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APPROXIMATION OF $\mathrm{SSub}_\varphi(\Omega)$ STOCHASTIC PROCESSES IN THE SPACE $L_p(\mathbb{T})$

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Abstract. A bound for the distributions of norms is obtained for $\mathrm{Sub}_\varphi(\Omega)$ stochastic processes in the space $L_p(\mathbb{T})$. This bound is used to construct an approximation of strictly φ-sub-Gaussian processes by random broken lines in the space $L_p(\mathbb{T})$ with a given accuracy and reliability

1. Introduction

We consider an approximation of strictly φ-sub-Gaussian stochastic processes by random broken lines in the space $L_p(\mathbb{T})$. We obtain an upper bound for the norm of $\mathrm{Sub}_\varphi(\Omega)$ processes and use it to construct an approximation of a stochastic process with a given accuracy and reliability.

We recall some definitions and results concerning $\mathrm{Sub}_\varphi(\Omega)$ and $\mathrm{SSub}_\varphi(\Omega)$ stochastic processes.

The notion of the $\mathrm{Sub}_\varphi(\Omega)$ space is introduced in the paper [3]. The definition and properties of $\mathrm{Sub}_\varphi(\Omega)$ random variables are studied in [2].

Let (Ω, B, P) be a standard probability space.

Definition 1.1 ([1])**.** A continuous even convex function $u = (u(x), x \in \mathbf{R})$ is called an N-Orlicz function if it increases in the domain $x > 0$, $u(0) = 0$, and if

$$\frac{u(x)}{x} \to 0 \quad \text{as } x \to 0 \quad \text{and} \quad \frac{u(x)}{x} \to \infty \quad \text{as } x \to \infty.$$

Condition Q ([2])**.** We say that a function N satisfies condition **Q** if

$$\varliminf_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0.$$

Remark 1.1. The case of $c = \infty$ in Condition **Q** also fits Definition 1.1.

Definition 1.2 ([2])**.** Let φ be an N-Orlicz function satisfying condition **Q**. We say that a centered random variable ξ belongs to $\mathrm{Sub}_\varphi(\Omega)$ (to the space of φ-sub-Gaussian random variables) if there exists a constant $r_\xi \geq 0$ such that

$$\mathsf{E}\exp(\lambda\xi) \leq \exp(\varphi(r_\xi\lambda))$$

for all $\lambda \in \mathbf{R}$.

Theorem 1.1 ([1]–[3])**.** $\mathrm{Sub}_\varphi(\Omega)$ *is a Banach space with respect to the norm*

$$\tau_\varphi(\xi) = \inf\{a \geq 0\colon\ \mathsf{E}\exp(\lambda\xi) \leq \exp(\varphi(a\lambda)), \lambda \in \mathbf{R}\}.$$

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Definition 1.3 ([1])**.** Let $\mathbb{T}$ be a set of parameters. We say that a stochastic process $\{X(t), t \in \mathbb{T}\}$ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ for all $t \in \mathbb{T}$ and $\sup_{t\in\mathbb{T}} \tau_\varphi(X(t)) < \infty$.

Definition 1.4 ([4])**.** A family Δ of random variables ξ of the space $\mathrm{Sub}_\varphi(\Omega)$ is called a strictly $\mathrm{Sub}_\varphi(\Omega)$ family if there exists a constant $C_\Delta > 0$ such that

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\xi_i\right) \le C_\Delta\left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{1/2}$$

for all finite sets I, for all collections $\xi_i \in \Delta$, $i \in I$, and for all $\lambda_i \in \mathbf{R}^1$.

The number C_Δ is called a defining constant.

Definition 1.5 ([4])**.** A φ-sub-Gaussian stochastic process $\{X(t), t \in \mathbb{T}\}$ is called a strictly $\mathrm{Sub}_\varphi(\Omega)$ process if the family of random variables $\{X(t), t \in \mathbb{T}\}$ is a strictly $\mathrm{Sub}_\varphi(\Omega)$ family.

Definition 1.6 ([1])**.** Let $f = (f(x), x \in \mathbf{R})$ be a real function. The function

$$f^* = (f^*(x), x \in \mathbf{R})$$

defined by $f^*(x) = \sup_{y\in\mathbf{R}}(xy - f(y))$ is called the Young–Fenchel transform of the function f or the conjugate function to f.

Let $X = \{X(t), t \in \mathbb{T}\}$, $\mathbb{T} = [0, T]$, be a given strictly φ-sub-Gaussian stochastic process with a defining constant C_Δ.

Let $S := \{t_k\}_{k=0}^{k=N} = \{Tk/N, k = 0, \dots, N\}$ be a partition of the interval $\mathbb{T}$ into N parts $[t_k, t_{k+1}]$. We want to approximate the process $\{X(t), t \in \mathbb{T}\}$ by the line $X_N(t)$ constructed from the points $(t_k, X(t_k))$, $k = 0, \dots, N$, as follows:

$$X_N(t) = \alpha_1 X(t_k) + \alpha_2 X(t_{k+1}), \qquad t \in [t_k, t_{k+1}],\ k = 0, \dots, N-1,$$

where $\alpha_1 = 1 - (t - t_k)N/T$ and $\alpha_2 = (t - t_k)N/T$.

Definition 1.7. We say that a broken line $X_N(t)$ approximates the process $X(t)$ in the space $L_p(\mathbb{T})$ with a given accuracy $\varepsilon > 0$ and reliability $1 - \delta$, $0 < \delta < 1$, if

$$\mathsf{P}\left\{\left(\int_{\mathbb{T}} |X(t) - X_N(t)|^p \, dt\right)^{1/p} > \varepsilon\right\} \le \delta.$$

The problem considered in this paper is to reconstruct the process $\{X(t)\}$ from the broken line $\{X_N(t), t \in \mathbb{T}\}$ with a given accuracy $\varepsilon > 0$ and reliability $1 - \delta$, $0 < \delta < 1$, in the norm of the space $L_p(\mathbb{T})$ if the values of the process at the corresponding points $\{Tk/N, k = 0, \dots, N\}$ are known.

Let $Y_N(t) := X(t) - X_N(t)$, $t \in \mathbb{T}$, be the deviation process.

Let the process $\{X(t), t \in \mathbb{T}\}$ be such that

$$\mathsf{E}\,|X(t+h) - X(t)|^2 \le b^2(h) \quad \text{for all } h > 0 \quad \text{and for all } t \in \mathbb{T},$$

where $b(h)$ increases and $b(h) \to 0$ as $h \to 0$.

2. A bound for norms in $L_p(\mathbb{T})$ for sub-Gaussian stochastic processes

Theorem 2.1. *Let $\{\mathbb{T}, \Lambda, M\}$ be a measurable space, $X = \{X(t), t \in \mathbb{T}\}$ a measurable stochastic process belonging to the space $\mathrm{Sub}_\varphi(\Omega)$, and let $\tau_\varphi(t) := \tau_\varphi(X(t))$. Further, let the Lebesgue integral $\int_{\mathbb{T}} (\tau_\varphi(t))^p \, d\mu(t) < \infty$ be well defined for $p \ge 1$. Then the integral $\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) < \infty$ exists with probability 1 and*

$$\mathsf{P}\left\{\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) > \varepsilon\right\} \le 2\exp\left\{-\varphi^*\left(\left(\frac{\varepsilon}{c}\right)^{1/p}\right)\right\} \tag{1}$$

for all $\varepsilon > c\left(f\left(c^{1/p}p/\varepsilon^{1/p}\right)\right)^p$, *where* $c = \int_{\mathbb{T}} (\tau_\varphi(t))^p \, d\mu(t)$, f *is a function such that* $\varphi(u) = \int_0^u f(v)\,dv$ *for* $u > 0$, *and where* φ^* *is the Young–Fenchel transform of the function* φ.

Proof. If ξ is a random variable of the space $\mathrm{Sub}_\varphi(\Omega)$, then

$$\mathsf{E}\,|\xi|^\alpha \le 2\left(\frac{\alpha}{e}\right)^\alpha (\tau_\varphi(\xi))^\alpha \cdot \exp\{\varphi(\lambda) - \alpha \ln \lambda\} \tag{2}$$

for arbitrary $\alpha > 0$ and $\lambda > 0$ (see [2]). Inequality (2) together with Fubini's theorem implies that

$$\mathsf{E}\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) = \int_{\mathbb{T}} \mathsf{E}\,|X(t)|^p \, d\mu(t) \le 2\left(\frac{p}{e}\right)^p \int_{\mathbb{T}} (\tau_\varphi(t))^p \, d\mu(t) \exp\{\varphi(\lambda) - p\ln\lambda\} < \infty$$

for all $p \ge 1$ and $\lambda > 0$; that is, $\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) < \infty$ with probability 1.

Let $r \ge 1$. Then the generalized Minkowski inequality implies that

$$\begin{aligned}\left\|\int_{\mathbb{T}} |X(t)|^p \, d\mu(t)\right\|_{L_r} &= \left(\mathsf{E}\left|\int_{\mathbb{T}} |X(t)|^p \, d\mu(t)\right|^r\right)^{1/r} \le \int_{\mathbb{T}} \left\| |X(t)|^p \right\|_{L_r} d\mu(t) \\ &= \int_{\mathbb{T}} \left(\mathsf{E}\,|X(t)|^{pr}\right)^{1/r} \, d\mu(t).\end{aligned}$$

Now we obtain from inequality (2) that

$$\begin{aligned}\left\|\int_{\mathbb{T}} |X(t)|^p \, d\mu(t)\right\|_{L_r} &\le \int_{\mathbb{T}} \left(2\,(\tau_\varphi(t))^{pr} \left(\frac{pr}{e}\right)^{pr} \exp\{\varphi(\lambda)\}\lambda^{-pr}\right)^{1/r} d\mu(t) \\ &= \left(2\left(\frac{pr}{e}\right)^{pr} \exp\{\varphi(\lambda)\}\lambda^{-pr}\right)^{1/r} \int_{\mathbb{T}} (\tau_\varphi(t))^p \, d\mu(t)\end{aligned} \tag{3}$$

for all $\lambda > 0$. Let $\varepsilon > 0$, $r \ge 1$, and $\lambda > 0$. Using the Chebyshev inequality we derive from (3) that

$$\begin{aligned}\mathsf{P}\left\{\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) > \varepsilon\right\} &\le \frac{\left(\mathsf{E}\int_{\mathbb{T}} |X(t)|^p \, d\mu(t)\right)^r}{\varepsilon^r} \\ &\le 2\left(\frac{pr}{e}\right)^{pr} \frac{\lambda^{-pr}\exp\{\varphi(\lambda)\}}{\varepsilon^r} \left(\int_{\mathbb{T}} (\tau_\varphi(t))^p \, d\mu(t)\right)^r.\end{aligned} \tag{4}$$

Letting $c := \int_{\mathbb{T}} (\tau_\varphi(t))^p \, d\mu(t)$, we deduce from inequality (4) that

$$\mathsf{P}\left\{\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) > \varepsilon\right\} \le 2\left(\frac{c}{\varepsilon}\frac{p^p}{(\lambda e)^p}\right)^r \cdot r^{pr}\exp\{\varphi(\lambda)\} = 2a^r r^{pr}\exp\{\varphi(\lambda)\},$$

where $a = cp^p/(\varepsilon(\lambda e)^p)$.

Let $\psi(r) = a^r r^{pr}$. Then

$$(\ln\psi(r))' = (r\ln a + pr\ln r)' = \ln a + p\ln r + p.$$

Thus $(\ln\psi(r))' = 0$ if $r = a^{-1/p}e^{-1}$. For this point r,

$$\psi(a^{-1/p}e^{-1}) = \exp\{-pa^{-1/p}e^{-1}\},$$

whence

$$\begin{aligned}\mathsf{P}\left\{\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) > \varepsilon\right\} &\le 2\exp\left\{-\left(\frac{pa^{-1/p}}{e} - \varphi(\lambda)\right)\right\} \\ &= 2\exp\left\{-\left(\lambda\left(\frac{\varepsilon}{c}\right)^{1/p} - \varphi(\lambda)\right)\right\}\end{aligned} \tag{5}$$

if $a^{-1/p}e^{-1} > 1$, that is, if $\varepsilon^{1/p}\lambda/(c^{1/p}p) \geq 1$ or $\lambda \geq c^{1/p}p/\varepsilon^{1/p} = z$. Inequality (5) implies that

$$\mathsf{P}\left\{\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) > \varepsilon\right\} \leq 2\exp\left\{-\sup_{\lambda \geq z}\left(\lambda\left(\frac{\varepsilon}{c}\right)^{1/p} - \varphi(\lambda)\right)\right\}. \tag{6}$$

Let $\varphi^*(u)$ be the Young–Fenchel transform of the function $\varphi(u)$ for $u > 0$. This means that

$$\varphi^*(u) = \sup_{\lambda > 0}\left(u\lambda - \varphi(\lambda)\right). \tag{7}$$

The supremum in equality (7) is attained at the point

$$\lambda = f^{(-1)}(u)$$

(see [1]), where $f^{(-1)}(u)$ is the generalized inverse function to $f(u)$ and $f(u)$ is the density of $\varphi(u)$, that is, $\varphi(u) = \int_0^u f(v)\,dv$.

Thus inequality (6) implies that

$$\mathsf{P}\left\{\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) > \varepsilon\right\} \leq 2\exp\left\{-\varphi^*\left(\left(\frac{\varepsilon}{c}\right)^{1/p}\right)\right\}$$

for

$$f^{(-1)}\left(\left(\frac{\varepsilon}{c}\right)^{1/p}\right) \geq z. \tag{8}$$

Note that inequality (8) holds if

$$\left(\frac{\varepsilon}{c}\right)^{1/p} \geq f\left(\frac{c^{1/p}p}{\varepsilon^{1/p}}\right),$$

that is, if

$$\varepsilon \geq c\left(f\left(\frac{c^{1/p}p}{\varepsilon^{1/p}}\right)\right)^p. \qquad \square$$

Corollary 2.1. *If $\varphi(x) = x^2/2$, that is, if $X(t)$ is a sub-Gaussian stochastic process, then inequality* (1) *becomes of the following form:*

$$\mathsf{P}\left\{\int_{\mathbb{T}} |X(t)|^p \, d\mu(t) > \varepsilon\right\} \leq 2\exp\left\{-\frac{\varepsilon^{2/p}}{2c^{2/p}}\right\}$$

for $\varepsilon > cp^{p/2}$, where

$$c = \int_{\mathbb{T}} \left(\tau_{x^2/2}(t)\right)^p \, d\mu(t).$$

3. Accuracy and reliability of the approximation

Lemma 3.1. *Let $X(t)$ be a $\mathrm{SSub}_\varphi(\Omega)$ process with a defining constant C_Δ,*

$$\mathsf{E}\,|X(t+h) - X(t)|^2 \leq b^2(h) \quad \textit{for all } h > 0 \textit{ and for all } t \in \mathbb{T},$$

where $b(h)$ increases and $b(h) \to 0$ as $h \to 0$. Then the deviation process $\{Y_N(t), t \in \mathbb{T}\}$ is such that

$$\tau_\varphi(Y_N(t)) \leq C_\Delta b\left(\frac{T}{N}\right) \quad \textit{for all } t \in \mathbb{T}. \tag{9}$$

Proof. We apply the inequality $\tau_\varphi(Y_N(t)) \le C_\Delta(\mathsf{E}\,Y_N^2(t))^{1/2}$. Taking into account the representation

$$X_N(t) = \alpha_1 X(t_k) + \alpha_2 X(t_{k+1}), \qquad t \in [t_k, t_{k+1}], \quad k = 0, \dots, N-1,$$

where $\alpha_1 = 1 - (t-t_k)N/T$ and $\alpha_2 = (t-t_k)N/T$, we obtain

$$\begin{aligned}
\mathsf{E}\,Y_N^2(t) &= \mathsf{E}\,|X(t) - X_N(t)|^2 = \mathsf{E}\,|X(t) - \alpha_1 X(t_k) - \alpha_2 X(t_{k+1})|^2 \\
&= \mathsf{E}\,|X(t)|^2 + \mathsf{E}\,\alpha_1^2 X^2(t_k) + \mathsf{E}\,\alpha_2^2 X^2(t_{k+1}) - 2\alpha_1\,\mathsf{E}\,X(t)X(t_k) \\
&\quad - 2\alpha_2\,\mathsf{E}\,X(t)X(t_{k+1}) + 2\alpha_1\alpha_2\,\mathsf{E}\,X(t_k)X(t_{k+1}) \\
&= B(t,t) + \left(1 - 2\frac{N}{T}(t-t_k) + \frac{N^2}{T^2}(t-t_k)^2\right) B(t_k,t_k) \\
&\quad + (t-t_k)^2\frac{N^2}{T^2}B(t_{k+1},t_{k+1}) - 2B(t,t_k)\left(1 - (t-t_k)\frac{N}{T}\right) \\
&\quad - 2B(t,t_{k+1})(t-t_k)\frac{N}{T} + 2\left(1-(t-t_k)\frac{N}{T}\right)(t-t_k)\frac{N}{T}B(t_k,t_{k+1}) \\
&= \left(1-(t-t_k)\frac{N}{T}\right)\big(B(t,t) - 2B(t,t_k) + B(t_k,t_k)\big) \\
&\quad + (t-t_k)\frac{N}{T}\big(B(t,t) - 2B(t,t_{k+1}) + B(t_{k+1},t_{k+1})\big) \\
&\quad + \left(\frac{N^2}{T^2}(t-t_k)^2 - \frac{N}{T}(t-t_k)\right) B(t_k,t_k) \\
&\quad + \left(\frac{N^2}{T^2}(t-t_k)^2 - \frac{N}{T}(t-t_k)\right) B(t_{k+1},t_{k+1}) \\
&\quad + 2\left(1-(t-t_k)\frac{N}{T}\right)\frac{N}{T}(t-t_k)B(t_k,t_{k+1}) \\
&= \left(1-(t-t_k)\frac{N}{T}\right)\mathsf{E}\,|X(t)-X(t_k)|^2 + \frac{N}{T}(t-t_k)\,\mathsf{E}\,|X(t)-X(t_{k+1})|^2 \\
&\quad + \frac{N}{T}(t-t_k)\left(1-(t-t_k)\frac{N}{T}\right)\big(-B(t_k,t_k) - B(t_{k+1},t_{k+1}) + 2B(t_k,t_{k+1})\big) \\
&\le \left(1-(t-t_k)\frac{N}{T}\right) b^2(t-t_k) + \frac{N}{T}(t-t_k)b^2(t_{k+1}-t) \\
&\quad - \frac{N}{T}(t-t_k)\left(1-(t-t_k)\frac{N}{T}\right)\mathsf{E}\,|X(t_k)-X(t_{k+1})|^2 \\
&\le \left(1-(t-t_k)\frac{N}{T}\right) b^2\left(\frac{T}{N}\right) + (t-t_k)\frac{N}{T}b^2\left(\frac{T}{N}\right) - 0 = b^2\left(\frac{T}{N}\right),
\end{aligned}$$

whence inequality (9) follows. $\square$

Theorem 3.1. *Let $X = \{X(t), t \in \mathbb{T}\}$ be the stochastic process introduced in Lemma* 3.1. *Let a random broken line $X_N(t)$ approximate the process $X(t)$ with a given accuracy ε and reliability $1-\delta$. Then the number N satisfies the following system of inequalities:*

$$\begin{cases} 2\exp\left\{-\varphi^*\left(\frac{\varepsilon^{1/p}}{(\mu(\mathbb{T}))^{1/p}C_\Delta b(T/N)}\right)\right\} \le \delta, \\ \varepsilon > (\mu(\mathbb{T}))^{1/p} C_\Delta b\left(\frac{T}{N}\right) f\left(\frac{(\mu(\mathbb{T}))^{1/p}C_\Delta b(T/N)p}{\varepsilon}\right). \end{cases}$$

Proof. This result is an easy corollary of Theorem 2.1 and Lemma 3.1. $\square$

Example. Let $\varphi(x) = x^\alpha/\alpha$, $|x| > 1$, $T = [0,1]$, and $C_\Delta = 1$. Then $\varphi^*(x) = x^\beta/\beta$ for $|x| > 1$, where $\alpha^{-1} + \beta^{-1} = 1$.

Let $b(t,h) = b(h) = h$. Then Theorem 3.1 implies the following inequalities determining N:

$$\begin{cases} 2\exp\left\{-\frac{\varepsilon^\beta N^\beta}{\beta}\right\} \le \delta, \\ N > \frac{p^{(\alpha-1)/\alpha}}{\varepsilon}. \end{cases}$$

Let, for example, $p = 3$, $\alpha = 3$, $\varepsilon = 0.01$, and $\delta = 0.01$. Then the number of points in a partition of the interval $\mathbb{T} = [0,1]$ used for constructing an approximation of a given process $X(t)$ with a given accuracy ε and reliability $1 - \delta$ is such that $N \ge 7948$.

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