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CONDITIONS FOR THE EXISTENCE AND SMOOTHNESS OF THE DISTRIBUTION DENSITY OF THE ORNSTEIN–UHLENBECK PROCESS WITH LÉVY NOISE

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Abstract. Some sufficient conditions are found for the distribution of the Ornstein–Uhlenbeck process with Lévy noise to be absolutely continuous or to have a smooth density. These conditions are necessary for one-dimensional processes with a nondegenerate drift coefficient. We also give a multidimensional analog of the condition that the drift parameter is nondegenerate.

1. Introduction

The theory of stochastic differential equations with a jump noise has been developed intensively over several decades. This is explained by the needs of several sciences ranging from climatology (see [1]) to finance mathematics (see [2, 3]) where such equations arise as natural models. One of the central problems of this theory is to study the local properties of distributions of solutions of such equations. For example, the existence of the distribution density of a solution helps one to study effectively ergodic properties of a process (see [4] and the references therein) which in turn allows one to perform a statistical analysis as well as to solve the problems of filtration and optimal control for such processes.

There exists an extensive literature devoted to studies of properties of solutions of stochastic differential equations with a jump noise (see, for example, [5]–[14]). The properties depend essentially on the structure of an equation and on its coefficients as well as on the characteristics of the jump noise (of its Lévy measure, in other words). The papers [5]–[14] contain a wide range of sufficient conditions; however, the answers to the questions on the local properties of distributions of solutions of stochastic differential equations are not completely adequate. These sufficient conditions are hard to compare to each other, on the one hand, and it is not clear at all how close they are to the necessary conditions, on the other hand. Therefore an actual and, at the same time, complicated problem concerns general sufficient conditions (being close to the necessary conditions) for the existence or for the smoothness of the distribution density of a solution of a stochastic differential equation with a jump noise.

In this paper, we solve this problem for the simplest class of equations, namely for linear stochastic differential equations with a jump noise. The solutions of such equations are often called Ornstein–Uhlenbeck processes with Lévy noise.

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2. Setting of the problem

Consider a linear stochastic differential equation in $\mathbb{R}^m$,

$$X(t) = X(0) + \int_0^t AX(s)\,ds + Z(t), \tag{1}$$

where $X(0) \in \mathbb{R}^m$, A is a $m \times m$ matrix, Z is a Lévy process assuming values in $\mathbb{R}^m$ (that is, Z is a stochastically continuous homogeneous process with independent increments). It is known (see [15]) that Z possesses the following representation:

$$Z(t) = Z(0) + at + BW(t) + \int_0^t \int_{\|u\|_{\mathbb{R}^m} > 1} u\,\nu(ds, du) + \int_0^t \int_{\|u\|_{\mathbb{R}^m} \le 1} u\,\tilde{\nu}(ds, du), \tag{2}$$

where $a \in \mathbb{R}^m$ and $B \in \mathbb{R}^{m \times m}$ are a nonrandom vector and matrix, respectively, W is a Wiener process in $\mathbb{R}^m$, ν is a random Poisson point measure on $\mathbb{R}^+ \times \mathbb{R}^m$ with the intensity measure $dt \times \Pi(du)$ (here Π is Lévy's measure of the measure ν), and where $\tilde{\nu}(ds, du) = \nu(ds, du) - ds\,\Pi(du)$ is the corresponding compensated measure (W and ν are independent).

Equation (1) can be naturally treated as a set of Volterra type equations parameterized with the probability parameter ω. Thus equation (1) with an arbitrary measurable process Z whose trajectories are locally bounded with probability 1 has a unique solution whose trajectories also are locally bounded with probability 1. Note that every Lévy process has a modification that satisfies the above assumptions imposed on the process Z and therefore a solution of (1) is well defined. Moreover this solution admits the following explicit representation:

$$\begin{aligned} X(t) = e^{tA}X(0) &+ \int_0^t e^{(t-s)A} a\,ds + \int_0^t e^{(t-s)A} B\,dW(s) \\ &+ \int_0^t \int_{\|u\|_{\mathbb{R}^m} > 1} e^{(t-s)A}\,u\nu(ds, du) + \int_0^t \int_{\|u\|_{\mathbb{R}^m} \le 1} e^{(t-s)A} u\,\tilde{\nu}(ds, du), \\ & t \ge 0, \end{aligned} \tag{3}$$

where

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^k}{k!}, \qquad t \ge 1,$$

is a solution of the matrix differential equation $dE(t) = AE(t)\,dt$, where $E(0) = I_{\mathbb{R}^m}$ is the unit $m \times m$ matrix. Equality (3) can be checked explicitly by using Itô's formula.

The first two terms in (3) are nonrandom and therefore do not influence the existence or smoothness of the distribution density of the random vector $X(t)$. In what follows we assume that $X(0) = 0$ and $a = 0$. Moreover, all the terms in (3) are independent and the next to the last term vanishes on the event

$$\big\{\nu([0,t] \times \{\|u\|_{\mathbb{R}^m} > 1\}) = 0\big\}$$

whose probability is positive. Thus $X(t)$ has a (smooth) distribution density if and only if so does the sum where the term mentioned above is omitted. Taking this consideration into account, we always assume that $\Pi(\|u\| > 1) = 0$ when studying the questions on the existence or smoothness of the distribution density of $X(t)$.

3. One-dimensional equation

In the one-dimensional case, A and B are real numbers and the third term in (3) is a Gaussian random variable with variance $B^2 \int_0^t e^{2(t-s)A}\,ds$. Since the terms in (3) are independent, the distribution of $X(t)$ is a convolution of some distribution with a nondegenerate Gaussian distribution if $B \neq 0$. Thus the distribution density is smooth. Throughout this section we assume that $B = 0$.

If $A = 0$, then $X(t) - X(0) = Z(t) - Z(0)$ and thus the question on the properties of the distribution of a solution of (1) is the question on the conditions of the existence and smoothness of the distribution density of a Lévy process without diffusion component. The complete answer to this question is not yet known. Below we provide two sufficient conditions.

Proposition 1. 1. [16] *Put* $\mu(du) = (u^2 \wedge 1)\,\Pi(du)$. *If* $\Pi(\mathbb{R}) = +\infty$ *and, for some* $n \in \mathbb{N}$, *the convolution of* n *copies of the measure* μ *is absolutely continuous, then the distribution of* $Z(t)$ *is absolutely continuous for an arbitrary* $t > 0$.

2. [17] *If* $\left[\varepsilon^2 \ln(1/\varepsilon)\right]^{-1} \int_{-\varepsilon}^{\varepsilon} u^2\,\Pi(du) \to +\infty$ *as* $\varepsilon \to 0+$, *then the distribution of* $Z(t)$ *possesses the density belonging to the class* C_b^∞ *for an arbitrary* $t > 0$.

Here and in what follows the symbol C_b^∞ denotes the class of infinitely differentiable functions whose derivatives are bounded. Throughout below, the conditions of assertions 1 and 2 of Proposition 1 are called the Sato and Kallenberg conditions, respectively. We stress again that every one of these conditions is sufficient and is not necessary. It turns out that one can find necessary and sufficient conditions for both the existence and smoothness of the distribution density if the drift coefficient of an equation is *nondegenerate*.

Proposition 2. *Let* $B = 0$ *and* $A \neq 0$. *Then the distribution of the random variable* $X(t)$ *is absolutely continuous for all* $t > 0$ *if and only if* $\Pi(\mathbb{R}) = +\infty$.

The necessity of condition $\Pi(\mathbb{R}) = +\infty$ is obvious: if $\Pi(\mathbb{R}) = Q < +\infty$, then the distribution of the random variable $X(t)$ contains an atom of weight e^{-tQ}. The sufficiency follows from more general results, namely from Theorem 4.3 of [12] or Theorem A of [14].

Theorem 1. *Let* $B = 0$ *and* $A \neq 0$. *The following three assertions are equivalent:*

(i) *for all* $t > 0$, *the random variable* $X(t)$ *has a distribution density belonging to the class* C_b^∞;

(ii) *for all* $t > 0$, *the random variable* $X(t)$ *has a bounded distribution density;*

(iii) $\left[\varepsilon^2 \ln(1/\varepsilon)\right]^{-1} \int_{\mathbb{R}} (u^2 \wedge \varepsilon^2)\,\Pi(du) \to +\infty$ *as* $\varepsilon \to 0+$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. We prove the implication (ii) $\Rightarrow$ (iii). Put $\rho(\varepsilon) = \left[\varepsilon^2 \ln(1/\varepsilon)\right]^{-1} \int_{\mathbb{R}} (u^2 \wedge \varepsilon^2)\,\Pi(du)$. Fix $\varepsilon \in (0,1)$. Then

$$X(t) = \int_0^t \int_{|u| \le \varepsilon} e^{(t-s)A} u\,\tilde{\nu}(ds, du) + \int_0^t \int_{|u| \in (\varepsilon, 1]} e^{(t-s)A} u\,\tilde{\nu}(ds, du).$$

The variance of the first term is estimated by $t \exp\{2|A|t\} \int_{-\varepsilon}^{\varepsilon} u^2\,\Pi(du)$. By the Chebyshev inequality, the probability that the absolute value of the first term does not exceed $\sqrt{\varepsilon}$ is bounded from below by

$$1 - \varepsilon^{-1} t e^{2|A|t} \int_{-\varepsilon}^{\varepsilon} u^2\,\Pi(du) \ge 1 - t e^{2|A|t} \left[\varepsilon \ln \frac{1}{\varepsilon}\right] \rho(\varepsilon).$$

The second term is equal to

$$M(t, \varepsilon) = \int_0^t \int_{|u| \in (\varepsilon, 1]} e^{(t-s)A} u\,\Pi(du)\,ds$$

with the probability being greater than or equal to

$$\mathsf{P}\big(\nu\big((0,t) \times \{|u| \in (\varepsilon, 1]\}\big) = 0\big) = \exp[-t\Pi(|u| \in (\varepsilon, 1])].$$

The last term is estimated from below by the function $\exp[-t\ln\frac{1}{\varepsilon}\rho(\varepsilon)] = \varepsilon^{t\rho(\varepsilon)}$. Thus

$$\mathsf{P}\big(X(t)\in[M(t,\varepsilon)-\sqrt{\varepsilon}, M(t,\varepsilon)+\sqrt{\varepsilon}]\big) \geq \varepsilon^{t\rho(\varepsilon)} - te^{2|A|t}\left[\varepsilon\ln\frac{1}{\varepsilon}\right]\rho(\varepsilon). \tag{4}$$

Assume that assertion (iii) does not hold, that is, that there exists a sequence $\varepsilon_n \to 0+$ such that $\rho(\varepsilon_n) \leq C < +\infty$. Then, for $t < 1/(2C)$, $x_n = M(t,\varepsilon_n) - \sqrt{\varepsilon}_n$, and

$$y_n = M(t,\varepsilon_n) + \sqrt{\varepsilon}_n,$$

bound (4) implies the convergence

$$\frac{\mathsf{P}(X(t)\in[x_n,y_n])}{y_n - x_n} \to +\infty, \qquad n\to+\infty, \tag{5}$$

whence we conclude that assertion (ii) does not hold.

Now we prove the implication (iii) $\Rightarrow$ (i). According to general properties of the Fourier transform, the existence of a density of the distribution of an m-dimensional random vector belonging to the class C_b^∞ follows from the following condition imposed on the characteristic function ϕ of this vector:

$$\text{for all } n\geq 0, \qquad \|z\|^n_{\mathbb{R}^m}|\phi(z)| \to 0, \quad \|z\|_{\mathbb{R}^m} \to \infty. \tag{6}$$

This is a standard condition, sometimes called condition (C) (see [17]).

The process $X(t)$ is an integral of a nonrandom function with respect to the compensated Poisson point measure. Hence its characteristic function is given by the following explicit relation:

$$\phi_{X(t)}(z) = \exp\left\{\int_0^t\int_{\mathbb{R}}\left[\exp\left\{ize^{(t-s)A}u\right\} - 1 - ize^{(t-s)A}u\right]\Pi(du)\,ds\right\}. \tag{7}$$

Without loss of generality, we assume that $A>0$. Below $t>0$ is fixed. Choose a number $\beta>0$ such that $\beta e^{(t-s)A}\leq 1$ for $s\in[0,t]$ (for example, $\beta = e^{-At}$ fits this assumption). Put

$$I_1(s,z) = \int_{\{|uz|\leq\beta\}}\left[\cos\left(e^{(t-s)\alpha}uz\right) - 1\right]\Pi(du),$$
$$I_2(s,z) = \int_{\{|uz|>\beta\}}\left[\cos\left(e^{(t-s)\alpha}uz\right) - 1\right]\Pi(du).$$

According to (7),

$$|\phi_{X(t)}(z)| = \exp\left\{\int_0^t I_1(s,z)\,ds + \int_0^t I_2(s,z)\right\}, \qquad z\in\mathbb{R}.$$

Let $C = 1-\cos 1$. It is easy to check that $\cos x - 1 \leq -Cx^2$ for $|x|\leq 1$. Thus

$$I_1(s,z) \leq -C\int_{|uz|\leq\beta}\left(e^{(t-s)A}uz\right)^2\Pi(du) = -Cz^2e^{2(t-s)A}\int_{|uz|\leq\beta}u^2\,\Pi(du),$$

whence

$$\int_0^t I_1(s,z)\,ds \leq -C_1z^2\int_{|uz|\leq\beta}u^2\,\Pi(du),$$

where

$$C_1 = C\frac{e^{2tA}-1}{2A} > 0.$$

Further

$$\int_0^t I_2(s,z)\,ds = \int_{|uz|>\beta}\int_0^t\left[\cos\left(e^{(t-s)A}uz\right) - 1\right]\,ds\,\Pi(du).$$

Using the change of variables $y = e^{(t-s)A}uz$ we get

$$\begin{aligned}\int_0^t I_2(s,z)\,ds &= \frac{1}{A}\int_{|uz|>\beta}\int_{uz}^{e^{\,tA}uz}\frac{\cos y - 1}{y}\,dy\,\Pi(du)\\ &\le \int_{|uz|>\beta}\frac{1}{A|uz|}\int_{|uz|}^{e^{\,tA}|uz|}(\cos y - 1)\,dy\,\Pi(du)\\ &= \frac{1}{A}\int_{|uz|>\beta}\left(\frac{\sin(e^{\,tA}|uz|)-\sin(|uz|)}{|uz|} - (e^{\,tA}-1)\right)\Pi(du),\end{aligned}$$

since the function $x \mapsto \cos x - 1$ is even. Put

$$\gamma = \sup_{|y|>\frac{(e^{tA}-1)\beta}{2}}\left|\frac{\sin y}{y}\right| < 1.$$

Then

$$\frac{\sin(e^{tA}x)-\sin x}{x} = \left(e^{tA}-1\right)\frac{\sin\left(\frac{(e^{tA}-1)x}{2}\right)}{\frac{e^{tA}-1}{2}}\cos\left(\frac{\left(e^{tA}+1\right)x}{2}\right) \le \gamma(e^{tA}-1)$$

for $|x| > \beta$. Thus

$$\int_0^t I_2(s,z)\,ds \le -C_2\Pi(\{|uz|>\beta\}), \quad \text{where } C_2 = \frac{1-\gamma}{A}(e^{tA}-1) > 0.$$

Let $C_3 = \min(C_1\beta^2, C_2)$. The bounds obtained above for the integral $\int_0^t I_{1,2}(s,z)\,ds$ imply that

$$|\phi_{X(t)}(z)| \le \left(\frac{\beta}{|z|}\right)^{-C_3\rho\left(\frac{\beta}{|z|}\right)}, \qquad z \in \mathbb{R},$$

whence condition (6) follows by (iii). Thus (i) holds. The theorem is proved. □

Remark 1. The implication (ii) ⇒ (iii) can be completed with the following assertion. Assume that

$$\liminf_{\varepsilon\to 0+}\left[\varepsilon^2\ln\frac{1}{\varepsilon}\right]^{-1}\int_{\mathbb{R}}\left(u^2\wedge\varepsilon^2\right)\,\Pi(du) = 0.$$

Then the random variable $X(t)$ does not have a distribution density belonging to the class $L_p(\mathbb{R})$ for all $t>0$ and $p>1$. To prove this assertion, one can choose

$$\alpha = \frac{1}{2} + \frac{1}{2p} \in (0,1)$$

and the sequence ε_n such that $\rho(\varepsilon_n) \to 0$. Then similar bounds hold for

$$x_n = M(t,\varepsilon_n) - \varepsilon_n^\alpha$$

and $y_n = M(t,\varepsilon_n) + \varepsilon_n^\alpha$, and this implies the convergence

$$\frac{\mathsf{P}(X(t)\in(x_n,y_n))}{(y_n-x_n)^{2-2\alpha}} \to +\infty.$$

The latter relation together with the Hölder inequality shows that $X(t)$ does not have a distribution density belonging to the class $L_{\frac{1}{2\alpha-1}}(\mathbb{R}) = L_p(\mathbb{R})$.

Condition (iii) looks similar to the Kallenberg condition. The following example shows nevertheless that these two conditions are essentially different.

Example 1. Let $\Pi = \sum_{n\ge 1} n\delta_{\frac{1}{n!}}$. Then

$$\liminf_{\varepsilon\to 0+}\rho(\varepsilon) \ge \liminf_{\varepsilon\to 0+}\left\{\left[\ln\frac{1}{\varepsilon}\right]^{-1}\Pi(|u|>\varepsilon)\right\} \ge \liminf_{N\to+\infty}\frac{1}{\ln N!}\sum_{n\le N-1} n$$
$$\ge \liminf_{N\to+\infty}\frac{N(N-1)}{2N\ln N} = +\infty$$

and condition (iii) holds. One can check that the Kallenberg condition does not hold, but we show even more; namely, we show that the distribution of $Z(t)$ is singular for an arbitrary t. This result follows from $\mathsf{E}\, e^{izZ(t)} \not\to 0$ as $z\to\infty$. The latter relation can be proved by

$$\lim_{N\to+\infty}\left|\mathsf{E}\, e^{i2\pi N!\, Z(t)}\right| = \lim_{N\to+\infty}\prod_{n>N}\left|\exp\left\{tn\left(e^{\frac{i2\pi N!}{n!}} - 1 - \frac{i2\pi N!}{n!}\right)\right\}\right| = 1.$$

Therefore we have an interesting phenomenon: the distributions of the process $Z(t)$ are singular but the distributions of a solution of equation (1) with a nondegenerate drift ($A\ne 0$) possess densities belonging to the class C_b^∞. One can describe this situation by saying that the process Z has a "hidden smoothness" that does not appear in the distributions of the process itself but appears if the process is involved as a noise into an equation with a nondegenerate drift.

Such a phenomenon occurs because of the difference between the Kallenberg condition and condition (iii).

Proposition 2 and Theorem 1 imply the general conclusion that the conditions for the *existence* of the distribution density of $X(t)$ and those for the *smoothness* of this density are essentially different. We demonstrate this difference in the following example.

Example 2. Let $\Pi = \sum_{n\ge 1}\delta_{\frac{1}{n!}}$. Then $\Pi(\mathbb{R}) = +\infty$ and

$$\rho(\varepsilon)\to 0, \qquad \varepsilon\to 0.$$

Thus if $A\ne 0$, then a solution of equation (1) has the density, but this density is rather irregular; namely, it does not belong to any of the spaces $L_p(\mathbb{R})$ for $p>1$. This follows from Proposition 2 and Remark 1. Note also that the distribution of Z is singular for this example (this can be proved in the same way as in Example 1). Therefore our example exhibits another possible phenomenon of the "regularization" of a Lévy process under an action of a stochastic differential equation with a nondegenerate drift coefficient.

4. Multidimensional equation

Consider the following auxiliary construction. Let a σ-finite measure Π be defined on $\mathfrak{B}(\mathbb{R}^d)$ for some $d\in\mathbb{N}$. Consider the class

$$\mathfrak{L}_\Pi = \left\{L \text{ is a linear subspace of } \mathbb{R}^d \text{ such that } \Pi(\mathbb{R}^d\backslash L) < +\infty\right\}.$$

It is clear that $L_1\cap L_2\in\mathfrak{L}_\Pi$ if $L_1, L_2\in\mathfrak{L}_\Pi$. This implies that there exists a subspace $L_\Pi\in\mathfrak{L}_\Pi$ such that $L_\Pi\subset L$ for all $L\in\mathfrak{L}_\Pi$.

Definition 1. A subspace L_Π is called an essential linear support of the measure Π. A measure Π is called essentially linearly nondegenerate if $L_\Pi = \mathbb{R}^d$.

Yamazato [18] was the first to provide a condition for a measure Π to be essentially linearly nondegenerate. Since then it is often called the Yamazato condition.

In this section, we study local properties of the distribution of a solution of the following equation:

$$X(t) = \int_0^t AX(s)\,ds + BW(t) + DZ(t), \qquad t \geq 0, \tag{8}$$

where A, B, and D are $m \times m$, $m \times k$, and $m \times d$ matrices, respectively, W is a Wiener process in $\mathbb{R}^k$, the process Z is of the form

$$Z(t) = \int_0^t \int_{\|u\|_{\mathbb{R}^d} > 1} u\,\nu(ds, du) + \int_0^t \int_{\|u\|_{\mathbb{R}^d} \leq 1} u\,\tilde{\nu}(ds, du), \qquad t \geq 0,$$

and the Lévy measure of the process Z is essentially linearly nondegenerate. We have seen above that a solution X of equation (1) depends on Z linearly; namely, if $Z = Z_1 + Z_2$, then $X = X_1 + X_2$, where $X_{1,2}$ are solutions of equation (1) in which $Z_{1,2}$ is substituted for Z. If Z_1 and Z_2 are independent and Z_1 is a Lévy process without a diffusion component whose Lévy measure Π_1 is finite, then the distribution of $X_1(t)$ has an atom. Hence the existence or smoothness of the density of the distribution of $X(t)$ is equivalent to the existence or smoothness of the distribution of $X_2(t)$. This reasoning allows one to omit the insignificant (in the sense of Definition 1) jump component of the Lévy process Z. Namely, let ν_1 be the restriction of the measure ν to $\mathbb{R}^+ \times (\mathbb{R}^m \setminus L_\Pi)$ and let Z_1 be defined by equality (2), where $a = 0$, $B = 0$, and ν is replaced by ν_1. It is easy to see that equation (1) where Z_1 is substituted for Z is of the form (8) for $k = m$ and $d = \dim L_\Pi$. Therefore the study of equation (1) is reduced to that of equation (8).

If $D = 0$ in equation (8), then the well-known Kalman controllability condition is necessary and sufficient for the existence of a smooth distribution density for $X(t)$, $t > 0$ (see, for example, [20]); namely,

$$\operatorname{Rank}[B, AB, \dots, A^{m-1}B] = m.$$

Here $[B, AB, \dots, A^{m-1}B]$ denotes a $m \times mk$ block matrix whose blocks are the matrices $B, \dots, A^{m-1}B$. We write an analogous condition for equation (8):

(**H1**) $\operatorname{Rank}[B, AB, \dots, A^{m-1}B, D, AD, \dots, A^{m-1}D] = m$,

where $[B, AB, \dots, A^{m-1}B, D, AD, \dots, A^{m-1}D]$ is an $m \times m(k + d)$ matrix constructed from the matrices $B, \dots, A^{m-1}B, D, AD, \dots, A^{m-1}D$.

In what follows let $S^d = \{l \in \mathbb{R}^d, \|l\|_{\mathbb{R}^d} = 1\}$ be the unit sphere in $\mathbb{R}^d$. Consider the following multidimensional analog of the Kallenberg condition:

$$\left[\varepsilon^2 \ln \frac{1}{\varepsilon}\right]^{-1} \inf_{l \in S^d} \int_{|(u,l)_{\mathbb{R}^d}| \leq \varepsilon} (u, l)^2_{\mathbb{R}^d}\, \Pi(du) \to +\infty, \qquad \varepsilon \to 0+\,. \tag{9}$$

Note that this condition did not appear in the literature until now.

Theorem 2. *Let a Lévy process Z satisfy* (9). *If condition* (**H1**) *holds, then the distribution of $X(t)$, $t > 0$, has the density belonging to the class C_b^∞.*

Proof. Similarly to the proof of Theorem 1 we check that the characteristic function of $X(t)$ satisfies condition (6). Without loss of generality we assume that $\Pi(\|u\|_{\mathbb{R}^d} > 1) = 0$. The value of $X(t)$ is represented as a sum of integrals over the (independent) Wiener process and the compensated Poisson point measure. Thus the corresponding characteristic function admits the following representation:

$$\begin{aligned}\phi_{X(t)}(z) = \exp\Bigg\{\int_0^t \Bigg(-\frac{1}{2}\left\|B^* e^{(t-s)A^*} z\right\|^2_{\mathbb{R}^k} \\ + \int_{\mathbb{R}^d} \Big[\exp\Big\{i\left(e^{(t-s)A} Du, z\right)_{\mathbb{R}^m}\Big\} - 1 \\ - i\left(e^{(t-s)A} Du, z\right)_{\mathbb{R}^m}\Big]\Pi(du)\Bigg)ds\Bigg\}, \quad z \in \mathbb{R}^m,\end{aligned} \tag{10}$$

where the symbol $*$ denotes conjugation. Then

$$\begin{aligned}|\phi_{X(t)}(z)| = \exp\Bigg\{\int_0^t \Bigg(-\frac{1}{2}\left\|B^* e^{(t-s)A^*} z\right\|^2_{\mathbb{R}^k} \\ + \int_{\mathbb{R}^d} \Big[\cos\left(e^{(t-s)A} Du, z\right)_{\mathbb{R}^m} - 1\Big]\Pi(du)\Bigg)ds\Bigg\}.\end{aligned} \tag{11}$$

Put $B(s,z) = B^* e^{sA^*} z$ and $D(s,z) = D^* e^{sA^*} z$. Reducing the integration area with respect to the variable u to the set $\{|(D(s,z),u)_{\mathbb{R}^d}| \le 1\}$ and using the inequality

$$1 - \cos x \ge Cx^2, \qquad |x| \le 1,\ C = 1 - \cos 1 > \frac{1}{2},$$

we get

$$|\phi_{X(t)}(z)| \le \exp\left\{-\frac{1}{2}\int_0^t \left(\|B(s,z)\|^2_{\mathbb{R}^k} + \int_{|(D(s,z),u)_{\mathbb{R}^d}| \le 1} (D(s,z),u)^2_{\mathbb{R}^d}\,\Pi(du)\right)ds\right\}. \tag{12}$$

Put

$$\Phi(r) = r^2 \inf_{l \in S^m} \int_{|(u,l)_{\mathbb{R}^d}| \le 1/r} (u,l)^2_{\mathbb{R}^d}\,\Pi(du), \qquad r > 0.$$

Note that condition (9) is equivalent to the convergence $\Phi(r)/\ln r \to +\infty$ as $r \to +\infty$. The notation introduced above allows one to rewrite (12) as follows:

$$|\phi_{X(t)}(z)| \le \exp\left\{-\frac{1}{2}\int_0^t \left(\|B(s,z)\|^2_{\mathbb{R}^k} + \Phi(\|D(s,z)\|_{\mathbb{R}^d})\right)ds\right\}. \tag{13}$$

Lemma 1. *Assume that condition* (**H1**) *holds. For a given $t > 0$, there are α, β, and $\gamma > 0$ such that*

$$\text{for all } l \in S^m, \qquad \lambda\big\{0 \le s \le t \colon \|B(s,l)\|_{\mathbb{R}^k} > \alpha \text{ or } \|D(s,l)\|_{\mathbb{R}^d} > \beta\big\} \ge \gamma,$$

where λ is Lebesgue measure on $\mathbb{R}$.

Proof. We prove Lemma 1 by contradiction. Let the statement of the lemma be false. Then there exists a sequence $l_n \in S^m$, $n \ge 1$, such that

$$\lambda\left\{0 \le s \le t \colon \|B(s,l_n)\|_{\mathbb{R}^k} > \frac{1}{n} \text{ or } \|D(s,l_n)\|_{\mathbb{R}^d} > \frac{1}{n}\right\} < \frac{1}{n}, \qquad n \ge 1,$$

whence we conclude that both sequences of functions $\{\|B(\cdot,l_n)\|_{\mathbb{R}^k}\}$ and $\{\|D(\cdot,l_n)\|_{\mathbb{R}^d}\}$ converge in the Lebesgue measure to the function vanishing throughout. Since S^m is a compact set, we assume without loss of generality that $l_n \to l \in S^m$. On the other hand, for all $s \in [0,t]$, the mappings $B(s,\cdot)$ and $D(s,\cdot)$ are linear and continuous. Thus the functions $\|B(\cdot,l)\|$ and $\|D(\cdot,l)\|$ are equal to zero almost everywhere with respect to the Lebesgue measure. Since these functions are continuous,

$$B^* e^{sA^*} l = 0, \qquad D^* e^{sA^*} l = 0, \qquad s \in [0,t]. \tag{14}$$

Differentiating equality (14) $m-1$ times with respect to s and considering the values of the functions $B(s,l)$ and $D(s,l)$ and those of their derivatives at $s=0$, we obtain

$$B^*l = B^*A^*l = \cdots = B^*(A^*)^{m-1}l = 0, \qquad D^*l = D^*A^*l = \cdots = D^*(A^*)^{m-1}l = 0.$$

The latter equality is equivalent to the condition that the rows of the matrix

$$[B, AB, \ldots, A^{m-1}B, D, AD, \ldots, A^{m-1}D]$$

are linearly dependent and the coefficients of this dependence are equal to the elements of the vector l. This contradicts condition (**H1**). The lemma is proved. $\square$

Now we can finish the proof of Theorem 2. Given a vector $z \in \mathbb{R}^m$, put $l(z) = z/\|z\|_{\mathbb{R}}^m$. Then

$$\begin{aligned}\lambda\{0 \le s \le t\colon \|B(s,z)\|_{\mathbb{R}^k} > \alpha\|z\|_{\mathbb{R}^m} \text{ or } \|D(s,z)\|_{\mathbb{R}^d} > \beta\|z\|_{\mathbb{R}^m}\} \\ = \lambda\big\{0 \le s \le t\colon \|B(s,l(z))\|_{\mathbb{R}^k} > \alpha \text{ or } \|D(s,l(z))\|_{\mathbb{R}^d} > \beta\big\} \ge \gamma.\end{aligned}$$

The latter inequality and bound (13) imply that

$$|\phi_{X(t)}(z)| \le \exp\Big\{-\frac{\gamma}{2}\min\Big(\alpha\|z\|^2_{\mathbb{R}^m}, \Phi(\beta\|z\|^2_{\mathbb{R}^m})\Big)\Big\}. \tag{15}$$

This bound together with (9) justifies condition (6). The theorem is proved. $\square$

Remark 2. One can extend Theorem 2 by describing the asymptotic behavior of derivatives of the density $p_{X(t)}$ as $\|x\|_{\mathbb{R}^m} \to \infty$ in more detail. We study this behavior below in Section 5.

Remark 3. A result similar to Theorem 2 is given in [21] (Theorem 1.3). However conditions imposed on the Lévy measure in [21] (namely, Assumption 1.3) are too restrictive and less precise than the multidimensional Kallenberg condition (9) used in this paper.

It is proved in [21] (Theorem 1.1) that condition (**H1**) implies the absolute continuity of the distribution of a solution of equation (8) if the jump noise satisfies an analog of the Sato condition (the case of $B=0$ is considered in [21]). This result and Theorem 1 of this paper show that (**H1**) is a natural condition imposed on the coefficients of the equation that guarantee the "conservation of the smoothness" presented in the noise (W,Z). On the other hand, this condition is satisfied, for example, with the matrices $A=0$, $B=0$, and $D=I_{\mathbb{R}^m}$, $d=m$, for which $X(t)-X(0)=Z(t)-Z(0)$. This makes it clear that condition (**H1**) does not imply the "regularization" similar to that obtained for one-dimensional equations with a nondegenerate drift studied in the latter section.

A kind of "regularization" mentioned above follows from the following condition:

(**H2**) $\operatorname{Rank}[AD, \ldots, A^mD] = m$

(at least, this condition implies the existence of the distribution density). This condition involves both matrices A and D; nevertheless, it can naturally be treated as an analog of the condition for the nondegeneracy of the drift coefficient. Note that this condition is new, too.

Theorem 3. *The following assertions are equivalent:*

(i) *condition* (**H2**) *holds;*

(ii) *for an arbitrary equation of the form* (8) *where the process Z satisfies the Yamazato condition, the distribution of the random vector $X(t)$ is absolutely continuous for all $t>0$.*

Proof. We prove the implication (i) $\Rightarrow$ (ii) assuming, without loss of generality, that $B=0$ and $\Pi(\|u\|_{\mathbb{R}^d} > 1) = 0$. As explained above, this assumption does not restrict our consideration, indeed, since the solution depends on the noise in a linear way.

Now we use a sufficient condition for the absolute continuity of the distribution of a solution of a stochastic differential equation with a jump noise given in Theorem 1.1 of [13]. In turn, this assumption uses the construction described in [12]. In [12] and [13], the general statements are formulated under an additional moment condition on the Lévy measure (condition (1.1) in [13]). For equation (8), the latter condition becomes $\int_{\|u\|\le 1}\|Du\|\,\Pi(du)<+\infty$. This condition is used in [12, 13] just to prove the differentiability of a solution $X(t)$ with respect to a certain group of transformations of a Poisson point measure. This differentiability holds for equations with an additive noise without any conditions (see [14, 19]). Thus one can apply the results of [13] to equation (8) dropping the moment condition (1.1) of [13].

Statement A of Theorem 1.1 in [13] is stated in terms of a certain subspace generated by a sequence of vector fields associated with the initial equation. This assertion can be reformulated in the following form for the particular case of the linear equation (8). For $u\in\mathbb{R}^d$, put

$$\Delta(u)=ADu,\qquad \mathfrak{L}(u)=\operatorname{span}\Big\{\Lambda^k\Delta(u),k\in\mathbb{Z}_+\Big\},\qquad \Lambda v\stackrel{\mathrm{df}}{=}-Av.$$

If

$$\Pi\Big(u\colon l \text{ is not orthogonal to } \mathfrak{L}(u)\Big)=+\infty \tag{16}$$

for an arbitrary $l\in S^m$, then the distribution of a solution of equation (8) is absolutely continuous. If condition (**H2**) holds and $l\in S^m$ is arbitrary, then there exists a proper linear subspace $L_l\subset\mathbb{R}^d$ such that

$$u\notin L_l\Rightarrow \exists k\in\{1,\dots,m\}\colon\quad A^kDu\not\perp l.$$

Thus

$$\Pi\Big(u\colon l \text{ is not orthogonal to } \mathfrak{L}(u)\Big)\ge\Pi\Big(\mathbb{R}^d\setminus L_l\Big)$$

and this together with the Yamazato condition implies (16). The implication (i) $\Rightarrow$ (ii) is proved.

Now we prove the implication (ii) $\Rightarrow$ (i). Let $B=0$. We show that there exists a nonzero vector $l\in\mathbb{R}^m$ such that

$$(X(t),l)_{\mathbb{R}^m}=(Z(t)-Z(0),D^*l)_{\mathbb{R}^d},\qquad t\ge 0. \tag{17}$$

If $D=0$, then (17) obviously holds for every $l\in\mathbb{R}^m$. Further we consider the case of $D\ne 0$. In this case, $\operatorname{Ker}D^*$ is a proper subspace of $\mathbb{R}^m$. If condition (**H2**) does not hold, then there exists a nonzero vector $l\in\mathbb{R}^m$ such that

$$D^*A^*l=\dots=D^*(A^*)^ml=0; \tag{18}$$

that is, the vectors $A^*l,\dots,(A^*)^ml$ belong to the subspace $\operatorname{Ker}D^*$. Since the dimension of this subspace does not exceed $m-1$, there are $k\le m$ and $c_1,\dots,c_{k-1}\in\mathbb{R}$ such that

$$(A^*)^kl=\sum_{j=1}^{k-1}c_j(A^*)^jl. \tag{19}$$

Multiplying (19) on the left by $(A^*)^{m+1-k}$ and taking into account that

$$(A^*)^{m+1-k+j}\in\operatorname{Ker}D^*,\qquad j\le k-1,$$

we obtain $(A^*)^{m+1}l\in\operatorname{Ker}D^*$. Repeating this reasoning we prove that

$$(A^*)^nl\in\operatorname{Ker}D^*,\qquad n\in\mathbb{N},$$

whence $e^{(t-s)A^*}l - l \in \operatorname{Ker} D^*$, $0 \le s \le t$. Thus

$$(X(t), l)_{\mathbb{R}^m} = \int_0^t \int_{\|u\|_{\mathbb{R}^d} \le 1} \left(u, D^* e^{(t-s)A^*} l\right)_{\mathbb{R}^d} \tilde{\nu}(ds, du) = \int_0^t \int_{\|u\|_{\mathbb{R}^d} \le 1} (u, D^* l)_{\mathbb{R}^d} \tilde{\nu}(ds, du)$$

and this proves (17).

If $D^* l = 0$, then (17) implies that the distribution of $X(t)$ is singular for an arbitrary Z.

Further we consider the case of $D^* l \ne 0$. Choose an orthonormal basis $e_1, \dots, e_d$ in $\mathbb{R}^d$ such that e_1 is collinear to $D^* l$. Denote by $\gamma(r)$, $r \ge 0$, the point of $\mathbb{R}^d$ whose coordinates in this basis are $r, r^2, \dots, r^d$. The standard reasoning with the help of the Vandermonde determinant proves that the curve $\{\gamma(r), r \in \mathbb{R}^+\}$ has at most d intersection points with an arbitrary hyperplane of $\mathbb{R}^d$. Thus the measure $\Pi = \sum_{k \in \mathbb{N}} \delta_{\gamma(\frac{1}{k!})}$ satisfies the Yamazato condition. This means that the distribution of the random variable $(X(t) - e^{At} X(0), l)_{\mathbb{R}^m}$ coincides, up to a multiplicative constant, with the distribution of the random variable $Z(t)$ introduced in Example 2 and therefore is singular. Thus the distribution of the vector $X(t)$ is also singular. The theorem is proved. $\square$

Remark 4. Condition (**H1**) involves all three matrices A, B, and D. Thus the smoothness of the distribution density of a solution of equation (8) depends on both the diffusion and jump noises in contrast to the statement (ii) of Theorem 3 where both the process Z and the matrix B are arbitrary. The following condition is an analog of (**H2**) for an equation with a fixed matrix B:

(**H2′**) $\operatorname{Rank}[B, \dots, A^{m-1}B, AD, \dots, A^m D] = m$.

The proof of the necessity of this condition is completely analogous to that of the necessity of condition (**H2**) given in Theorem 3. For the proof of the sufficiency, we cannot rely on the results of the papers [12, 13], since equations with a diffusion component are not considered in [12, 13]. For *linear* stochastic differential equations, the reasoning presented in [12, 13] can be extended without any essential change to equations with a diffusion component. In doing so, one should prove that condition (**H2′**) is sufficient. We see no need to provide a detailed proof here, since this would require repeating a big part of the papers [12, 13].

Example 3 ([13, Example 1.1]). Consider the following system of stochastic differential equations:

$$\begin{cases} dX_1(t) = X_1(t)\,dt + dZ(t), \\ dX_2(t) = X_1(t)\,dt. \end{cases}$$

If W is substituted for Z in this system of equations, then we obtain the well-known Kolmogorov example of a two-dimensional diffusion with a smooth distribution density generated by a one-dimensional Brownian motion. The above system is of the form (8) with $m = 2$, $d = 1$, $B = 0$, and

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad [D, AD] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad [AD, A^2 D] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Condition (**H1**) holds in this case but condition (**H2**) does not. This means that the property of "preservation of the smoothness" is presented in Kolmogorov's example, while the property of the "regularization" is not.

We modify the Kolmogorov example by considering the following system of stochastic differential equations:

$$\begin{cases} dX_1(t) = X_2(t)\,dt + dZ(t), \\ dX_2(t) = X_1(t)\,dt. \end{cases}$$

This system is of the form (8) with $m = 2$, $d = 1$, $B = 0$, and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad [AD, A^2D] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Condition (**H2**) holds for this system. Thus, for an arbitrary process Z with an infinite Lévy measure (that is, for a process with an infinite number of jumps in every time interval), the distribution of $X(t) = (X_1(t), X_2(t))$ has the density with respect to the Lebesgue measure in $\mathbb{R}^2$.

5. Asymptotic properties of the derivatives of the distribution density

Along with the question concerning the existence and differentiability of the distribution density

$$p_{X(t)}(x), \qquad x \in \mathbb{R}^m,$$

one can ask the question on the limit behavior of this density as $\|x\|_{\mathbb{R}^m} \to +\infty$. In [21] (Remark 3.1), the question on the integrability of the derivatives of the density $p_{X(t)}$ appears in the studies of smoothing properties of the semigroup generated by the process X. In this section, we provide a stronger result than Theorem 2 that answers completely the above question.

In what follows we denote by $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of infinitely differentiable functions $f\colon \mathbb{R}^m \to \mathbb{R}$ such that the rate of convergence to zero as $\|x\|_{\mathbb{R}^m} \to \infty$ of any derivative of a function $f \in \mathcal{S}(\mathbb{R}^m)$ is higher than $\|x\|_{\mathbb{R}^m}^{-n}$ for all n.

Theorem 4. *Consider equation* (8). *If conditions* (9) *and* (**H1**) *hold, then for all* $j_1, \dots, j_r \in \{1, \dots, m\}$, $r \in \mathbb{N}$, *and* $t > 0$,

$$\frac{\partial^r}{\partial x_{j_1} \cdots \partial x_{j_m}} p_{X(t)} \in L_1(\mathbb{R}^m). \tag{20}$$

Moreover, if the Lévy measure of the process Z is such that

$$\int_{\|u\|_{\mathbb{R}^d} > 1} \|u\|_{\mathbb{R}^d}^n \, \Pi(du) < +\infty, \qquad n \in \mathbb{N}, \tag{21}$$

then $p_{X(t)} \in \mathcal{S}(\mathbb{R}^m)$ *for* $t > 0$.

Proof. First we consider the case where the Lévy measure of the process Z satisfies condition (21). The Fourier transform is a bijective transform $\mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m)$ (see, for example, [22, §6.1]). In order to prove the theorem, we thus need to show that the rate of convergence to zero as $\|z\|_{\mathbb{R}^m} \to \infty$ of every derivative of the characteristic function $\phi_{X(t)}(z)$, $z \in \mathbb{R}^m$, is higher than $\|z\|_{\mathbb{R}^m}^{-n}$ for all n. The characteristic function $\phi_{X(t)}$ admits a representation similar to (10), namely $\phi_{X(t)} = \exp[\psi_{X(t)}]$, where

$$\begin{aligned}\psi_{X(t)}(z) = \int_0^t \Bigg(&-\frac{1}{2} \left\| B^* e^{(t-s)A^*} z \right\|^2_{\mathbb{R}^k} \\ &+ \int_{\|u\|_{\mathbb{R}^d}>1} \left[\exp\left\{ i \left(e^{(t-s)A} Du, z \right)_{\mathbb{R}^m} \right\} - 1 \right] \Pi(du) \\ &- \int_{\|u\|_{\mathbb{R}^d}\le 1} \Bigg[\exp\left\{ i \left(e^{(t-s)A} Du, z \right)_{\mathbb{R}^m} \right\} - 1 \\ &\qquad - i \left(e^{(t-s)A} Du, z \right)_{\mathbb{R}^m} \Bigg] \Pi(du) \Bigg) \, ds, \qquad z \in \mathbb{R}^m.\end{aligned}$$

Then every derivative of the function $\phi_{X(t)}$ is of the form $R \cdot \phi_{X(t)}$, where R is some polynomial of the derivatives of $\psi_{X(t)}$. Conditions (9) and (**H1**) imply that

$$\phi_{X(t)}(z) = o\left(\|z\|^{-n}_{\mathbb{R}^m} \right), \qquad \|z\|_{\mathbb{R}^m} \to \infty, \ n \in \mathbb{N}.$$

(This result is proved in Theorem 2.) Thus, in order to obtain the result of Theorem 4, we need to check that every derivative of the function $\psi_{X(t)}$ has at most the polynomial rate as $\|z\|_{\mathbb{R}^m} \to \infty$. We have

$$\begin{aligned}&\frac{\partial}{\partial z_j} \psi_{X(t)}(z) \\ &\quad = -\int_0^t \left(B^* e^{(t-s)A^*} z, B^* e^{(t-s)A^*} e_j \right)_{\mathbb{R}^k} ds \\ &\qquad + \int_0^t \int_{\|u\|_{\mathbb{R}^d}\le 1} i \left(e^{(t-s)A} Du, e_j \right)_{\mathbb{R}^m} \left[\exp\left\{ i \left(e^{(t-s)A} Du, z \right)_{\mathbb{R}^m} \right\} - 1 \right] \Pi(du)\, ds \\ &\qquad + \int_0^t \int_{\|u\|_{\mathbb{R}^d}>1} i \left(e^{(t-s)A} Du, e_j \right)_{\mathbb{R}^m} \exp\left\{ i \left(e^{(t-s)A} Du, z \right)_{\mathbb{R}^m} \right\} \Pi(du)\, ds, \\ &\qquad\qquad j = 1, \dots, m,\end{aligned}$$

where e_j denotes the jth vector in a basis of $\mathbb{R}^m$. Since $|e^{iz} - 1| \le |z|$, we get

$$\begin{aligned}&\left| \frac{\partial}{\partial z_j} \psi_{X(t)}(z) \right| \\ &\quad \le C_1 \left(\|z\|_{\mathbb{R}^m} + \|z\|_{\mathbb{R}^m} \int_{\|u\|_{\mathbb{R}^d}\le 1} \|u\|^2_{\mathbb{R}^d} \Pi(du) + \int_{\|u\|_{\mathbb{R}^d}>1} \|u\|_{\mathbb{R}^d} \Pi(du) \right).\end{aligned} \tag{22}$$

Here and in what follows, C_r, $r = 1, 2, \dots$, denote some constants determined by the coefficients A, B, D and t. Further,

$$\begin{aligned}\frac{\partial^2}{\partial z_{j_1} \partial z_{j_2}} \psi_{X(t)}(z) = &-\int_0^t \left(B^* e^{(t-s)A^*} e_{j_1}, B^* e^{(t-s)A^*} e_{j_2} \right)_{\mathbb{R}^k} ds \\ &+ \int_0^t \int_{\mathbb{R}^d} i^2 \left(e^{(t-s)A} Du, e_{j_1} \right)_{\mathbb{R}^m} \left(e^{(t-s)A} Du, e_{j_2} \right)_{\mathbb{R}^m} \\ &\qquad \times \exp\left\{ i \left(e^{(t-s)A} Du, z \right)_{\mathbb{R}^m} \right\} \Pi(du)\, ds,\end{aligned}$$

$j_{1,2} = 1, \dots, m$. This implies that

$$\left| \frac{\partial^2}{\partial z_{j_1} \partial z_{j_2}} \psi(z)_{X(t)}(z) \right| \le C_2 \left(1 + \|z\|^2_{\mathbb{R}^m} \int_{\mathbb{R}^d} \|u\|^2_{\mathbb{R}^d} \Pi(du) \right), \qquad j_{1,2} = 1, \dots, m. \tag{23}$$

Finally, the partial derivatives of order r, $r \ge 3$, are of the form

$$\frac{\partial^r}{\partial z_{j_1}\cdots\partial z_{j_r}}\psi_{X(t)}(z) = \int_0^t\int_{\mathbb{R}^d} i^r \prod_{l=1}^r \left(e^{(t-s)A}Du, e_{j_l}\right)_{\mathbb{R}^m} \exp\left\{i\left(e^{(t-s)A}Du, z\right)_{\mathbb{R}^m}\right\} \Pi(du)\, ds,$$

$j_{1,\dots,r} = 1,\dots,m$, whence we derive the following bound:

$$\left|\frac{\partial^r}{\partial z_{j_1}\cdots\partial z_{j_r}}\psi_{X(t)}(z)\right| \le C_r\left(\int_{\|u\|_{\mathbb{R}^d}\le 1}\|u\|^2_{\mathbb{R}^d}\,\Pi(du) + \int_{\|u\|_{\mathbb{R}^d}>1}\|u\|^r_{\mathbb{R}^d}\,\Pi(du)\right). \tag{24}$$

Now relations (22)–(24) imply that the first derivatives of the function $\psi_{X(t)}$ are bounded from above by a polynomial of the first degree, while all higher derivatives are bounded. Thus $\phi_{X(t)} \in \mathcal{S}(\mathbb{R}^m)$, whence $p_{X(t)} \in \mathcal{S}(\mathbb{R}^m)$.

Now we consider the general case. We have $W = W_1 + W_2$, $Z = Z_1 + Z_2$, $W_2 = 0$, and $Z_2(t) = \int_0^t\int_{\|u\|_{\mathbb{R}^d}>1} u\,\nu(ds,du)$. Let $X_{1,2}$ be solutions of an equation of the form (8), where W and Z are changed by $W_{1,2}$ and $Z_{1,2}$, respectively. Then a solution of equation (8) is such that $X = X_1 + X_2$ and moreover X_1 and X_2 are independent. Thus the distribution density $p_{X(t)}$ is equal to

$$p_{X(t)}(x) = \int_{\mathbb{R}^m} p_{X_1(t)}(x-y)\,\mu_{X_2(t)}(dy), \qquad x \in \mathbb{R}^m,$$

where $\mu_{X_2(t)}$ is the distribution of $X_2(t)$. According to the case of the theorem proved above, $p_{X_1(t)} \in \mathcal{S}(\mathbb{R}^m)$, whence

$$\frac{\partial^r}{\partial x_{j_1}\cdots\partial x_{j_m}}p_{X(t)} = \int_{\mathbb{R}^m}\frac{\partial^r}{\partial x_{j_1}\cdots\partial x_{j_m}}p_{X_1(t)}(\cdot - y)\,\mu_{X_2(t)}(dy) \in L_1(\mathbb{R}^m),$$
$$j_{1,\dots,r} = 1,\dots,m, \quad r \in \mathbb{N}, \quad t > 0.$$

The theorem is proved. □

Remark 5. If (21) does not hold, then $\mathsf{E}\,\|Z(t)\|^n_{\mathbb{R}^d} = +\infty$ for some $n \in \mathbb{N}$. A stable process of index $\alpha \in (0,2)$ is a typical example where the latter case holds. Setting $d = m$, $A = 0$, $B = 0$, and $D = I_{\mathbb{R}^m}$ we get

$$\mathsf{E}\,\|X(t)\|^n_{\mathbb{R}^m} = +\infty.$$

Therefore condition (21) is, in fact, necessary for the distribution density of a solution $X(t)$ to belong to the Schwartz space $\mathcal{S}(\mathbb{R}^m)$.

Concluding remarks

Conditions obtained in this paper allow one to conclude that there are essential differences between two notions that are close to each other at first glance, namely between the *existence* of the distribution density and its *smoothness*. These two properties appear in a natural way when studying the local properties of distributions of solutions of stochastic differential equations with a jump noise. It turns out that the existence of a smooth density is closely related to the behavior of the Lévy measure of the jump noise in a neighborhood of the point 0 (the Kallenberg condition and its analog (9), condition (iii) of Theorem 1). In general, conditions (either necessary or sufficient) for the existence of the density are much weaker than those for its smoothness. Moreover, equations with a nondegenerate (in a certain sense) drift coefficient differ from the general case. Namely, in contrast to the general case, it is possible to provide the *criteria* for the existence and smoothness of the distribution density for equations with a nondegenerate drift coefficient. In addition, the nondegeneracy of the drift coefficient makes possible the "regularization" phenomenon of the distribution of the Lévy noise under an action of a stochastic differential equation.

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