Теорія Ймовір. та Матем. Статист.
Вип. 79, 2008

Theor. Probability and Math. Statist.
No. 79, 2009, Pages 57–72
S 0094-9000(09)00780-7
Article electronically published on December 29, 2009

ASYMPTOTIC NORMALITY OF L_p-ESTIMATORS IN NONLINEAR REGRESSION MODELS WITH WEAK DEPENDENCE

UDC 519.21

O. V. IVANOV AND I. V. ORLOVS'KIĬ

Abstract. A theorem on asymptotic normality is proved, and the limit distribution is found for L_p-estimators of a vector parameter in a nonlinear regression model with continuous time and weakly dependent random noise.

Introduction

We obtain conditions of asymptotic normality for L_p-estimators of an unknown parameter of nonlinear regression model with random noise satisfying a weak dependence condition.

L_p-estimators belong to the class of the so-called M-estimators. The number of papers related to M-estimators is rather large. Applications of M-estimators in linear regression models with independent observation errors are considered in the pioneering papers by Huber [22, 13]. Several asymptotic results for M-estimators of parameters of linear and nonlinear regression models with independent observation errors are proved by Hampel et al. [21], Chen and Wu [16], Jurečková [29], Liese and Vajda [38, 39, 40, 41, 42], Müller [43], Liese [37], Arcones [15], Wu and Zen [47], van de Geer [17], Orlovskiĭ [12], Ivanov and Orlovskiĭ [10, 27], and by many other authors.

Asymptotic properties of M-estimators of parameters of linear and nonlinear regression models with random noise satisfying a strong dependence condition are studied by Koul [30, 31], Koul and Mukherjee [33], Giraitis et al. [19], Koul and Surgailis [34, 35, 36], Giraitis and Koul [18], Koul et al. [32] in the case of discrete time, and by Ivanov and Leonenko [25] and Ivanov and Orlovskiĭ [11] in the case of continuous time.

Orlovskiĭ [44] and Ivanov and Orlovskiĭ [11] consider asymptotic properties of M-estimators of parameters of nonlinear regression models with continuous time and a weakly dependent random noise.

The most studied in the class of L_p-estimators are the least squares estimator (the case of $p = 2$) and least modules estimator (the case of $p = 1$). Asymptotic properties of least squares estimators and least modules estimators of parameters of nonlinear regression models are studied by many authors. We only mention the monographs by Ivanov and Leonenko [9] and Ivanov [23], where a rather complete bibliography concerning this question can be found.

Asymptotic properties of L_p-estimators of parameters of linear and nonlinear regression models with independent observation errors are considered by Huber [13], Ronner [46], Ivanov [8], Bardadym and Ivanov [1, 2]; those for nonlinear models with random

2000 *Mathematics Subject Classification.* Primary 62J02; Secondary 62J99.
Key words and phrases. L_p-estimators, asymptotic normality, nonlinear regression models, weak dependence.

noise satisfying a condition of either strong or weak dependence are considered in the papers by Ivanov and Orlovskiĭ [26] and Ivanov [24].

We deal with the L_p-estimators, $1 < p < 2$, in this paper.

1. Conditions for and statement of the main result

Consider the following regression model:

$$X(t) = g(t,\theta) + \varepsilon(t), \qquad t \in [0,T], \tag{1}$$

where $g(t,\theta)\colon [0;\infty) \times \Theta^c \to \mathbf{R}^1$ is a real continuous function, Θ^c a closure in $\mathbf{R}^q$ of a bounded open set $\Theta \subset \mathbf{R}^q$, and $\varepsilon(t)$, $t \in \mathbf{R}^1$, a stochastic process satisfying the following condition:

Condition A1. Let $\varepsilon(t)$, $t \in \mathbf{R}^1$, be a real mean square continuous measurable stationary Gaussian process with zero mean and the covariance function $B(t) = \mathsf{E}\,\varepsilon(0)\varepsilon(t)$, $B(0) = 1$.

Definition 1. Any random vector $\widehat{\theta}_T = \widehat{\theta}_T(X(t), t \in [0,T]) \in \Theta^c$, such that

$$Q_{p,T}\big(\widehat{\theta}_T\big) = \inf_{\tau \in \Theta^c} Q_{p,T}(\tau), \qquad Q_{p,T}(\tau) = \int_0^T |X(t) - g(t,\tau)|^p \, dt$$

is called an L_p-estimator of the unknown parameter $\theta \in \Theta$ constructed from observations $X(t)$, $t \in [0,T]$, described by the model (1).

Note that the estimator $\widehat{\theta}_T$ exists for $p > 0$ under the conditions introduced above (see, for example, [28, 45, 14]). The case of $p \in [1,2]$ is the most interesting one.

Let $\rho(x) = |x|^p$, $p \in (1,2)$. Then $\rho'(x) = \psi(x) = p|x|^{p-1}\operatorname{sgn} x$, $\psi'(x) = p(p-1)|x|^{p-2}$, and $\psi''(x) = p(p-1)(p-2)|x|^{p-3}\operatorname{sgn} x$ for $x \neq 0$. We also assume that $\psi'(0) = \infty$ and $|\psi''(0)| = \infty$.

The most technically complicated part in the proof of the asymptotic (as $T \to \infty$) properties of L_p-estimators for $p \in (1,2)$ is related to the observation that the derivatives ψ' and ψ'' are not bounded in a neighborhood of the origin if $\rho(x) = |x|^p$ (this differs from the case of a number of standard M-estimators obtained when minimizing the functionals $\int_0^T \rho(X(t) - g(t,\tau))\,dt$).

Assume that $g(t,\tau)$ is a twice continuously differentiable function with respect to $\tau \in \Theta^c$. Put

$$g_i(t,\tau) = \frac{\partial}{\partial \tau_i} g(t,\tau), \qquad g_{il}(t,\tau) = \frac{\partial^2}{\partial \tau_i \partial \tau_l} g(t,\tau), \qquad i,l = 1,\dots,q,$$
$$d_T^2(\theta) = \operatorname{diag}\left(d_{iT}(\theta)\right)_{i=1}^q,$$

where

$$d_{iT}^2(\theta) = \int_0^T g_i^2(t,\theta)\,dt, \qquad \underline{\lim}\, T^{-1} d_{iT}^2(\theta) > 0, \qquad T \to \infty, \quad i = 1,\dots,q.$$

Our approach works even in the case where the latter limits are infinite. Also let

$$d_{il,T}^2(\tau) = \int_0^T g_{il}^2(t,\tau)\,dt, \qquad \tau \in \Theta^c, \quad i,l = 1,\dots,q.$$

By the symbol k with various indices, we denote positive constants. Assume that, for all sufficiently large T ($T > T_0$), the following condition holds.

Condition B1.

$$\sup_{t\in[0,T]}\sup_{\tau\in\Theta^c}\frac{|g_i(t,\tau)|}{d_{iT}(\theta)} \le k^i T^{-1/2}, \qquad i=1,\dots,q, \tag{2}$$

$$\sup_{t\in[0,T]}\sup_{\tau\in\Theta^c}\frac{|g_{il}(t,\tau)|}{d_{il,T}(\theta)} \le k^{il} T^{-1/2}, \qquad i,l=1,\dots,q, \tag{3}$$

$$\sup_{\tau\in\Theta^c}\frac{d_{il,T}(\tau)}{d_{iT}(\theta)d_{lT}(\theta)} \le \widetilde{k}^{il} T^{-1/2}, \qquad i,l=1,\dots,q. \tag{4}$$

Put

$$J_T(\theta) = \left(J_{il,T}(\theta)\right)_{i,l=1}^q, \qquad J_{il,T}(\theta) = d_{iT}^{-1}(\theta)d_{lT}^{-1}(\theta)\int_0^T g_i(t,\theta)g_l(t,\theta)\,dt,$$
$$\Lambda_T(\theta) = \left(\Lambda_T^{il}(\theta)\right)_{i,l=1}^q = J_T^{-1}(\theta).$$

By $\lambda_{\min}(A)(\lambda_{\max}(A))$ we denote the minimal (maximal) eigenvalue of a positive definite matrix A.

Condition B2. For some $\lambda_* > 0$ and $T > T_0$,

$$\lambda_{\min}(J_T(\theta)) \ge \lambda_*.$$

We have

$$\mathsf{E}\,\psi(\varepsilon(0)) = p\int_{-\infty}^{\infty}|x|^{p-1}\operatorname{sgn} x\,\varphi(x)\,dx = 0,$$

where $\varphi(x) = (2\pi)^{-1/2}\exp\{-x^2/2\}$. On the other hand, we get for $p\in(1,2)$ that

$$\mathsf{E}\,\psi^2(\varepsilon(0)) = \frac{p^2 2^{p-1}}{\sqrt{\pi}}\Gamma\left(p-\frac{1}{2}\right), \qquad \mathsf{E}\,\psi'(\varepsilon(0)) = \frac{p2^{p/2}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right) > 0.$$

Moreover

$$\mathsf{E}\big(\psi'(\varepsilon(0))\big)^2 = \frac{p^2(p-1)^2 2^{p-2}}{\sqrt{\pi}}\Gamma\left(p-\frac{3}{2}\right) < \infty$$

if $p > 3/2$.

Therefore $\mathsf{E}\,\psi^2(\varepsilon(0)) < \infty$ and $\mathsf{E}\left(\psi'(\varepsilon(0))\right)^2 < \infty$ for $p\in(\frac{3}{2},2)$. In this case, the functions $\psi(\varepsilon(t))$ and $\psi'(\varepsilon(t))$, $t\in\mathbf{R}^1$, can be expanded in the series in the Hilbert space $L_2(\mathbf{R}^1,\varphi(x)\,dx)$, namely

$$\begin{aligned}\psi(x) &= \sum_{k=0}^{\infty}\frac{C_k(\psi)}{k!}H_k(x), \qquad C_k(\psi) = \int_{-\infty}^{\infty}\psi(x)H_k(x)\varphi(x)\,dx,\\ \psi'(x) &= \sum_{k=0}^{\infty}\frac{C_k(\psi')}{k!}H_k(x), \qquad C_k(\psi') = \int_{-\infty}^{\infty}\psi'(x)H_k(x)\varphi(x)\,dx,\end{aligned} \tag{5}$$

where H_k are the Hermite–Chebyshev polynomials

$$H_k(u) = (-1)^k e^{u^2/2}\frac{d^k}{du^k}e^{-u^2/2}, \qquad k\ge 0.$$

Consider the following condition of weak dependence.

Condition A2. Let $\varepsilon(t)$, $t\in\mathbf{R}^1$, be a stochastic process such that

$$\alpha(r) = \sup_{A\in\sigma(-\infty,s],B\in\sigma[s+r,\infty)}|\mathsf{P}(AB)-\mathsf{P}(A)\,\mathsf{P}(B)| = O\big(r^{-1-\varepsilon}\big), \qquad r\to\infty,$$

for some $\varepsilon > 0$, where $\sigma(I)$ denotes the σ-algebra generated by the random variables $\{\varepsilon(t), t\in I\}$.

Condition **A2** implies that $B(\cdot) \in L_1(\mathbf{R}^1)$ and that the process $\varepsilon(t)$ has a bounded continuous spectral density $f(\lambda)$, $\lambda \in \mathbf{R}^1$.

It is easy to see that if $\varepsilon(t)$ satisfies the weak dependence condition, then $\psi(\varepsilon(t))$ also satisfies this condition. For all Borel sets G,

$$\{\psi(\varepsilon(t)) \in G\} = \big\{\varepsilon(t) \in \psi^{-1}(G)\big\},$$

and $\psi^{-1}(G)$ is also a Borel set, whence

$$\alpha_\psi(r) = \sup_{A \in \sigma_\psi(-\infty,t],\, B \in \sigma_\psi[t+r,\infty)} |\mathsf{P}(AB) - \mathsf{P}(A)\,\mathsf{P}(B)| \le \alpha(r),$$

where $\sigma_\psi(I)$ is the σ-algebra generated by the random variables $\{\psi(\varepsilon(t)), t \in I\}$.

Since the stochastic process $\psi(\varepsilon(t))$, $t \in \mathbf{R}^1$, can be expanded in the series (5) with respect to the Hermite–Chebyshev polynomials,

$$\mathsf{E}\, H_m(\varepsilon(t)) H_k(\varepsilon(s)) = \delta_m^k m!\, B^m(t-s),$$

and $\mathsf{E}\,\psi(\varepsilon(0)) = 0$, we get

$$\operatorname{cov}\big(\psi(\varepsilon(t)), \psi(\varepsilon(s))\big) = \sum_{k=1}^{\infty} \frac{C_k^2(\psi)}{k!} B^k(t-s).$$

In view of $|B(t)| \le 1$, $t \in \mathbf{R}^1$, we obtain

$$\big|\operatorname{cov}\big(\psi(\varepsilon(t)), \psi(\varepsilon(s))\big)\big| \le \sum_{k=1}^{\infty} \frac{C_k^2(\psi)}{k!} |B(t-s)| = \mathsf{E}\,\psi^2(\varepsilon(0)) |B(t-s)|. \tag{6}$$

Therefore the stationary stochastic process $\psi(\varepsilon(t))$, $t \in \mathbf{R}^1$, also has a bounded and continuous spectral density $f_\psi(x)$.

Consider the matrix measure $\mu_T(dx;\theta)$ on $(\mathbf{R}^1, \mathcal{B}^1)$ with the following matrix density:

$$\big(\mu_T^{jl}(x;\theta)\big)_{j,l=1}^q,$$

$$\mu_T^{jl}(x;\theta) = g_T^j(x,\theta)\overline{g_T^l(x,\theta)} \left(\int_{\mathbf{R}^1} \big|g_T^j(x,\theta)\big|^2 dx \int_{\mathbf{R}^1} \big|g_T^l(x,\theta)\big|^2 dx \right)^{-1/2},$$

$$g_T^j(x,\theta) = \int_0^T e^{ixt} g_j(t,\theta)\, dt, \qquad j,l = 1,\dots,q.$$

Note that $d_{jT}^2(\theta) = (2\pi)^{-1} \int_{\mathbf{R}^1} |g_T^j(x,\theta)|^2\, dx$.

Condition B3. The family of measures $\mu_T(\cdot;\theta)$ weakly converges to the measure $\mu(\cdot;\theta)$ as $T \to \infty$ and

$$\int_{\mathbf{R}^1} f_\psi(x)\, \mu(dx;\theta)$$

is a positive definite matrix.

Definition 2. The matrix measure $\mu(\cdot;\theta)$ is called the spectral measure of the regression function $g(t,\theta)$ [9, 6, 20].

Conditions **B2** and **B3** imply that

$$\int_{\mathbf{R}^1} \mu(dx;\theta) = \left(\int_{\mathbf{R}^1} \mu^{jl}(dx;\theta) \right)_{j,l=1}^q$$

is a nonsingular matrix.

Let

$$\sigma(\theta) = 2\pi\gamma^2 \left(\int_{\mathbf{R}^1} \mu(dx;\theta) \right)^{-1} \left(\int_{\mathbf{R}^1} f_\psi(x)\, \mu(dx;\theta) \right) \left(\int_{\mathbf{R}^1} \mu(dx;\theta) \right)^{-1}, \tag{7}$$

where

$$\gamma = \frac{1}{\mathsf{E}\,\psi'(\varepsilon(0))}. \tag{8}$$

Consider the following condition for the asymptotic uniqueness of a solution of the system of "normal" equations defining the L_p-estimator.

Condition C. For all $\varepsilon > 0$ and $R > 0$, the probability that the system of equations (10) with $T > T_0$ has a unique solution in the ball $v^c(R)$ is not less than $1 - \varepsilon$.

Sufficient conditions for Condition **C** can be found in [24].

Now we state the main result of the paper.

Theorem. *Let Conditions* **A1, A2**, **B1**–**B3**, *and* **C** *hold. If* $p \in \left(\frac{3}{2}; 2\right)$, *then the distribution of the normalized* L_p*-estimator*

$$\widehat{u}_T = \widehat{u}_T(\theta) = d_T(\theta)\big(\widehat{\theta}_T - \theta\big) \tag{9}$$

converges as $T \to \infty$ *to the Gaussian* $N(0, \sigma(\theta))$ *distribution.*

2. Auxiliary results

Consider the change of variables $u = d_T(\theta)(\tau - \theta)$ corresponding to the normalization (9). Applying this change to the regression function and its derivatives, we get

$$g(t,\tau) = g\left(t, \theta + d_T^{-1}(\theta)u\right) = h(t,u),$$
$$g_i(t,\tau) = g_i\left(t, \theta + d_T^{-1}(\theta)u\right) = h_i(t,u), \qquad i = 1,\dots,q,$$
$$g_{il}(t,\tau) = g_{il}\left(t, \theta + d_T^{-1}(\theta)u\right) = h_{il}(t,u), \qquad i,l = 1,\dots,q.$$

We also use the notation

$$H(t;u_1,u_2) = h(t,u_1) - h(t,u_2),$$
$$H_i(t;u_1,u_2) = h_i(t,u_1) - h_i(t,u_2), \qquad i = 1,\dots,q.$$

Consider the vectors

$$M_T(u) = \left(M_T^i(u)\right)_{i=1}^q = \left(\gamma \int_0^T \psi(X(t) - h(t,u)) \frac{h_i(t,u)}{d_{iT}(\theta)}\,dt\right)_{i=1}^q$$

and

$$\Psi_T(u) = \left(\Psi_T^i(u)\right)_{i=1}^q = \left(\gamma \int_0^T \psi(\varepsilon(t)) \frac{h_i(t,u)}{d_{iT}(\theta)}\,dt + \int_0^T H(t;0,u)\frac{h_i(t,u)}{d_{iT}(\theta)}\,dt\right)_{i=1}^q,$$

where γ is defined by (8). The vectors $M_T(u)$ and $\Psi_T(u)$ are defined for $u \in U_T^c(\theta)$,

$$U_T(\theta) = d_T(\theta)(\Theta - \theta).$$

Our assumptions mean that the sets $U_T(\theta)$ are extending to $\mathbf{R}^q$ as $T \to \infty$. Then, for all $R > 0$,

$$v^c(R) = \{u \in \mathbf{R}^q \colon \|u\| \le R\} \subset U_T(\theta)$$

for $T > T_0(R)$.

The statistical meaning of the vectors $M_T(u)$ and $\Psi_T(u)$ is clear. Consider the functional $\gamma Q_T\big(\theta + d_T^{-1}(\theta)u\big)$. Then the normalized L_p-estimator $\widehat{u}_T$ satisfies the following system of equations:

$$M_T(u) = 0. \tag{10}$$

Let

$$\eta(t) = \gamma\psi(\varepsilon(t)), \qquad t \in \mathbf{R}^1, \tag{11}$$

and let the observations be of the following form:

$$Y(t) = g(t,\theta) + \eta(t), \qquad t \in [0,T]. \tag{12}$$

Then $\Psi_T(u) = 0$ is the system of normal equations determining the least squares estimator

$$\breve{u}_T = \breve{u}_T(\theta) = d_T(\theta)\big(\breve{\theta}_T - \theta\big)$$

of the unknown parameter θ of the auxiliary nonlinear regression model (12).

Lemma 1. *Let Conditions* **A1**, **A2**, *and* **B1** *hold. Then, for arbitrary $R > 0$ and $r > 0$,*

$$\mathsf{P}\left\{\sup_{u \in v^c(R)} \|M_T(u) - \Psi_T(u)\| > r\right\} \underset{T\to\infty}{\longrightarrow} 0. \tag{13}$$

Proof. For a fixed i, consider the difference

$$\begin{aligned} &M_T^i(u) - \Psi_T^i(u) \\ &\quad = \gamma \int_0^T \frac{h_i(t,u)}{d_{iT}(\theta)} \big[\psi\left(\varepsilon(t) + H(t;0,u)\right) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t)) H(t;0,u)\big]\, dt \\ &\qquad + \gamma \int_0^T H(t;0,u) \frac{h_i(t,u)}{d_{iT}(\theta)} \zeta(t)\, dt \\ &\quad = I_1(u) + I_2(u), \end{aligned} \tag{14}$$

$$\zeta(t) = \psi'(\varepsilon(t)) - \mathsf{E}\,\psi'(\varepsilon(t)), \qquad t \in \mathbf{R}^1.$$

One needs to prove that $I_1(u)$ and $I_2(u)$ converge to zero in probability uniformly with respect to $u \in v^c(R)$. Let $u \in v^c(R)$ be fixed. Then

$$\mathsf{E}\, I_2^2(u) = \gamma^2 \int_0^T \int_0^T H(t;0,u) H(s;0,u) \frac{h_i(t,u)}{d_{iT}(\theta)} \frac{h_i(s,u)}{d_{iT}(\theta)} \operatorname{cov}(\zeta(t),\zeta(s))\, dt\, ds. \tag{15}$$

We have

$$\sup_{t\in[0,T]} |H(t;0,u)| = \sup_{t\in[0,T]} \left| \sum_{i=1}^q \frac{h_i(t,u_t^*)}{d_{iT}\theta)} u_i \right| \le \|u\| \sup_{t\in[0,T]} \left(\sum_{i=1}^q \left[\frac{h_i(t,u_t^*)}{d_{iT}(\theta)}\right]^2 \right)^{1/2},$$

where $\|u_t^*\| \le \|u\|$. Using (2), we obtain

$$\sup_{t\in[0,T]} |H(t;0,u)| \le T^{-1/2} \|k\| \cdot \|u\|, \tag{16}$$

where $k = (k^1, \dots, k^q)$ are the constants involved in inequality (2). Applying once more inequalities (16) and (2) to integral (15), we get

$$\mathsf{E}\, I_2^2(u) \le \gamma^2 \|k\|^2 \left(k^i\right)^2 R^2 \frac{1}{T^2} \int_0^T \int_0^T |\operatorname{cov}(\zeta(t),\zeta(s))|\, dt\, ds.$$

We show that

$$\frac{1}{T^2} \int_0^T \int_0^T |\operatorname{cov}(\zeta(t),\zeta(s))|\, dt\, ds \to 0 \tag{17}$$

as $T \to \infty$. Similarly to (6),

$$\big|\operatorname{cov}\big(\psi'(\varepsilon(t)), \psi'(\varepsilon(s))\big)\big| \le \operatorname{Var} \psi'(\varepsilon(0)) |B(t-s)|$$

and

$$\frac{1}{T^2}\int_0^T\int_0^T |\operatorname{cov}(\zeta(t),\zeta(s))|\,dt\,ds \le \frac{\operatorname{Var}\psi'(\varepsilon(0))}{T^2}\int_0^T\int_0^T |B(t-s)|\,dt\,ds$$
$$\le \frac{2\operatorname{Var}\psi'(\varepsilon(0))}{T}\int_0^T |B(u)|\,du \to 0.$$

This implies that $I_2(u) \xrightarrow{\mathsf{P}} 0$ pointwise for $u \in v^c(R)$.

For $u_1, u_2 \in v^c(R)$, consider the difference

$$I_2(u_1) - I_2(u_2) = \gamma\int_0^T H(t;0,u_1)\frac{H_i(t;u_1,u_2)}{d_{iT}(\theta)}\zeta(t)\,dt - \gamma\int_0^T H(t;u_1,u_2)\frac{h_i(t,u_2)}{d_{iT}(\theta)}\zeta(t)\,dt$$
$$= I_3(u_1,u_2) + I_4(u_1,u_2).$$

For all $h>0$ and $r>0$, we write

$$\begin{aligned}\mathsf{P}\left\{\sup_{\|u_1-u_2\|\le h}|I_3(u_1,u_2)| > r\right\} &\le r^{-1}\,\mathsf{E}\sup_{\|u_1-u_2\|\le h}|I_3(u_1,u_2)| \\ &\le 2r^{-1}T\sup_{\substack{t\in[0,T]\\ u\in v^c(R)}}|H(t;0,u)|\sup_{\|u_1-u_2\|\le h}\sup_{t\in[0,T]}\frac{|H_i(t;u_1,u_2)|}{d_{iT}(\theta)},\end{aligned}\tag{18}$$

$$\begin{aligned}\sup_{\|u_1-u_2\|\le h}\sup_{t\in[0,T]}\frac{|H_i(t;u_1,u_2)|}{d_{iT}(\theta)} &\le h\sup_{t\in[0,T]}\left[\sum_{l=1}^q\left(\sup_{u\in v^c(R)}\frac{|h_{il}(t,u)|}{d_{il,T}(\theta)}\right)\frac{d_{il,T}(\theta)}{d_{iT}(\theta)d_{lT}(\theta)}\right] \\ &\le \sum_{l=1}^q k^{il}\widetilde{k}^{il}hT^{-1}\end{aligned}\tag{19}$$

by using conditions (3) and (4). Applying (16) and (19) in (18), we get

$$\mathsf{P}\left\{\sup_{\|u_1-u_2\|\le h}|I_3(u_1,u_2)| > r\right\} \le k_1 r^{-1}T^{-1/2}h,\tag{20}$$

where $k_1 = 2R\|k\|\bigl(\sum_{l=1}^q k^{il}\widetilde{k}^{il}\bigr)$. Similarly, we use inequality (2) and obtain

$$\begin{aligned}\mathsf{P}\left\{\sup_{\|u_1-u_2\|\le h}|I_4(u_1,u_2)| > r\right\} &\le r^{-1}\,\mathsf{E}\sup_{\|u_1-u_2\|\le h}|I_4(u_1,u_2)| \\ &\le 2r^{-1}T\sup_{\substack{t\in[0,T]\\ u\in v^c(R)}}\frac{|h_i(t,u)|}{d_{iT}(\theta)}\sup_{\|u_1-u_2\|\le h}\sup_{t\in[0,T]}|H(t;u_1,u_2)| \le k_2 r^{-1}h,\end{aligned}\tag{21}$$

where $k_2 = 2k^i\|k\|$. It follows from bounds (20) and (21) that

$$\mathsf{P}\left\{\sup_{\|u_1-u_2\|\le h}|I_2(u_1)-I_2(u_2)| > r\right\} \le 2r^{-1}h\left(k_1T^{-1/2}+k_2\right) \le k_3r^{-1}h.\tag{22}$$

Denote by N_h a finite h-net of the ball $v^c(R)$. Then

$$\sup_{u\in v^c(R)}|I_2(u)| \le \sup_{\|u_1-u_2\|\le h}|I_2(u_1)-I_2(u_2)| + \max_{u\in N_h}|I_2(u)|.\tag{23}$$

From (22) and (23) we obtain for all $r>0$ that

$$\mathsf{P}\left\{\sup_{u\in v^c(R)}|I_2(u)| > r\right\} \le 2k_3r^{-1}h + \mathsf{P}\left\{\max_{u\in N_h}|I_2(u)| > \frac{r}{2}\right\}.$$

Let $\varepsilon > 0$ and $h = \varepsilon r/(4k_3)$. In view of the pointwise convergence in probability of $I_2(u)$ to zero we prove that

$$\mathsf{P}\left\{\max_{u\in N_{\varepsilon r/(4k_3)}} |I_2(u)| > r/2\right\} \le \frac{\varepsilon}{2}$$

for $T > T_0$, whence

$$\mathsf{P}\left\{\sup_{u\in v^c(R)} |I_2(u)| > r\right\} \le \varepsilon.$$

On the other hand, if $u_T^* \in v^c(R)$ is a random variable, then

$$\sup_{u\in v^c(R)} |I_1(u)| \le \gamma \int_0^T \left|\frac{h_i(t,u_T^*)}{d_{iT}(\theta)}\right| \left|\left(\psi(\varepsilon(t)+H(t;0,u_T^*)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t;0,u_T^*)\right)\right| dt. \tag{24}$$

Let $\chi_T(t)$ be the indicator of the random event $\{|\varepsilon(t)| \le 2\|k\|RT^{-1/2}\}$ and let

$$\overline{\chi}_T(t) = 1 - \chi_T(t).$$

Using inequality (2) again, we continue the estimation in (24) as follows:

$$\begin{aligned}
\sup_{u\in v^c(R)} &|I_1(u)| \\
&\le \gamma k^i T^{-1/2} \int_0^T \left|\psi(\varepsilon(t)+H(t;0,u_T^*)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t;0,u_T^*)\right|\chi_T(t)\,dt \\
&\quad + \gamma k^i T^{-1/2} \int_0^T \left|\psi(\varepsilon(t)+H(t;0,u_T^*)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t;0,u_T^*)\right|\overline{\chi}_T(t)\,dt \\
&= \Delta_1(T) + \Delta_2(T).
\end{aligned}$$

Bound (16) implies that

$$\begin{aligned}
\mathsf{E}\,\Delta_1(T) &\le \gamma k^i p\left(3^{p-1}+2^{p-1}\right)(\|k\|R)^{p-1}T^{-p/2}\int_0^T \left(\int_{\{|x|\le 2\|k\|RT^{-1/2}\}} \varphi(x)\,dx\right) dt \\
&\qquad + \gamma k^i p(p-1)\|k\|R\frac{1}{T}\int_0^T \left(\int_{\{|x|\le 2\|k\|RT^{-1/2}\}} |x|^{p-2}\varphi(x)\,dx\right) dt \\
&\le k_4 T^{(1-p)/2},
\end{aligned}$$

where $k_4 = 4\gamma k^i(2\pi)^{-1/2}p(\|k\|R)^p\left(3^{p-1}+2^{p-1}+2^{p-2}\right)$. Therefore $\mathsf{E}\,\Delta_1(T) \to 0$ as $T \to \infty$.

Now we estimate $\mathsf{E}\,\Delta_2(T)$. Note that

$$\begin{aligned}
&\left|\psi(\varepsilon(t)+H(t;0,u_T^*)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t;0,u_T^*)\right| \\
&\qquad = \frac{1}{2}|\psi''(\varepsilon(t)+\delta_t H(t;0,u_T^*))|H^2(t;0,u_T^*) \\
&\qquad \le \frac{1}{2}p(p-1)(2-p)\frac{1}{|\varepsilon(t)+\delta_t H\,(t;0,u_T^*)\,|^{3-p}}\|k\|^2R^2T^{-1}
\end{aligned} \tag{25}$$

for some $\delta_t \in (0,1)$. Let $k_5 = \frac{1}{2}\gamma k^i p(p-1)(2-p)\|k\|^2 R^2$. Taking into account (25), we obtain

$$\begin{aligned} \mathsf{E}\,\Delta_2(T) &\le k_5 T^{-3/2} \int_0^T \left(\int_{\{|x| \ge 2\|k\| R T^{-1/2}\}} \frac{\varphi(x)\,dx}{(|x| - \|k\| R T^{-1/2})^{3-p}} \right) dt \\ &\le \frac{2k_5}{\sqrt{2\pi}} T^{-1/2} \int_{2\|k\|RT^{-1/2}}^{\infty} \frac{dx}{(x - \|k\| R T^{-1/2})^{3-p}} = k_6 T^{(1-p)/2}, \end{aligned} \tag{26}$$

where $k_6 = \gamma k^i (2\pi)^{-1/2} p(p-1)\|k\|^p R^p$, that is, $\mathsf{E}\,\Delta_2(T) \to 0$.

Bounds (25) and (26) yield

$$\sup_{u \in v^c(R)} |I_1(u)| \xrightarrow{\mathsf{P}} 0 \quad \text{as } T \to \infty.$$

Lemma 1 is proved. $\square$

Consider the random vector

$$L_T(u) = \left(L_T^i(u)\right)_{i=1}^q = \left(\int_0^T \left(\eta(t) - \sum_{l=1}^q \frac{g_l(t,\theta)}{d_{lT}(\theta)} u_l \right) \frac{g_i(t,\theta)}{d_{iT}(\theta)} dt \right)_{i=1}^q \tag{27}$$

corresponding to the auxiliary linear regression model

$$Z(t) = \sum_{i=1}^q g_i(t,\theta)\beta_i + \eta(t), \qquad t \in [0,T],$$

where $\eta(t)$ is defined by equality (11).

The system of normal equations

$$L_T(u) = 0 \tag{28}$$

determines a normalized linear least squares estimator $\widetilde{\beta}_T$ of the parameter $\beta \in \mathbf{R}^q$; namely,

$$\widetilde{u}_T = \widetilde{u}_T(\theta) = d_T(\theta)\big(\widetilde{\beta}_T - \beta\big). \tag{29}$$

Lemma 2. *If Conditions* **A1**, **A2**, *and* **B1** *hold, then*

$$\mathsf{P}\left\{ \sup_{u \in v^c(R)} \|\Psi_T(u) - L_T(u)\| > r \right\} \to 0 \quad \text{as } T \to \infty \tag{30}$$

for all $R > 0$ and $r > 0$.

Proof. For an arbitrary $i \in \{1, \dots, q\}$,

$$\begin{aligned} \Psi_T^i(u) - L_T^i(u) = &\int_0^T \eta(t) \frac{h_i(t,u)}{d_{iT}(\theta)}\,dt + \int_0^T H(t;0,u) \frac{h_i(t,u)}{d_{iT}(\theta)}\,dt - \int_0^T \eta(t) \frac{g_i(t,\theta)}{d_{iT}(\theta)}\,dt \\ &+ \int_0^T \frac{g_i(t,\theta)}{d_{iT}(\theta)} \sum_{l=1}^q \frac{g_l(t,\theta)}{d_{lT}(\theta)} u_l\,dt \\ = &\int_0^T \eta(t) \frac{H_i(t;u,0)}{d_{iT}(\theta)}\,dt + \int_0^T H(t;0,u) \frac{H_i(t;u,0)}{d_{iT}(\theta)}\,dt \\ &+ \int_0^T \frac{g_i(t,\theta)}{d_{iT}(\theta)} \left[H(t;0,u) + \sum_{l=1}^q \frac{g_l(t,\theta)}{d_{lT}(\theta)} u_l \right] dt \\ = &\, I_5(u) + I_6(u) + I_7(u). \end{aligned}$$

Let $u \in v^c(R)$ be fixed. Using inequalities (6) and (19), we obtain

$$\begin{aligned}\mathsf{E}\, I_5^2(u) &= \int_0^T\int_0^T \operatorname{cov}(\eta(t),\eta(s)) \frac{H_i(t;u,0)}{d_{iT}(\theta)} \frac{H_i(s;u,0)}{d_{iT}(\theta)}\, dt\, ds \\ &\le \left(\sum_{l=1}^q k^{il}\widetilde{k}^{il}\right)^2 R^2 \frac{1}{T^2}\int_0^T\int_0^T |\operatorname{cov}(\eta(t),\eta(s))|\, dt\, ds \to 0 \quad \text{as} \quad T\to\infty;\end{aligned}$$

that is, $I_5(u) \xrightarrow{\mathsf{P}} 0$ as $T\to\infty$ pointwise for $u\in v^c(R)$. On the other hand,

$$\mathsf{E} \sup_{\|u_1-u_2\|\le h} |I_5(u_1) - I_5(u_2)| \le |\gamma|\, \mathsf{E}\,|\psi(\varepsilon(0))| \left(\sum_{l=1}^q k^{il}\widetilde{k}^{il}\right) h$$

in view of (19). Similarly to the proof of the convergence of $I_2(u)$ in Lemma 1, one can show that $I_5(u)$ converges uniformly to zero in probability with respect to $u\in v^c(R)$.

Taking into account inequalities (16) and (19) we get

$$\sup_{u\in v^c(R)} |I_6(u)| \le \|k\| \left(\sum_{l=1}^q k^{il}\widetilde{k}^{il}\right) R^2 T^{-1/2} \to 0 \quad \text{as} \quad T\to\infty.$$

The term $I_7(u)$ can be represented as follows:

$$I_7(u) = -\frac{1}{2}\sum_{j,l=1}^q \left(\int_0^T \frac{h_{jl}(t,u_T^*)}{d_{jT}(\theta)d_{lT}(\theta)} \frac{g_i(t,\theta)}{d_{iT}(\theta)}\, dt\right) u_j u_l$$

for some $u_T^* \in v(R)$. Under Condition **B1** we have

$$|I_7(u)| \le \frac{k^i}{2}\sum_{j,l=1}^q \left(k^{jl}\widetilde{k}^{jl}|u_j||u_l|\right) T^{-1/2} \le \frac{qk^i}{2} \max_{j,l=1,\dots,q}\left[k^{jl}\widetilde{k}^{jl}\right] \|u\|^2 T^{-1/2},$$

whence $\sup_{u\in v^c(R)} |I_7(u)| \to 0$. Lemma 2 is proved. □

Relations (13) and (30) imply the following corollary.

Corollary. *Let Conditions* **A1**, **A2**, *and* **B1** *hold. Then*

$$\mathsf{P}\left\{\sup_{u\in v^c(R)} \|M_T(u) - L_T(u)\| > r\right\} \to 0 \quad \textit{as} \quad T\to\infty \tag{31}$$

for all $R>0$ and $r>0$.

Using (27) and (28) one can evaluate $\widetilde{u}_T$ in an explicit form (see (29)) if Condition **B2** holds; namely,

$$\widetilde{u}_T = \Lambda_T(\theta)\int_0^T \eta(t) d_T^{-1}(\theta)\nabla g(t,\theta)\, dt, \tag{32}$$

where $\Lambda_T(\theta) = J_T^{-1}(\theta)$ and $\nabla g(t,\theta)$ is the gradient of the function $g(t,\theta)$:

$$\nabla g(t,\theta) = \begin{bmatrix} g_1(t,\theta) \\ \vdots \\ g_q(t,\theta)\end{bmatrix}.$$

Note that the covariance matrix of the vector $\widetilde{u}_T$ is given by

$$\sigma_T(\theta) = 2\pi\gamma^2 \left(\int_{\mathbf{R}^1} \mu_T(dx;\theta)\right)^{-1} \left(\int_{\mathbf{R}^1} f_\psi(x)\, \mu_T(dx;\theta)\right) \left(\int_{\mathbf{R}^1} \mu_T(dx;\theta)\right)^{-1}.$$

Consider the random event

$$A_T = \{\widetilde{u}_T \in v^c(R-r)\}. \tag{33}$$

Lemma 3. *Let Conditions* **B2** *and* (2) *hold. Then, for arbitrary* $\varepsilon > 0$ *and* $r > 0$*, there exists* $R > r$ *such that* $\mathsf{P}\{\overline{A}_T\} \le \varepsilon$ *for* $T > T_0$*.*

Proof. Fix $\varepsilon > 0$. Analogously to the proof of convergence (17) we derive that

$$\begin{aligned}\mathsf{P}\{\overline{A}_T\} &= \mathsf{P}\{\|\widetilde{u}_T\| > R-r\} \le \frac{\mathsf{E}\,\|\widetilde{u}_T\|^2}{(R-r)^2} \\ &\le \frac{1}{\lambda_*^2(R-r)^2}\sum_{i=1}^{q}\int_0^T\int_0^T \left|\operatorname{cov}(\eta(t),\eta(s))\frac{g_i(t,\theta)}{d_{iT}(\theta)}\frac{g_i(s,\theta)}{d_{iT}(\theta)}\right| dt\,ds \\ &\le \frac{\|k\|^2}{T\lambda_*^2(R-r)^2}\int_0^T\int_0^T |\operatorname{cov}(\eta(t),\eta(s))|\,dt\,ds \le \frac{k_6}{(R-r)^2},\end{aligned}$$

where $k_6 = 2(\lambda_*)^{-2}\gamma^2\|k\|^2\,\mathsf{E}(\psi(\varepsilon(0)))^2\int_0^\infty |B(u)|\,du$.

Setting $R = r + \sqrt{k_6/\varepsilon}$ we get the desired result. □

We need the following results for the proof of the theorem.

Proposition 1. *Let* $F\colon v^c(R) \to v^c(R)$ *be a continuous mapping. Then there exists* $x_0 \in v^c(R)$ *such that* $F(x_0) = x_0$*.*

This proposition is a special case of the Brouwer fixed point theorem (see, for example, [4]).

Proposition 2. *Let Conditions* **A1***,* **A2***,* **B2***, and* **B3** *hold. Then the random vector* $\widetilde{u}_T$ *defined by* (32) *is asymptotically normal with parameters* 0 *and* $\sigma(\theta)$*, where the covariance function* $\sigma(\theta)$ *is defined by* (7)*.*

The latter result is a multivariate central limit theorem for the integral of a weighted stationary stochastic process satisfying the weak dependence condition. A more general result of this kind as well as its proof can be found in [9] (Theorem 1.7.5).

Let $\mathcal{B}^q$ be the σ-algebra of Borel sets of $\mathbf{R}^q$. For $C \in \mathcal{B}^q$ and $\varepsilon > 0$, let

$$C_\varepsilon = \{x\colon x \in \mathbf{R}^q, d(x,C) < \varepsilon\},$$

where $d(x,C) = \inf_{y\in C}\|x-y\|$ and $C_{-\varepsilon} = \mathbf{R}^q \setminus (\mathbf{R}^q \setminus C)_\varepsilon$.

Proposition 3. *Let* $\nu \ge 0$ *be a differentiable function defined on* $[0,\infty)$ *and such that*

$$b = \int_0^\infty |\nu'(\lambda)|\lambda^{q-1}\,d\lambda < \infty, \qquad \lim_{\lambda\to\infty}\nu(\lambda) = 0.$$

Then

$$\int_{C_\varepsilon\setminus C_{-\delta}} \nu\left(\|x\|\right)\,dx \le b\left(\frac{2\pi^{q/2}}{\Gamma(q/2)}\right)(\varepsilon+\delta)$$

for an arbitrary convex set $C \in \mathcal{B}^q$ *and for all* $\varepsilon, \delta > 0$*.*

The proof of the latter result can be found in §3 of the book [3].

3. Proof of the theorem

One needs to prove that the distribution function $G_T(y,\theta)$ of the random vector $\widehat{u}_T$ defined by (9) converges as $T \to \infty$ to the Gaussian distribution function $\Phi_{0,\sigma(\theta)}(y)$. Note that $\sigma(\theta)$ is a positive definite matrix (7) in view of Conditions **B2** and **B3**.

We show that

$$\Delta_T(r) = \mathsf{P}\{\|\widehat{u}_T - \widetilde{u}_T\| > r\} \to 0, \qquad T \to \infty, \tag{34}$$

for all $r > 0$.

Let A_T be the random event defined by equality (33) and let R be such that $\mathsf{P}\{\overline{A}_T\} \le \frac{\varepsilon}{3}$ (this inequality holds by Lemma 3), where $\varepsilon > 0$ is a fixed but small number.

We also consider the following random events:

$$B_T = \left\{ \sup_{u \in v^c(R)} \|\Lambda_T(\theta)(M_T(u) - L_T(u))\| \le r \right\},$$

$$C_T = \{\text{system of equations (10) has a unique solution in the ball } v^c(R)\}.$$

Conditions **B2** and **C** together with the corollary to Lemmas 1 and 2 imply for $T > T_0$ that

$$\mathsf{P}\{\overline{B}_T\} \le \mathsf{P}\left\{ \sup_{u \in v^c(R)} \|M_T(u) - L_T(u)\| > \lambda_* r \right\} \le \frac{\varepsilon}{3},$$

$$\mathsf{P}\{\overline{C}_T\} \le \frac{\varepsilon}{3},$$

whence

$$\mathsf{P}\{A_T \cap B_T \cap C_T\} > 1 - \varepsilon \tag{35}$$

for $T > T_0$.

Taking into account relations (27) and (32) we get $\Lambda_T(\theta) L_T(u) = \widetilde{u}_T - u$. If the event $A_T \cap B_T \cap C_T$ occurs, then

$$\|u + \Lambda_T(\theta) M_T(u)\| \le \|\Lambda_T(\theta)(M_T(u) - L_T(u))\| + \|\widetilde{u}_T\| \le r + (R - r) = R$$

for $u \in v^c(R)$; that is, $F_T(u) = u + \Lambda_T(\theta) M_T(u)$ is a continuous mapping of $v^c(R)$ to $v^c(R)$. To prove (34) we apply the Brouwer fixed point theorem (see Proposition 1).

Applying this theorem to $F_T(u)$, we prove that there exists a point $u_T^0 \in v^c(R)$ such that $F_T(u_T^0) = u_T^0$. Since $\Lambda_T(\theta)$ is nonsingular, we also have $M_T(u_T^0) = 0$. According to Condition **C**, the normalized L_p-estimator $\widehat{u}_T$ is a unique solution of the system of equations $M_T(u) = 0$. Therefore $A_T \cap B_T \cap C_T \subset \{\widehat{u}_T \in v^c(R)\}$ and hence

$$\mathsf{P}\{\widehat{u}_T \in v^c(R)\} > 1 - \varepsilon.$$

Note also that inequality (35) yields

$$\begin{aligned} 1 - \varepsilon &< \mathsf{P}\{\{\widehat{u}_T \in v^c(R)\} \cap B_T\} \\ &\le \mathsf{P}\left\{\|\Lambda_T(\theta)(M_T(\widehat{u}_T) - L_T(\widehat{u}_T))\| \le r\right\} = \mathsf{P}\left\{\|\widehat{u}_T - \widetilde{u}_T\| \le r\right\}. \end{aligned} \tag{36}$$

Relation (36) for all $\varepsilon > 0$ is equivalent to (34).

Put $\Pi(-\infty, y \pm \vec{\varepsilon}) = (-\infty, y_1 \pm \varepsilon) \times \cdots \times (-\infty, y_q \pm \varepsilon)$, $\varepsilon \ge 0$. Considering relation (34), we prove for the distribution function $G_T(y,\theta) = \mathsf{P}\{\widehat{u}_T \in \Pi(-\infty, y)\}$ that

$$-\Delta_T(\varepsilon) + \mathsf{P}\{\widetilde{u}_T \in \Pi(-\infty, y - \vec{\varepsilon})\} \le G_T(y,\theta) \le \mathsf{P}\{\widetilde{u}_T \in \Pi(-\infty, y + \vec{\varepsilon})\} + \Delta_T(\varepsilon).$$

Proposition 2 implies that

$$\left|\mathsf{P}\{\widetilde{u}_T \in \Pi(-\infty, y \pm \vec{\varepsilon})\} - \Phi_{0,\sigma(\theta)}(y \pm \vec{\varepsilon})\right| \to 0 \tag{37}$$

as $T \to \infty$.

Let $\varphi(y,\theta)$ be the Gaussian density corresponding to the distribution function

$$\Phi_{0,\sigma(\theta)}(y).$$

Since $\lambda_{\min}(\sigma(\theta)) = \underline{\lambda} > 0$ and $\lambda_{\max}(\sigma(\theta)) = \overline{\lambda} < \infty$, we get

$$\varphi(y,\theta) \le (2\pi\underline{\lambda})^{-q/2} \exp\{-\|y\|^2/2\overline{\lambda}\} = \nu(\|y\|).$$

If $A = \Pi(-\infty, y)$, then $A_{-\varepsilon} = \Pi(-\infty, y-\vec{\varepsilon}]$ and $(\Pi(-\infty, y+\vec{\varepsilon}))_{-\varepsilon} = \Pi(-\infty, y] = A^c$. Applying Proposition 3 to the function $\nu(\|y\|)$ we conclude that

$$\begin{gathered}\left|\Phi_{0,\sigma(\theta)}(y) - \Phi_{0,\sigma(\theta)}(y+\vec{\phi})\right| = \int_\Pi \varphi(y,\theta)\,dy \le b\left(\frac{2\pi^{q/2}}{\Gamma(q/2)}\right)|\phi|, \\ \Pi = \begin{cases} \Pi(-\infty, y+\vec{\phi}) \setminus A^c & \text{if } \phi > 0, \\ A \setminus A_\phi & \text{if } \phi < 0, \end{cases}\end{gathered} \tag{38}$$

for all $\phi \ne 0$.

Further

$$\begin{aligned} G_T(y,\theta) - \Phi_{0,\sigma(\theta)}(y) \le \Delta_T(\varepsilon) &+ \left|\mathsf{P}\{\widetilde{u}_T \in \Pi(-\infty, y+\vec{\varepsilon})\} - \Phi_{0,\sigma(\theta)}(y+\vec{\varepsilon})\right| \\ &+ |\Phi_{0,\sigma(\theta)}(y+\vec{\varepsilon}) - \Phi_{0,\sigma(\theta)}(y)|, \end{aligned} \tag{39}$$

$$\begin{aligned} \Phi_{0,\sigma(\theta)}(y) - G_T(y,\theta) \le |\Phi_{0,\sigma(\theta)}(y) &- \Phi_{0,\sigma(\theta)}(y-\vec{\varepsilon})| \\ &+ \left|\Phi_{0,\sigma(\theta)}(y-\vec{\varepsilon}) - \mathsf{P}\{\widetilde{u}_T \in \Pi(-\infty, y-\vec{\varepsilon})\}\right| + \Delta_T(\varepsilon) \end{aligned} \tag{40}$$

for all $y \in \mathbf{R}^q$ and $\varepsilon > 0$. Applying (37) and (38) to (39), (40) we obtain the desired result.

4. Example

Consider the regression model

$$X(t) = A_0 \cos\omega_0 t + B_0 \sin\omega_0 t + \varepsilon(t), \qquad t \in (0,\infty), \tag{41}$$

where $\varepsilon(t)$, $t \in \mathbf{R}^1$, is a stochastic process satisfying Conditions **A1** and **A2** and having the covariance function $B(t) = e^{-\varkappa|t|}$. The vector of parameters

$$\theta = (\theta_1, \theta_2, \theta_3) = (A_0, B_0, \omega_0)$$

belongs to an open bounded set and moreover $A_0^2 + B_0^2 > 0$ and $0 < \underline{\omega} < \omega_0 < \overline{\omega} < \infty$.

Consider the loss function $\rho(x) = |x|^p$, $p \in \left(\frac{3}{2}; 2\right)$.

It is easy to check that

$$d_{1T}^2(\theta) = \frac{T}{2} + \frac{1}{4\omega}\sin 2\omega t, \qquad d_{2T}^2(\theta) = \frac{T}{2} - \frac{1}{4\omega}\sin 2\omega t,$$

$$d_{3T}^2(\theta) = \frac{A^2+B^2}{6}T^3 + O\left(T^2\right), \qquad d_{11,T}^2(\theta) = d_{12,T}^2(\theta) = d_{21,T}^2(\theta) = d_{22,T}^2(\theta) = 0,$$

$$d_{13,T}^2(\theta) = d_{31,T}^2(\theta) = \frac{T^3}{6} + O\left(T^2\right), \qquad d_{23,T}^2(\theta) = d_{32,T}^2(\theta) = \frac{T^3}{6} + O\left(T^2\right),$$

$$d_{33,T}^2(\theta) = \frac{A^2+B^2}{10}T^5 + O\left(T^4\right).$$

Condition **B1** is easy to check, too. Further

$$\lim_{T\to\infty} J_T(\theta) = J(\theta) = \begin{pmatrix} 1 & 0 & \frac{B_0}{\sqrt{4(A_0^2+B_0^2)/3}} \\ 0 & 1 & -\frac{A_0}{\sqrt{4(A_0^2+B_0^2)/3}} \\ \frac{B_0}{\sqrt{4(A_0^2+B_0^2)/3}} & -\frac{A_0}{\sqrt{4(A_0^2+B_0^2)/3}} & 1 \end{pmatrix}.$$

The above matrix $J(\theta)$ is positive definite; that is, Condition **B2** holds.

It is known that the spectral measure for the regression function of the model (41) is given by

$$\mu(dx,\theta)=\begin{pmatrix}\delta_{\omega_0}(dx) & i\rho_{\omega_0}(dx) & \frac{B_0\delta_{\omega_0}(dx)-iA_0\rho_{\omega_0}(dx)}{\sqrt{\frac{4}{3}(A_0^2+B_0^2)}}\\ -i\rho_{\omega_0}(dx) & \delta_{\omega_0}(dx) & \frac{-A_0\delta_{\omega_0}(dx)-iB_0\rho_{\omega_0}(dx)}{\sqrt{\frac{4}{3}(A_0^2+B_0^2)}}\\ \frac{B_0\delta_{\omega_0}(dx)+iA_0\rho_{\omega_0}(dx)}{\sqrt{\frac{4}{3}(A_0^2+B_0^2)}} & \frac{-A_0\delta_{\omega_0}(dx)+iB_0\rho_{\omega_0}(dx)}{\sqrt{\frac{4}{3}(A_0^2+B_0^2)}} & \delta_{\omega_0}(dx)\end{pmatrix}$$

(see, for example, [7]) where the measure δ_{ω_0} and the charge ρ_{ω_0} are concentrated at the points $\pm\omega_0$ and

$$\delta_{\omega_0}(\{\pm\omega_0\})=\frac{1}{2},\qquad \rho_{\omega_0}(\{\pm\omega_0\})=\pm\frac{1}{2}.$$

This means that

$$\int_{\mathbf{R}} f_\psi(x)\,\mu(dx)=f_\psi(\omega_0)J(\theta).$$

If $f_\psi(\omega_0)\neq 0$, then $\int_{\mathbf{R}} f_\psi(x)\,\mu(dx)$ is a positive definite matrix and Condition **B3** holds.

It remains to note that the assumptions of the paper [24] are satisfied for model (41) and that they imply Condition **C**. Now one can apply our theorem. Using the above evaluation of $d_T(\theta)$, one can adjust the representation of the normalized L_p-estimator as follows:

$$\hat{u}_T=\begin{pmatrix}\sqrt{\frac{T}{2}}(\hat{A}-A_0)\\ \sqrt{\frac{T}{2}}(\hat{B}-B_0)\\ \sqrt{\frac{A_0^2+B_0^2}{2}}T^{3/2}(\hat{\omega}-\omega_0)\end{pmatrix} \tag{42}$$

(see [7] for more details). According to this normalization, the covariance function of the asymptotic normal distribution becomes of the following form:

$$\sigma(\theta)=\frac{4\pi\gamma^2 f_\psi(\omega_0)}{A_0^2+B_0^2}\begin{pmatrix}A_0^2+4B_0^2 & -3A_0B_0 & -6B_0\\ -3A_0B_0 & 4A_0^2+B_0^2 & 6A_0\\ -6B_0 & 6A_0 & 12\end{pmatrix}, \tag{43}$$

where γ is defined by (8).

Considering the expansion of the function $\psi(\varepsilon(t))$ in the series (5) with respect to the Hermite–Chebyshev polynomials, we obtain the following representation of the spectral density:

$$f_\psi(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda t}B_\psi(t)\,dt=\sum_{n=1}^{\infty}\frac{C_n^2(\psi)}{n!}f_n(\lambda), \tag{44}$$

where $f_n(\lambda)=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{-i\lambda t}B^n(t)\,dt$. By assumption,

$$f_n(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda t}e^{-n\varkappa|t|}\,dt=\frac{n\varkappa}{\pi(n^2\varkappa^2+\lambda^2)}.$$

Since ψ is an odd function, $C_{2k}(\psi)=0$; that is,

$$\sigma(\theta)=\frac{4\varkappa\gamma^2}{A_0^2+B_0^2}\left(\sum_{k=0}^{\infty}\frac{C_{2k+1}^2(\psi)}{(2k)!\,((2k+1)^2\varkappa^2+\omega_0^2)}\right)\begin{pmatrix}A_0^2+4B_0^2 & -3A_0B_0 & -6B_0\\ -3A_0B_0 & 4A_0^2+B_0^2 & 6A_0\\ -6B_0 & 6A_0 & 12\end{pmatrix}.$$

The coefficients $C_{2k+1}(\psi)$ are the values of known integrals (see, for example, relation 7.376.3 in the book [5]).

Bibliography

1. T. A. Bardadym and A. V. Ivanov, *Asymptotic normality of l_α-estimators of parameters of a nonlinear regression model*, Dokl. Akad. Nauk Ukrain. SSR Ser. A (1988), no. 8, 68–70, 88. (Russian) MR966270 (90h:62208)
2. T. A. Bardadym and A. V. Ivanov, *On the asymptotic normality of l_α-estimators of a parameter of a nonlinear regression model*, Teor. Imovir. Mat. Stat. **60** (1999), 1–10; English transl. in Theory Probab. Math. Statist. **60** (2000), 1–11. MR1826135
3. R. N. Bhattacharya and R. Ranga Rao, *Normal Approximation and Asymptotic Expansions*, John Wiley & Sons, New York–London–Sydney, 1976. MR0436272 (55:9219)
4. Yu. V. Goncharenko and S. I. Lyashko, *Brouwer's Theorem*, Kiĭ, Kiev, 2000. (Russian)
5. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, GIFML, Moscow, 1963; English transl., Academic Press, New York–London, 1965. MR0197789 (33:5952)
6. I. A. Ibragimov and Yu. A. Rozanov, *Gaussian Random Processes*, Nauka, Moscow, 1970; English transl., Springer-Verlag, New York–Berlin, 1978. MR543837 (80f:60038)
7. A. V. Ivanov, *A solution of the problem of detecting hidden periodicities*, Teor. Veroyatnost. i Mat. Statist. **20** (1979), 44–59; English transl. in Theory Probab. Math. Statist. **20** (1980), 51–68. MR529259 (80c:62107)
8. A. V. Ivanov, *On the consistency of l_α-estimators for the parameters of a regression function*, Teor. Veroyatnost. i Mat. Statist. **42** (1990), 42–48; English transl. in Theory Probab. Math. Statist. **42** (1991), 47–53. MR1069312 (92a:62043)
9. A. V. Ivanov and N. N. Leonenko, *Statistical Analysis of Random Fields*, Vyshcha Shkola, Kiev, 1986; English transl., Kluwer Academic Publishers Group, Dordrecht, 1989. MR917486 (89e:62125)
10. A. V. Ivanov and I. V. Orlovskiĭ, *Asymptotic normality of Koenker–Bassett estimators in nonlinear regression models*, Teor. Veroyatnost. i Mat. Statist. **72** (2005), 30–41; English transl. in Theory Probab. Math. Statist. **72** (2006), 33–45. MR2168134 (2007b:62029)
11. A. V. Ivanov and I. V. Orlovskiĭ, *The consistency of M-estimators in nonlinear regression models with continuous time*, Naukovi visti NTUU KPI (2005), no. 4 (42), 140–147. (Ukrainian)
12. I. V. Orlovskiĭ, *The consistency of Koenker–Bassett estimators in nonlinear regression models*, Naukovi visti NTUU KPI (2004), no. 3 (35), 144–150.
13. P. Huber, *Robust statistics*, John Wiley & Sons, Inc., New York, 1981. MR606374 (82i:62057)
14. L. Schmetterer, *Introduction to Mathematical Statistics*, Springer-Verlag, Berlin–New York, 1974; Translated from the second German edition. MR0359100 (50:11555)
15. M. A. Arcones, *Asymptotic theory of M-estimators over a convex kernel*, Econometric Theory **14** (1998), no. 4, 387–422. MR1650029 (99m:62029)
16. X. R. Chen and Y. H. Wu, *Strong consistency of M-estimates in linear models*, J. Multivariate Anal. **27** (1988), 116–130. MR971177 (90k:62140)
17. S. A. van de Geer, *Empirical Processes in M-Estimation*, Cambridge University Press, 2000.
18. L. Giraitis and H. L. Koul, *Estimation of the dependence parameter in linear regression with long-range-dependent errors*, Stochastic Process. Appl. **71** (1997), no. 2, 207–224. MR1484160 (99b:62145)
19. L. Giraitis, H. L. Koul, and D. Surgailis, *Asymptotic normality of regression estimators with long memory errors*, Stat. Probab. Letters **29** (1996), 317–335. MR1409327 (97h:62055)
20. U. Grenander and M. Rosenblatt, *Statistical Analysis of Stationary Time Series*, Almqvist and Wiksell, Stockholm, 1956. MR0084975 (18:959b)
21. F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw, and W. A. Stahel, *Robust Statistics. The Approach Based on Influence Functions*, Wiley, New York, 1986. MR829458 (87k:62054)
22. P. J. Huber, *Robust regression: asymptotics, conjectures and Monte-Carlo*, Ann. Statist. **1** (1973), no. 5, 799–821. MR0356373 (50:8843)
23. A. V. Ivanov, *Asymptotic Theory of Nonlinear Regression*, Kluwer, Dordrecht, 1997. MR1472234 (99h:62086)
24. A. V. Ivanov, *Asymptotic properties of L_p-estimators*, Theor. Stoch. Process. **14 (30)** (2008), no. 1, 60–68. MR2479706
25. A. V. Ivanov and N. N. Leonenko, *Asymptotic behavior of M-estimators in continuous-time non-linear regression with long-range dependent errors*, Random Oper. Stoch. Equ. **10** (2002), no. 3, 201–222. MR1923424 (2003k:62080)
26. A. V. Ivanov and I. V. Orlovsky, *L_p-estimates in nonlinear regression with long-range dependence*, Theor. Stoch. Process. **7 (23)** (2002), no. 3–4, 38–49.
27. A. V. Ivanov and I. V. Orlovsky, *Parameter estimators of nonlinear quantile regression*, Theor. Stoch. Process. **11 (27)** (2005), no. 3–4, 82–91. MR2330004 (2008j:62068)

28. R. I. Jennrich, *Asymptotic properties of non-linear least squares estimators*, Ann. Math. Statist. **40** (1969), 633–643. MR0238419 (38:6695)
29. J. Jurečková, *Consistency of M-estimators in a linear model, generated by nonmonotone and discontinuous ψ-functions*, Probab. Math. Statist. **10** (1989), 1–10. MR990395 (90g:62155)
30. H. L. Koul, *M-estimators in linear models with long range dependent errors*, Stat. Probab. Letters **14** (1992), 153–164. MR1173413 (93f:62124)
31. H. L. Koul, *Asymptotics of M-estimations in non-linear regression with long-range dependence errors*, Proc. Athens Conf. Appl. Probab. and Time Ser. Analysis (P. M. Robinson and M. Rosenblatt, eds.), Lecture Notes in Statistics, vol. II, Springer-Verlag, 1996, pp. 272–291. MR1466752 (98g:62124)
32. H. L. Koul, R. T. Baillie, and D. Surgailis, *Regression model fitting with a long memory covariance process*, Economic Theory **20** (2004), 485–512. MR2061725 (2005e:62124)
33. H. L. Koul and K. Mukherjee, *Regression quantiles and related processes under long range dependent errors*, J. Multivariate Anal. **51** (1994), 318–337. MR1321301 (96e:62153)
34. H. L. Koul and D. Surgailis, *Asymptotic expansion of M-estimators with long memory errors*, Ann. Statist. **25** (1997), 818–850. MR1439325 (98c:62172)
35. H. L. Koul and D. Surgailis, *Second order behavior of M-estimators in linear regression with long-memory errors*, J. Statist. Plann. Inference **91** (2000), 399–412. MR1814792 (2001m:62101)
36. H. L. Koul and D. Surgailis, *Robust estimators in regression models with long memory errors*, Theory and Application of Long-Range Dependence (P. Doukhan, G. Oppenheim, and M. S. Taqqu, eds.), Birkhäuser, Boston, 2003, pp. 339–353. MR1957498
37. F. Liese, *Necessary and sufficient conditions for consistency of approximate M-estimators in nonlinear models*, Proc. Prague Stochastics, 1998, pp. 357–360.
38. F. Liese and I. Vajda, *Asymptotic Normality of M-Estimators in Nonlinear Regression*, Research Report No. 1714, UTIA, Prague, 1991.
39. F. Liese and I. Vajda, *Consistency of M-Estimators in Nonlinear Regression*, Research Report No. 1713, UTIA, Prague, 1991.
40. F. Liese and I. Vajda, *Consistency of M-estimates in general regression models*, J. Multivariate Anal. **50** (1994), no. 1, 93–114. MR1292610 (95g:62133)
41. F. Liese and I. Vajda, *Necessary and sufficient conditions for consistency of generalized M-estimates*, Metrika **42** (1995), 291–324. MR1380211 (97b:62039)
42. F. Liese and I. Vajda, *A General Asymptotic Theory of M-Estimators*, Research Report No. 1951, UTIA, Prague, 1999.
43. Ch. H. Müller, *Robust Planning and Analysis of Experiments*, Lecture Notes in Statistics, Springer, New York, 1997. MR1454843 (98g:62140)
44. I. V. Orlovsky, *M-estimates in nonlinear regression with weak dependence*, Theor. Stoch. Process. **9(25)** (2003), no. 1–2, 108–122. MR2080017 (2005i:62099)
45. J. Pfanzagl, *On the measurability and consistency of minimum contrast estimates*, Metrika **14** (1969), 249–272.
46. A. E. Ronner, *Asymptotic normality of p-norm estimators in multiple regression*, Z. Wahrsch. verw. Gebiete. **66** (1984), 613–620. MR753816 (86a:62042)
47. Y. Wu and M. M. Zen, *A strongly consistent information criterion for linear model selection based on M-estimation*, Probab. Theory Related Fields **113** (1999), 599–625. MR1717532 (2000g:62167)

Department of Mathematical Analysis and Probability Theory, National Technical University of Ukraine, Kyiv Polytechnical Institute, Peremogy Avenue 37, Kyiv-56, 03056, Ukraine

E-mail address: ivanov@paligora.kiev.ua

Department of Mathematical Analysis and Probability Theory, National Technical University of Ukraine, Kyiv Polytechnical Institute, Peremogy Avenue 37, Kyiv-56, 03056, Ukraine

E-mail address: avalon@ukrpost.ua

Received 11/SEP/2008

Translated by OLEG KLESOV