition $\mathbf{C_2}$ implies that δ_1 can be chosen in such a way that, for some constant $L_6 = L_6(\delta_1) < \infty$, the following inequality holds for $\delta \le \delta_1$:

$$\mathsf{E}\exp\left\{\beta\left|Y^{(\delta)}(0)\right|\right\} \le L_6. \tag{9}$$

Note that $Y^{(\delta)}(t)$ is not a Markov process. Nevertheless, an analogue of the Kolmogorov inequality can be obtained for it by a slight modification of the standard proof for Markov processes (see, for example, Gikhman and Skorokhod (1971) [19]). We state this result in Lemma 3 below. Note that we do assume in Lemma 3 that the two-component process $Z^{(\delta)}(t)$ is a Markov process.

Lemma 3. *Let $a, b > 0$. Assume that*

$$\sup_{z\in\mathbf{Z}} \mathsf{P}_{z,t}\left\{\left|Y^{(\delta)}(t'') - Y^{(\delta)}(t)\right| \ge a\right\} \le L < 1, \qquad t' \le t \le t''.$$

Then, for any point $z_0 \in \mathbf{Z}$,

$$\begin{aligned} &\mathsf{P}_{z_0,t'}\left\{\sup_{t'\le t\le t''}\left|Y^{(\delta)}(t) - Y^{(\delta)}(t')\right| \ge a + b\right\} \\ &\qquad \le \frac{1}{1-L}\mathsf{P}_{z_0,t'}\left\{\left|Y^{(\delta)}(t'') - Y^{(\delta)}(t')\right| \ge b\right\}. \end{aligned} \tag{10}$$

We refer to the report [49], where one can find the proof of Lemma 3.

Now we use Lemma 3 to show that

$$\sup_{0\le t'\le t''\le t'+c\le T}\ \sup_{z\in\mathbf{Z}} \mathsf{E}_{z,t'} W_\beta^{(\delta)}[t', t''] \le L_7 \tag{11}$$

for $\delta \le \delta_1$, where

$$L_7 = \frac{e^\beta\left(e^\beta - 1\right)L_5}{1 - L_5} < \infty. \tag{12}$$

Relation (8) implies that

$$\begin{aligned}
&\sup_{0\le t'\le t\le t''\le t'+c\le T}\sup_{z\in\mathbf{Z}}\mathsf{P}_{z,t}\Big\{\big|Y^{(\delta)}(t'')-Y^{(\delta)}(t)\big|\ge 1\Big\}\\
&\qquad\le \sup_{0\le t'\le t\le t''\le t'+c\le T}\sup_{z\in\mathbf{Z}}\frac{\mathsf{E}_{z,t}\exp\big\{\beta\big|Y^{(\delta)}(t'')-Y^{(\delta)}(t)\big|\big\}}{e^{\beta}}\\
&\qquad\le \frac{\Delta_\beta\big(Y^{(\delta)}(\cdot),c,T\big)+1}{e^{\beta}}\le L_5<1
\end{aligned}\tag{13}$$

for all $\delta\le\delta_1$.

Applying Lemma 3, we get

$$\begin{aligned}
&\mathsf{P}_{z,t'}\left\{\sup_{t'\le t\le t''}\big|Y^{(\delta)}(t)-Y^{(\delta)}(t')\big|\ge 1+b\right\}\\
&\qquad\le\frac{1}{1-L_5}\mathsf{P}_{z,t'}\Big\{\big|Y^{(\delta)}(t'')-Y^{(\delta)}(t')\big|\ge b\Big\}
\end{aligned}\tag{14}$$

for all $\delta\le\delta_1$, $0\le t'\le t''\le t'+c\le T$, $z\in\mathbf{Z}$, and $b>0$.

To shorten the notation, we set

$$W=\big|Y^{(\delta)}(t'')-Y^{(\delta)}(t')\big|\quad\text{and}\quad W^{+}=\sup_{t'\le t\le t''}\big|Y^{(\delta)}(t)-Y^{(\delta)}(t')\big|.$$

Note that $e^{\beta W^{+}}=W^{(\delta)}_{\beta}[t',t'']$.

Relations (8) and (14) imply that

$$\begin{aligned}
\mathsf{E}_{z,t'}e^{\beta W^{+}}&=1+\beta\int_0^\infty e^{\beta b}\mathsf{P}_{z,t'}\{W^{+}\ge b\}\,db\\
&\le 1+\beta\int_0^1 e^{\beta b}\,db+\beta\int_1^\infty e^{\beta b}\mathsf{P}_{z,t'}\{W^{+}\ge b\}\,db\\
&=e^{\beta}+\beta\int_0^\infty e^{\beta(1+b)}\mathsf{P}_{z,t'}\{W^{+}\ge 1+b\}\,db\\
&\le e^{\beta}+\frac{\beta e^{\beta}}{1-L_5}\int_0^\infty e^{\beta b}\mathsf{P}_{z,t'}\{W\ge b\}\,db\\
&=e^{\beta}+\frac{\beta e^{\beta}}{1-L_5}\,\frac{\mathsf{E}_{z,t'}e^{\beta W}-1}{\beta}=\frac{e^{\beta}}{1-L_5}\left(\mathsf{E}_{z,t'}e^{\beta W}-L_5\right)\\
&\le\frac{e^{\beta}}{1-L_5}\Big(\Delta_\beta\big(Y^{(\delta)}(\cdot),c,T\big)+1-L_5\Big)\le\frac{e^{\beta}\left(e^{\beta}-1\right)L_5}{1-L_5}=L_7
\end{aligned}\tag{15}$$

for all $\delta\le\delta_1$, $0\le t'\le t''\le t'+c\le T$, and $z\in\mathbf{Z}$.

Since inequality (15) holds for all $\delta\le\delta_1$, $0\le t'\le t''\le t'+c\le T$, and $z\in\mathbf{Z}$, relation (11) follows.

Now we complete the proof of Lemma 1. Using condition $\mathbf{C_2}$, relations (7), (9)–(12), and the Markov property of the process $Z^{(\delta)}(t)$ we get

$$\begin{aligned}
\mathsf{E}S^{(\delta)}_{\beta}(v_{n,m})&\le\mathsf{E}\Big\{S^{(\delta)}_{\beta}(v_{n-1,m})\mathsf{E}\Big\{(W^{(\delta)}_{\beta}[v_{n-1,m},v_{n,m}]+1)\;/\;Z^{(\delta)}(v_{n-1,m})\Big\}\Big\}\\
&\le\mathsf{E}S^{(\delta)}_{\beta}(v_{n-1,m})(L_7+1)\le\cdots\le\mathsf{E}S^{(\delta)}_{\beta}(0)(L_7+1)^n\le L_6(L_7+1)^n
\end{aligned}\tag{16}$$

for all $\delta\le\delta_1$ and $m=[T/c]+1$, where $[x]$ denotes the integer part of x (in this case, $T/m\le c$), $n=1,\dots,m$.

Finally, we get

$$\mathsf{E}\exp\left\{\beta \sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right\} = \mathsf{E}S_\beta^{(\delta)}(v_{m,m}) \le L_6(L_7+1)^m \tag{17}$$

for $\delta \le \delta_1$.

Relation (17) obviously implies that inequality (4) of Lemma 1 holds for $\delta \le \delta_1$ and with the constant

$$L_1 = L_6(L_7+1)^m. \tag{18}$$

The proof of Lemma 1 is complete. □

Proof of Lemma 2. According to conditions $\mathbf{A_1(c)}$ and $\mathbf{A_1(d)}$ and since $\gamma_2+1\le\gamma$, we have

$$\begin{aligned} g^{(\delta)}\big(u, S^{(\delta)}(u)\big) &\le \int_0^{S^{(\delta)}(u)} \left|\frac{\partial g^{(\delta)}(u,s)}{\partial s}\right| ds + g^{(\delta)}(u,0) \\ &\le K_3 S^{(\delta)}(u) + \frac{K_4}{\gamma_2+1} S^{(\delta)}(u)^{\gamma_2+1} + K_5 \le L_8 \exp\left\{\gamma\left|Y^{(\delta)}(u)\right|\right\}, \end{aligned} \tag{19}$$

where

$$L_8 = K_3 + \frac{K_4}{\gamma_2+1} + K_5 < \infty \tag{20}$$

for $\delta \le \delta_0$.

Relation (6) together with inequality (19) implies that

$$\left(\sup_{0\le u\le T} g^{(\delta)}\big(u, S^{(\delta)}(u)\big)\right)^{\beta/\gamma} \le (L_8)^{\beta/\gamma}\exp\left\{\beta \sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right\}. \tag{21}$$

The bounds (4) and (21) obviously imply that inequality (5) holds for $\delta \le \delta_1$ and with the constant

$$L_2 = L_1(L_8)^{\beta/\gamma} < \infty. \tag{22}$$

The proof of Lemma 2 is complete. □

Relation (5) of Lemma 2 implies that

$$\Phi\left(\mathcal{M}_{\max,T}^{(\delta)}\right) \le \mathsf{E}\sup_{0\le u\le T} g^{(\delta)}\big(u, S^{(\delta)}(u)\big) \le (L_2)^{\gamma/\beta} < \infty \tag{23}$$

for $\delta \le \delta_1$.

Therefore, the functional $\Phi\big(\mathcal{M}_{\max,T}^{(\delta)}\big)$ is well defined for $\delta \le \delta_1$. In what follows we consider $\delta \le \delta_1$.

3. Skeleton approximations

In this section, we derive skeleton approximations for the reward functional

$$\Phi\left(\mathcal{M}_{\max,T}^{(\delta)}\right)$$

by using a similar functional for an imbedded discrete time model.

Let $\Pi = \{0 = t_0 < t_1 < \dots < t_N = T\}$ be a partition of the interval $[0,T]$. We consider the class $\hat{\mathcal{M}}_{\Pi,T}^{(\delta)}$ of all Markov moments of $\mathcal{M}_{\max,T}^{(\delta)}$, which only take the values $t_0, t_1, \dots, t_N$, and the class $\mathcal{M}_{\Pi,T}^{(\delta)}$ of all Markov moments $\tau^{(\delta)}$ of $\hat{\mathcal{M}}_{\Pi,T}^{(\delta)}$ such that $\big\{\omega\colon \tau^{(\delta)}(\omega) = t_k\big\} \in \sigma\big[Z^{(\delta)}(t_0), \dots, Z^{(\delta)}(t_k)\big]$ for $k = 0, \dots, N$. By definition,

$$\mathcal{M}_{\Pi,T}^{(\delta)} \subseteq \hat{\mathcal{M}}_{\Pi,T}^{(\delta)} \subseteq \mathcal{M}_{\max,T}^{(\delta)}. \tag{24}$$

Relations (23) and (24) imply that, under the assumptions of Lemma 2,

$$\Phi\left(\mathcal{M}^{(\delta)}_{\Pi,T}\right) \le \Phi\left(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}\right) \le \Phi\left(\mathcal{M}^{(\delta)}_{\max,T}\right) < \infty. \tag{25}$$

The reward functionals $\Phi(\mathcal{M}^{(\delta)}_{\max,T})$, $\Phi(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T})$, and $\Phi(\mathcal{M}^{(\delta)}_{\Pi,T})$ correspond to the models of an American type option in continuous time, Bermudan type option in continuous time, and American type option in discrete time, respectively.

In the first two cases, the underlying price process is a continuous time Markov type price process modulated by a stochastic index while the corresponding price process in the third case is a discrete time Markov type process modulated by a stochastic index.

Indeed, the random variables $Z^{(\delta)}(t_0), Z^{(\delta)}(t_1), \dots, Z^{(\delta)}(t_N)$ form a discrete time inhomogeneous Markov chain whose phase space, transition probabilities, and initial distribution are

$$\mathbf{Z}, \qquad P^{(\delta)}(t_n, z, t_{n+1}, A), \qquad P^{(\delta)}(A),$$

respectively.

Note that we have modified the standard definition of a discrete time Markov chain by considering the moments $t_0, \dots, t_N$ as the jump moments for the Markov chain $Z^{(\delta)}(t_n)$ instead of the moments $0, \dots, N$. This is done in order to synchronize the discrete and continuous time models.

Thus, the optimization problem (3) for the class $\mathcal{M}^{(\delta)}_{\Pi,T}$ is indeed the problem of finding an optimal expected reward for American type options in discrete time.

Now we are ready to formulate the first main result of the paper concerning the skeleton approximations of the reward functional in the continuous time model by the corresponding reward functional in the corresponding discrete time model. Note that the skeleton approximations have an asymptotically uniform form with respect to the perturbation parameter. This is important for using these approximations in the convergence theorems given in the second part of the paper.

We use the method developed in Kukush and Silvestrov (2001) [32]. However, we essentially improve the skeleton approximation obtained in this paper, where the difference $\Phi(\mathcal{M}^{(\delta)}_{\max,T}) - \Phi(\mathcal{M}^{(\delta)}_{\Pi,T})$ is estimated from above via the modulus of compactness in the uniform topology for the price processes. This estimate could only be used for continuous price processes. In the present paper, we obtain other estimates based on the exponential moment modulus of compactness $\Delta_\beta(Y^{(\delta)}(\cdot), c, T)$. These estimates can be effectively used for càdlàg price processes, too.

Theorem 1 develops the estimates mentioned above. The explicit expression for the constants L_3 and L_4 in the bound (26) will be given in the proof of the theorem.

Theorem 1. *Let conditions* $\mathbf{A_1}$*,* $\mathbf{C_1}$*, and* $\mathbf{C_2}$ *hold, and let* $\delta \le \delta_1$ *and* $d(\Pi) \le c$*, where the constants* c *and* δ_1 *are defined in relations* (8) *and* (9)*. Then there exist constants* $L_3, L_4 < \infty$ *such that the following skeleton approximation inequality holds:*

$$\Phi\left(\mathcal{M}^{(\delta)}_{\max,T}\right) - \Phi\left(\mathcal{M}^{(\delta)}_{\Pi,T}\right) \le L_3 d(\Pi) + L_4\left(\Delta_\beta\big(Y^{(\delta)}(\cdot), d(\Pi), T\big)\right)^{(\beta-\gamma)/\beta}. \tag{26}$$

Proof of Theorem 1. We start with the following result which plays an important role in the proof of Theorem 1.

Lemma 4. *For any partition* $\Pi = \{0 = t_0 < t_1 < \dots < t_N = T\}$ *of the interval* $[0, T]$*,*

$$\Phi\left(\mathcal{M}^{(\delta)}_{\Pi,T}\right) = \Phi\left(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}\right). \tag{27}$$

Proof of Lemma 4. A similar result is given in Kukush and Silvestrov (2000) [31], (2004) [33], and we shortly present a modified version of the corresponding proof.

The optimization problem (3) for the class $\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}$ can be considered as a problem of the optimal expected reward for American type options with discrete time. To see this let us add to the random variables Z_{t_n} additional components $\bar{Z}^{(\delta)}_n = \{Z^{(\delta)}(t), t_{n-1} < t \le t_n\}$ with the corresponding phase space $\bar{\mathbf{Z}}$ equipped with the corresponding cylindrical σ-field. As $\bar{Z}^{(\delta)}_0$ we can take an arbitrary point in $\bar{\mathbf{Z}}$. Consider the extended Markov chain $\tilde{Z}^{(\delta)}_n = (Z^{(\delta)}(t_n), \bar{Z}^{(\delta)}_n)$ with the phase space $\tilde{\mathbf{Z}} = \mathbf{Z} \times \bar{\mathbf{Z}}$. As above, we slightly modify the standard definition and treat $t_0, \dots, t_N$ as the moments of jumps for this Markov chain instead of the moments $0, \dots, N$ in order to synchronize the discrete and continuous time models.

By $\tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}$ we denote the class of all Markov moments $\tau^{(\delta)} \le t_N$ for the discrete time Markov chain $\tilde{Z}^{(\delta)}_n$. Consider the following reward functional:

$$\Phi\left(\tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}\right) = \sup_{\tau^{(\delta)} \in \tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}} \mathsf{E} g^{(\delta)}\left(\tau^{(\delta)}, S^{(\delta)}(\tau^{(\delta)})\right). \tag{28}$$

It is readily seen that the optimization problem (3) for the class $\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}$ is equivalent to the optimization problem (28), i.e.,

$$\Phi\left(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}\right) = \Phi\left(\tilde{\mathcal{M}}^{(\delta)}_{\Pi,T}\right). \tag{29}$$

As is known (see, for example, Shiryaev (1976) [43]), the optimal stopping moment $\tau^{(\delta)}$ exists in any discrete time Markov model, and the decision $\{\tau^{(\delta)} = t_n\}$ depends only on the value $\tilde{Z}^{(\delta)}_n$. Moreover the optimal Markov moment has the first hitting time structure; i.e., it is of the form $\tau^{(\delta)} = \min(t_n\colon \tilde{Z}^{(\delta)}_n \in \tilde{\mathbb{D}}^{(\delta)}_n)$, where $\tilde{\mathbb{D}}^{(\delta)}_n$, $n = 0, \dots, N$, are some measurable subsets of the phase space $\tilde{\mathbf{Z}}$. The optimal stopping domains are determined by the transition probabilities of the extended Markov chain $\tilde{Z}^{(\delta)}_n$.

However, the extended Markov chain $\tilde{Z}^{(\delta)}_n$ has transition probabilities depending only on the values of the first component $Z^{(\delta)}(t_n)$. As was shown in Kukush and Silvestrov (2004) [33], the optimal Markov moment has in this case the first hitting time structure of the form $\tau^{(\delta)} = \min(t_n\colon Z^{(\delta)}(t_n) \in \mathbb{D}^{(\delta)}_n)$, where $\mathbb{D}^{(\delta)}_n$, $n = 0, \dots, N$, are some measurable subsets of the phase space of the first component $\mathbf{Z}$.

Therefore, for the optimal stopping moment $\tau^{(\delta)}$, the decision $\{\tau^{(\delta)} = t_n\}$ depends only on the value $Z^{(\delta)}(t_n)$ and on $\tau^{(\delta)} \in \mathcal{M}^{(\delta)}_{\Pi,T}$. Hence,

$$\Phi(\mathcal{M}_{\Pi,T}) \ge \mathsf{E} g^{(\delta)}\left(\tau^{(\delta)}, S^{(\delta)}(\tau^{(\delta)})\right) = \Phi\left(\hat{\mathcal{M}}^{(\delta)}_{\Pi,T}\right). \tag{30}$$

Inequalities (25) and (30) imply equality (27). □

For any Markov moment $\tau^{(\delta)} \in \mathcal{M}^{(\delta)}_{\max,T}$ and a partition

$$\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\},$$

one can define the discretisation moment, namely

$$\tau^{(\delta)}[\Pi] = \begin{cases} 0 & \text{if } \tau^{(\delta)} = 0, \\ t_k & \text{if } t_{k-1} < \tau^{(\delta)} \le t_k,\ k = 1, \dots, N. \end{cases}$$

Let $\tau^{(\delta)}_\varepsilon$ be an ε-optimal Markov moment in the class $\mathcal{M}^{(\delta)}_{\max,T}$, i.e.,

$$\mathsf{E} g^{(\delta)}\left(\tau^{(\delta)}_\varepsilon, S^{(\delta)}(\tau^{(\delta)}_\varepsilon)\right) \ge \Phi\left(\mathcal{M}^{(\delta)}_{\max,T}\right) - \varepsilon. \tag{31}$$

Such an ε-optimal Markov moment always exists for any $\varepsilon > 0$, by definition of the reward functional $\Phi(\mathcal{M}^{(\delta)}_{\max,T})$.

By definition, the Markov moment $\tau_\varepsilon^{(\delta)}[\Pi]$ belongs to $\hat{\mathcal{M}}_{\Pi,T}^{(\delta)}$. This fact together with relation (27) of Lemma 4 implies that

$$\mathsf{E}g^{(\delta)}\left(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}\left(\tau_\varepsilon^{(\delta)}[\Pi]\right)\right) \le \Phi\left(\hat{\mathcal{M}}_{\Pi,T}^{(\delta)}\right) = \Phi\left(\mathcal{M}_{\Pi,T}^{(\delta)}\right) \le \Phi\left(\mathcal{M}_{\max,T}^{(\delta)}\right). \tag{32}$$

Put

$$d(\Pi) = \max\{t_k - t_{k-1}, k = 1, \dots, N\}.$$

Obviously,

$$\tau_\varepsilon^{(\delta)} \le \tau_\varepsilon^{(\delta)}[\Pi] \le \tau_\varepsilon^{(\delta)} + d(\Pi). \tag{33}$$

Now inequalities (31) and (32) imply the following skeleton approximation inequality:

$$\begin{aligned} 0 &\le \Phi\left(\mathcal{M}_{\max,T}^{(\delta)}\right) - \Phi\left(\mathcal{M}_{\Pi,T}^{(\delta)}\right) \\ &\le \varepsilon + \mathsf{E}g^{(\delta)}\left(\tau_\varepsilon^{(\delta)}, S^{(\delta)}\left(\tau_\varepsilon^{(\delta)}\right)\right) - \mathsf{E}g^{(\delta)}\left(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}\left(\tau_\varepsilon^{(\delta)}[\Pi]\right)\right) \\ &\le \varepsilon + \mathsf{E}\left|g^{(\delta)}\left(\tau_\varepsilon^{(\delta)}, S^{(\delta)}\left(\tau_\varepsilon^{(\delta)}\right)\right) - g^{(\delta)}\left(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}\left(\tau_\varepsilon^{(\delta)}[\Pi]\right)\right)\right|. \end{aligned} \tag{34}$$

To shorten the notation, we introduce the random variables

$$\begin{aligned} \tau' &= \tau_\varepsilon^{(\delta)}, \qquad \tau'' = \tau_\varepsilon^{(\delta)}[\Pi], \\ Y' &= Y^{(\delta)}(\tau'), \qquad Y'' = Y^{(\delta)}(\tau''). \end{aligned}$$

Also let

$$Y^+ = Y' \vee Y'', \qquad Y^- = Y' \wedge Y''.$$

By definition,

$$0 \le \tau' \le \tau'' \le T \quad \text{and} \quad Y^- \le Y^+.$$

Using this notation and condition $\mathbf{A_1'}$, we get the following inequalities:

$$\begin{aligned} &\left|g^{(\delta)}\left(\tau', e^{Y'}\right) - g^{(\delta)}\left(\tau'', e^{Y''}\right)\right| \\ &\quad\le \left|g^{(\delta)}\left(\tau', e^{Y'}\right) - g^{(\delta)}(\tau'', e^{Y'})\right| + \left|g^{(\delta)}\left(\tau'', e^{Y'}\right) - g^{(\delta)}\left(\tau'', e^{Y''}\right)\right| \\ &\quad\le \int_{\tau'}^{\tau''} \left|g_1^{(\delta)}\left(t, e^{Y'}\right)\right| dt + \int_{Y^-}^{Y^+} \left|g_2^{(\delta)}\left(\tau'', e^y\right)e^y\right| dy \\ &\quad\le \int_{\tau'}^{\tau''} \left(K_1 + K_2 e^{\gamma_1 Y'}\right) dt + \int_{Y^-}^{Y^+} \left(K_3 e^y + K_4 e^{(\gamma_2+1)y}\right) dy \\ &\quad\le \left(K_1 + K_2 e^{\gamma_1|Y'|}\right)(\tau'' - \tau') + \left(K_3 e^{|Y^+|} + K_4 e^{(\gamma_2+1)|Y^+|}\right)(Y^+ - Y^-) \\ &\quad\le (K_1 + K_2)\exp\left\{\gamma_1 \sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right\}(\tau'' - \tau') \\ &\qquad + (K_3 + K_4)\exp\left\{(\gamma_2 + 1)\sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right\}|Y' - Y''|. \end{aligned} \tag{35}$$

Recall that $0 \le \tau'' - \tau' \le d(\Pi)$ and $\gamma_1 \vee (\gamma_2 + 1) = \gamma < \beta$. Now, applying Hölder's inequality (with parameters $p = \beta/\gamma$ and $q = \beta/(\beta - \gamma)$) to the corresponding products of random variables on the right hand side of (35), and using inequality (4) of Lemma 1, we obtain the following estimate for the expectation on the right hand side in (34):

$$
\begin{aligned}
&\mathsf{E}\left|g^{(\delta)}\left(\tau_\varepsilon^{(\delta)}, S^{(\delta)}\big(\tau_\varepsilon^{(\delta)}\big)\right) - g^{(\delta)}\left(\tau_\varepsilon^{(\delta)}[\Pi], S^{(\delta)}\big(\tau_\varepsilon^{(\delta)}[\Pi]\big)\right)\right| \\
&\quad = \mathsf{E}\left|g^{(\delta)}\big(\tau', e^{Y'}\big) - g^{(\delta)}\big(\tau'', e^{Y''}\big)\right| \\
&\quad \le (K_1 + K_2)\mathsf{E}\exp\left\{\gamma \sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right\} d(\Pi) \\
&\qquad + (K_3 + K_4)\mathsf{E}\exp\left\{\gamma \sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right\} |Y' - Y''| \\
&\quad \le (K_1 + K_2)[L_1]^{\gamma/\beta} d(\Pi) + (K_3 + K_4)[L_1]^{\gamma/\beta}\left(\mathsf{E}|Y' - Y''|^{\frac{\beta}{\beta-\gamma}}\right)^{\frac{\beta-\gamma}{\beta}}
\end{aligned} \tag{36}
$$

for $\delta \le \delta_1$.

The next step in the proof is to show that

$$
\mathsf{E}|Y' - Y''|^{\frac{\beta}{\beta-\gamma}} = \mathsf{E}\left|Y^{(\delta)}\big(\tau_\varepsilon^{(\delta)}\big) - Y^{(\delta)}\big(\tau_\varepsilon^{(\delta)}[\Pi]\big)\right|^{\frac{\beta}{\beta-\gamma}} \le L_9 \Delta_\beta\left(Y^{(\delta)}(\cdot), d(\Pi), T\right) \tag{37}
$$

for $\delta \le \delta_1$, where

$$
L_9 = \sup_{y\ge 0} \frac{y^{\frac{\beta}{\beta-\gamma}}}{e^{\beta y} - 1} < \infty. \tag{38}
$$

In order to prove inequality (37), we employ the method of [47] for the estimation of moments of increments of stochastic processes stopped at Markov type moments.

By definition, $\tau'' = \tau' + f_\Pi(\tau')$, where $f_\Pi(t) = t - t_k$ for $t_k \le t < t_{k+1}$, $k = 0, \ldots, N-1$, and $f_\Pi(t) = 0$ for $t = t_N$. Obviously the function $f_\Pi(t)$ is continuous from the right in the interval $[0, T]$ and $0 \le f_\Pi(t) \le d(\Pi)$.

Consider again the partition $\tilde{\Pi}_m$ of the interval $[0, T]$ by the points $v_{n,m} = nT/m$, $n = 0, \ldots, m$, and introduce the random variables

$$
\tau'[\tilde{\Pi}_m] = \begin{cases} 0 & \text{if } \tau' = 0, \\ v_{k,m} & \text{if } v_{k-1,m} < \tau' \le v_{k,m},\ k = 1, \ldots, N. \end{cases}
$$

Obviously $\tau' \le \tau'[\tilde{\Pi}_m] \le \tau' + T/m$. Thus

$$
\tau'[\tilde{\Pi}_m] \xrightarrow{\text{a.s.}} \tau' \quad \text{as } m \to \infty
$$

("a.s." is the abbreviation for "almost surely"). Since $Y^{(\delta)}(t)$ is a càdlàg process, we get the following relation:

$$
\begin{aligned}
Q_m^{(\delta)} &= \left|Y^{(\delta)}(\tau'[\tilde{\Pi}_m]) - Y^{(\delta)}\left(\tau'[\tilde{\Pi}_m] + f_\Pi\big(\tau'[\tilde{\Pi}_m]\big)\right)\right|^{\frac{\beta}{\beta-\gamma}} \\
&\xrightarrow{\text{a.s.}} Q^{(\delta)} = \left|Y^{(\delta)}(\tau') - Y^{(\delta)}(\tau' + f_\Pi(\tau'))\right|^{\frac{\beta}{\beta-\gamma}} \quad \text{as } m \to \infty.
\end{aligned} \tag{39}
$$

Note also that $Q_m^{(\delta)}$ are nonnegative random variables and that

$$
\begin{aligned}
Q_m^{(\delta)} &\le \left(\left|Y^{(\delta)}(\tau'[\tilde{\Pi}_m])\right| + \left|Y^{(\delta)}\left(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m])\right)\right|\right)^{\frac{\beta}{\beta-\gamma}} \\
&\le 2^{\frac{\beta}{\beta-\gamma}-1}\left(\left|Y^{(\delta)}(\tau'[\tilde{\Pi}_m])\right|^{\frac{\beta}{\beta-\gamma}} + \left|Y^{(\delta)}\left(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m])\right)\right|^{\frac{\beta}{\beta-\gamma}}\right) \\
&\le 2^{\frac{\beta}{\beta-\gamma}}\left(\sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right)^{\frac{\beta}{\beta-\gamma}} \\
&\le 2^{\frac{\beta}{\beta-\gamma}} L_9 \exp\left\{\beta \sup_{0\le u\le T}\left|Y^{(\delta)}(u)\right|\right\}
\end{aligned} \tag{40}
$$

for all $m \ge 1$.

Taking into account inequality (4) of Lemma 1, we prove that the random variable on the right hand side of (40) has a finite expectation. Now relations (39) and (40) together with the Lebesgue theorem imply that

$$
\mathsf{E}Q_m^{(\delta)} \to \mathsf{E}Q^{(\delta)} \quad \text{as } m \to \infty \tag{41}
$$

for $\delta \le \delta_1$.

Now we estimate $\mathsf{E}Q_m^{(\delta)}$. To shorten the notation, we put

$$
Y'_{n+1} = Y^{(\delta)}(v_{n+1,m}), \qquad Y''_{n+1} = Y^{(\delta)}(v_{n+1,m} + f_\Pi(v_{n+1,m})).
$$

Recall that τ' is a Markov moment for the Markov process $Z^{(\delta)}(t)$. Thus the random variables

$$
\chi(v_{n,m} < \tau' \le v_{n+1,m}) \quad \text{and} \quad |Y'_{n+1} - Y''_{n+1}|^{\beta/(\beta-\gamma)}
$$

are conditionally independent with respect to the random variable $Z^{(\delta)}(v_{n+1,m})$. This together with the inequality $f_\Pi(v_{n+1,m}) \le d(\Pi)$ implies that

$$
\begin{aligned}
\mathsf{E}Q_m^{(\delta)} &= \mathsf{E}\left|Y^{(\delta)}(\tau'[\tilde{\Pi}_m]) - Y^{(\delta)}\left(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m])\right)\right|^{\frac{\beta}{\beta-\gamma}} \\
&= \sum_{n=0}^{m-1} \mathsf{E}|Y'_{n+1} - Y''_{n+1}|^{\frac{\beta}{\beta-\gamma}}\chi(v_{n,m} < \tau' \le v_{n+1,m}) \\
&= \sum_{n=0}^{m-1} \mathsf{E}\left\{\chi(v_{n,m} < \tau' \le v_{n+1,m})\mathsf{E}\left\{|Y'_{n+1,} - Y''_{n+1}|^{\frac{\beta}{\beta-\gamma}} \,/\, Z^{(\delta)}(v_{n+1,m})\right\}\right\} \\
&\le \sum_{n=0}^{m-1} \sup_{z\in\mathbf{Z}} \mathsf{E}_{z,v_{n+1,m}}|Y'_{n+1,} - Y''_{n+1}|^{\frac{\beta}{\beta-\gamma}}\mathsf{P}\{v_{n,m} < \tau' \le v_{n+1,m}\} \\
&\le \sum_{n=0}^{m-1} L_9 \sup_{z\in\mathbf{Z}} \mathsf{E}_{z,v_{n+1,m}} \exp\{\beta|Y'_{n+1,} - Y''_{n+1}|\}\mathsf{P}\{v_{n,m} < \tau' \le v_{n+1,m}\} \\
&\le \sum_{n=0}^{m-1} L_9\Delta_\beta\left(Y^{(\delta)}(\cdot), d(\Pi), T\right)\mathsf{P}\{v_{n,m} < \tau' \le v_{n+1,m}\} \\
&\le L_9\Delta_\beta\left(Y^{(\delta)}(\cdot), d(\Pi), T\right)
\end{aligned} \tag{42}
$$

for $\delta \le \delta_1$.

Relations (41) and (42) yield

$$
\mathsf{E}Q^{(\delta)} = \mathsf{E}\left|Y^{(\delta)}(\tau') - Y^{(\delta)}(\tau' + f_\Pi(\tau'))\right|^{\frac{\beta}{\beta-\gamma}} \le L_9\Delta_\beta\left(Y^{(\delta)}(\cdot), d(\Pi), T\right) \tag{43}
$$

for $\delta \le \delta_1$.

This inequality is equivalent to inequality (37), since

$$\left|Y^{(\delta)}(\tau') - Y^{(\delta)}(\tau' + f_{\Pi}(\tau'))\right| = \left|Y^{(\delta)}\left(\tau_{\varepsilon}^{(\delta)}\right) - Y^{(\delta)}\left(\tau_{\varepsilon}^{(\delta)}[\Pi]\right)\right|.$$

If bound (37) holds, then estimate (36) can be continued and transformed to the following form:

$$\begin{aligned} &\mathsf{E}\left|g^{(\delta)}\left(\tau_{\varepsilon}^{(\delta)}, S^{(\delta)}\left(\tau_{\varepsilon}^{(\delta)}\right)\right) - g^{(\delta)}\left(\tau_{\varepsilon}^{(\delta)}[\Pi], S^{(\delta)}\left(\tau_{\varepsilon}^{(\delta)}[\Pi]\right)\right)\right| \\ &\qquad \le L_3 d(\Pi_N) + L_4\left(\Delta_\beta\left(Y^{(\delta)}(\cdot), d(\Pi), T\right)\right)^{\frac{\beta-\gamma}{\beta}} \end{aligned} \tag{44}$$

for $\delta \le \delta_1$, where

$$L_3 = (K_1 + K_2)(L_1)^{\gamma/\beta}, \qquad L_4 = (K_3 + K_4)(L_1)^{\gamma/\beta}(L_9)^{\frac{\beta-\gamma}{\beta}}. \tag{45}$$

Note that the right hand side of inequality (44) does not depend on ε. Substituting this expression into (34) and then passing to the limit as $\varepsilon \to 0$ we prove inequality (26) of Theorem 1.

The proof of Theorem 1 is complete. □

Note that the skeleton approximations given in Theorem 1 have their own value and can be used not only in the convergence theorems (this application will be presented in the second part of the paper).

Indeed, one of the main approaches used to evaluate the reward functional for American type options is based on the Monte Carlo algorithms, which obviously require that the corresponding continuous time price processes are replaced by their more simple discrete time models usually constructed on the base of the corresponding skeleton approximations. Theorem 1 gives explicit estimates for the accuracy of the corresponding approximations of the reward functionals for continuous time price processes by the corresponding reward functionals for the skeleton type discrete time price processes.

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Mälardalen University, Västerås, Sweden
E-mail address: dmitrii.silvestrov@mdh.se

Eurandom, Eindhoven University of Technology, The Netherlands
E-mail address: jonsson.@eurandom.tue.nl

Nordea bank, Stockholm, Sweden
E-mail address: fredsten@kth.se

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