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ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF A LINEAR STOCHASTIC DIFFERENTIAL-DIFFERENCE EQUATION OF NEUTRAL TYPE

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I. V. MALYK, E. F. TSAR'KOV, AND V. K. YASYNS'KYĬ

Abstract. Necessary and sufficient conditions are found for the exponential mean square stability of a stochastic differential-difference linear equation of neutral type in the scalar case.

1. Introduction

The Lyapunov stability of solutions of deterministic differential-functional equations of neutral type is studied in many papers. A special place among these papers is occupied by Hale [15], Azbelev et al. [1], Kamenskiĭ et al. [3], Khusainov and Shatyrko [13, 12], Slyusarchuk [9].

Kolmanovskiĭ and Nosov [8], Tsar'kov [14], and Khusainov [13] studied the behavior of solutions of stochastic differential-functional equations. The existence of a strong solution for a stochastic differential-functional equation of neutral type and generalizations of the second Lyapunov method are discussed by Kolmanovskiĭ and Shaĭkhet in [2], Bereza and Yasyns'kyĭ [5], and others.

In this paper, we obtain necessary and sufficient conditions for the exponential mean square stability of a stochastic differential-difference linear equation of neutral type and study the nonstable as well as the critical case.

2. Main part

Let $(\Omega, F, \mathsf{P}, \Im)$ be a probability basis [7] where $\Im \equiv \{F_t, t \geq 0\}$ is a filtration. Let a strong solution [2, 16] $x(t) = x(t,\omega) \in \mathbf{R}^1$ of the stochastic differential-difference linear equation of neutral type

$$d\{Dx_t\} = \{Lx_t\}\,dt + \{Gx_t\}\,dw(t) \tag{1}$$

with the initial condition

$$x_0 = \varphi \tag{2}$$

be defined on the probability basis $(\Omega, F, \mathsf{P}, \Im)$. Here

$$x_t \equiv \{x(t+s), -h \leq s \leq 0\} \in C([-h,0]),$$

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$\varphi \in C([-h, 0])$ is an F_0-measurable stochastic process, $w(t) = w(t, \omega)$ is a one-dimensional Wiener process adapted to $\Im = \{F_t, t \geq 0\}$, and where D, L, and G are difference operators defined for $\psi \in C([-h, 0])$ by the following relations:

$$
\begin{aligned}
D\psi &\equiv \psi(0) + \sum_{k=1}^{n} \delta_k \psi(-\tau_k), \qquad 0 < \tau_1 < \tau_2 < \cdots < \tau_n \leq h, \\
L\psi &\equiv \alpha\psi(0) + \sum_{k=1}^{m} b_k \psi(-\lambda_k), \qquad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m \leq h, \\
G\psi &\equiv f\psi(0) + \sum_{k=1}^{q} g_k \psi(-\theta_k), \qquad 0 < \theta_1 < \theta_2 < \cdots < \theta_q \leq h.
\end{aligned} \tag{3}
$$

For the stochastic differential-difference linear equation of neutral type (1), (2), the existence and uniqueness (up to stochastic equivalence) hold in the case of a strong solution $x(t) \in \mathbf{R}^1$ such that $\mathsf{E}\left\{x^2(t)\right\} < \infty$ (see [5]). Along with equation (1) consider the deterministic differential-difference equation of neutral type [1, 15]

$$d\{Dy_t\} = \{Ly_t\}\, dt \tag{4}$$

with the initial condition

$$y(t) = \varphi(t), \qquad -h \leq t \leq 0. \tag{5}$$

First we mention some auxiliary results.

Lemma 2.1. *If*

$$\sum_{k=1}^{n} |\delta_k| < 1, \tag{6}$$

then the solution $y(t) \equiv 0$ *of the deterministic differential-difference equation of neutral type* (4), (5) *is exponentially stable if and only if the roots of the characteristic quasi-polynomial*

$$V(z) \equiv z\left(1 + \sum_{k=1}^{n} e^{-z\tau_k}\delta_k\right) - a - \sum_{l=1}^{m} e^{-z\lambda_l} b_l \tag{7}$$

belong to the left semi-plane of the complex plane $\mathbf{C}$*, that is,*

$$\exists \rho > 0, \quad \forall z \in \mathbf{C}\colon \qquad V(z) = 0 \Rightarrow \operatorname{Re} z < -\rho. \tag{8}$$

Consider the Cauchy function $X(t)$ [15] as a solution of equation (4) satisfying the initial condition

$$X(t) \equiv \mathbf{1}(t) = \begin{cases} 0, & -h \leq t < 0; \\ 1, & t = 0. \end{cases} \tag{9}$$

Then the Cauchy function $X(t)$ can be represented in terms of the characteristic quasi-polynomial [4, 15].

Lemma 2.2. *The Cauchy function* $X(t)$ *is of the form*

$$X(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \mu} e^{zt} V^{-1}(z)\, dz, \tag{10}$$

where $\mu > -\rho$.

The following assertion holds.

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Theorem 2.1. *A solution of the stochastic differential-difference linear equation of neutral type* (1), (2) *satisfies the stochastic integral equation*

$$x(t) = y(t) + \int_0^t X(t-s) G x_s \, dw(s), \tag{11}$$

where $y(t)$ is a solution of (4), (2).

Proof. Since the neutral difference operator D is linear, we get

$$D x_t = D y_t + \int_0^t D X_{t-s} G x_s \, dw(s). \tag{12}$$

Substitute (12) into equation (1) and take into account the linearity of the difference operator L. Then we obtain

$$\begin{aligned} dDx_t &= dDy_t + d\int_0^t DX_{t-s} G x_s \, dw(s) \\ &= Ly_t \, dt + DX_0 G x_t \, dw(t) + \int_0^t d_t DX_{t-s} G x_s \, dw(s) \\ &= Ly_t \, dt + G x_t \, dw(t) + \int_0^t (LX_{t-s} \, dt) \, G x_s \, dw(s) \\ &= Ly_t \, dt + G x_t \, dw(t) + L\left\{ \int_0^t X(t-s) \, dt G x_s \, dw(s) \right\} dt \\ &= L\left\{ y_t + \int_0^t X(t-s) \, dt G x_s \, dw(s) \right\} dt + G x_t \, dw(t) = Lx_t \, dt + G x_t \, dw(t). \end{aligned}$$

The actions of the difference operator L and the integration can be interchanged indeed, since L is linear. Thus the stochastic process defined by (11) satisfies a stochastic differential-difference linear equation of neutral type. Theorem 2.1 is proved. □

Now we define the notion of exponential mean square stability of a solution of a stochastic differential-difference linear equation of neutral type (1), (2).

Definition 2.1. The trivial solution $x(t) \equiv 0$ of the problem (1), (2) is called exponentially mean square stable if there are constants $M > 0$ and $c > 0$ such that

$$\mathsf{E}\,|x(t)|^2 \le M e^{-ct}\,\mathsf{E}\,\|\varphi\|^2 \tag{13}$$

for all $t \ge 0$ and $\varphi \in C([-h,0])$, where $\|\varphi\| \equiv \max_{-h \le t \le 0} |\varphi(t)|$ and $\mathsf{E}\{\cdot\}$ means the mathematical expectation.

We need the following notation and related results to find necessary and sufficient conditions for the exponential stability of the trivial solution $x(t) \equiv 0$ of a stochastic differential-difference linear equation of neutral type (1), (2).

Using the integral representation (11) of the strong solution of equation (1), and the representation (10) of the Cauchy function $X(t)$, we write an integral Volterra equation of neutral type for the stochastic process $\gamma(t) \equiv G x_t$:

$$\gamma(t) \equiv G y_t + \int_0^t H(t-s) \gamma(t) \, dw(s), \tag{14}$$

where

$$H(t) \equiv G X_t = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \mu} e^{zt} G_1(z) V^{-1}(z) \, dz \tag{15}$$

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and

$$G_1(z) \equiv f + \sum_{i=1}^{q} g_i e^{-z\theta_i}. \tag{16}$$

To derive the latter equation, we apply the difference operator G to both sides of equation (10) and obtain

$$GX_t = G\left\{\frac{1}{2\pi i}\int_{\operatorname{Re} z=\mu} e^{zt}V^{-1}(z)\,dz\right\} = \frac{1}{2\pi i}\int_{\operatorname{Re} z=\mu}\left\{fe^{zt} + \sum_{k=1}^{q} g_k e^{z(t-\theta_k)}\right\}V^{-1}(z)\,dz$$
$$= \frac{1}{2\pi i}\int_{\operatorname{Re} z=\mu}\left\{f + \sum_{k=1}^{q} g_k e^{-z\theta_k}\right\}e^{zt}V^{-1}(z)\,dz = \frac{1}{2\pi i}\int_{\operatorname{Re} z=\mu} G_1(z)e^{zt}V^{-1}(z)\,dz,$$

whence (14) follows.

Introduce the following notation:

$$M(t,\varphi) \equiv \mathsf{E}\left\{|x(t)|^2 \;/\; F^0\right\} = \mathsf{E}\left\{|x(t)|^2\right\},$$
$$\Gamma(t,\varphi) \equiv \mathsf{E}\left\{|\gamma(t)|^2 \;/\; F^0\right\} = \mathsf{E}\left\{|\gamma(t)|^2\right\}.$$

The conditional mathematical expectations above are equal to the corresponding unconditional mathematical expectations, since the stochastic processes $x(t)$ and $\gamma(t)$ are independent of the initial σ-algebra F_0 [7].

Further we square the integral representation (11) of a strong solution of the stochastic differential-difference linear equation of neutral type and evaluate the mathematical expectations to obtain the following equation for the second moment $M(t,\varphi)$:

$$M(t,\varphi) = |y(t)|^2 + \int_0^t |X(t-s)|^2\,\Gamma(s,\varphi)\,ds. \tag{17}$$

In deriving this equality we take into account that the mathematical expectation of a Wiener–Itô integral is zero and that the mathematical expectation of the square of a Wiener–Itô integral is equal to the usual Riemann integral [7].

Analogously we get

$$\Gamma(t,\varphi) = |Gy_t|^2 + \int_0^t |H(t-s)|^2\Gamma(s,\varphi)\,ds. \tag{18}$$

Next we prove the following auxiliary result.

Theorem 2.2. *Let conditions* (6) *and* (8) *hold. Then a solution of the integral equation* (18) *is such that*

$$\exists C_1 > 0, \quad \exists C_2 > 0, \quad \forall \varphi \in C([-h,0])\colon \qquad \int_0^\infty \Gamma(t,\varphi)e^{tC_2}\,dt < C_1\|\varphi\|^2 \tag{19}$$

if and only if

$$B \equiv \int_0^\infty |H(t)|^2 dt < 1. \tag{20}$$

Proof. Conditions (6) and (8) imply that there exists a constant $M > 0$ such that

$$|H(t)| \le Me^{-\rho t}, \qquad |Gy_t| \le Me^{-\rho t}\|\varphi\| \tag{21}$$

for $\varphi \in C([-h,0])$. This means that one can apply the Laplace transform to (18) and obtain the following equation:

$$\int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt = \int_0^\infty |Gy_t|^2 e^{-zt}\,dt + \int_0^\infty |H(t)|^2 e^{-zt}\,dt \int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt, \tag{22}$$

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which is equivalent to the equation

$$\int_0^\infty \Gamma(t,\varphi)e^{-\lambda t}\,dt = \int_0^\infty |Gy_t|^2 e^{-zt}\,dt\left(1-\int_0^\infty |H(t)|^2 e^{-\lambda t}\,dt\right)^{-1} \tag{23}$$

if $\operatorname{Re} z$ is sufficiently large. Without loss of generality we assume that $\operatorname{Im} z = 0$. Note that the function $R(z) = \int_0^\infty |H(t)|^2 e^{-zt}\,ds$, as a function of $z \in \mathbf{R}$, is continuous and that $R(z)$ is a decreasing function in the interval $(-\rho,\infty)$ if $\operatorname{Im} z = 0$.

The sufficiency. Let relation (20) hold. Then there exists a number $\varepsilon > 0$ such that

$$\int_0^\infty |H(t)|^2 e^{-zt}\,dt < 1$$

for $z \in [-\varepsilon,\infty)$. Then

$$\begin{aligned}\int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt &< \int_0^\infty |Gy_t|^2 e^{-zt}\,dt\left(1-\int_0^\infty |H(t)|^2 e^{\varepsilon}\,dt\right)^{-1}\\ &\le \int_0^\infty |Gy_t|^2 e^{\varepsilon}\,dt\left(1-\int_0^\infty |H(t)|^2 e^{\varepsilon}\,dt\right)^{-1}\\ &\le M^2\|\varphi\|^2\,(\rho-\varepsilon)^{-1}\left(1-\int_0^\infty |H(t)|^2 e^{\varepsilon}\,dt\right)^{-1}\end{aligned} \tag{24}$$

for $z \in [-\varepsilon,\infty)$. Inequality (24) implies condition (19) if we let $C_2 = \varepsilon$ and

$$C_1 = M^2(\rho-\varepsilon)^{-1}\left(1-\int_0^\infty |H(t)|^2 e^{\varepsilon}\,dt\right)^{-1}.$$

The necessity. Let condition (19) hold and $\operatorname{Im} z = 0$. We prove (20) by contradiction. Assume that (20) does not hold, that is,

$$\int_0^\infty |H(t)|^2\,dt \ge 1.$$

Then $\int_0^\infty |H(t)|^2 e^{-zt}\,dt > 1$ for $z < 0$. This means that

$$\int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt = \int_0^\infty |Gy_t|^2\,e^{-zt}\,dt\left(1-\int_0^\infty |H(t)|^2 e^{-zt}\,dt\right)^{-1} < 0, \tag{25}$$

whence $\Gamma(t,\varphi) < 0$ for $\operatorname{Re} z < 0$. This is a contradiction, since $\int_0^\infty \Gamma(t,\varphi)e^{-\lambda t}\,dt > 0$ by definition. Theorem 2.2 is proved. $\square$

Lemma 2.3. *The integral on the left hand side of condition* (20) *is equal to*

$$B \equiv \int_0^\infty |H(t)|^2\,dt = \frac{1}{\pi}\int_0^\infty |G(is)|^2 |V(is)|^{-2}\,ds.$$

Proof. Using integral representation (15) in place of $H(t)$ we have

$$H(t) = \frac{1}{2\pi i}\int_{-\infty}^\infty G_1(is)e^{zt}V^{-1}(is)\,ds. \tag{26}$$

Note that $G_1(is)e^{zt}V^{-1}(is)$ is the image of $H(t)$ under the Laplace transform, that is,

$$G_1(is)e^{zt}V^{-1}(is) = \int_0^\infty e^{-ist}H(t)\,dt. \tag{27}$$

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Then the Plancherel–Parseval equality [4] implies that

$$\begin{aligned}\int_0^\infty |H(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^\infty |G(is)|^2|V(is)|^{-2}\,ds\\ &= \frac{1}{2\pi}\left(\int_{-\infty}^0 |G(is)|^2|V(is)|^{-2}\,ds + \int_0^\infty |G(is)|^2|V(is)|^{-2}\,ds\right)\\ &= \frac{1}{2\pi}\left(\int_0^\infty |G(-is)|^2\,|V(-is)|^{-2}\,ds + \int_0^\infty |G(is)|^2|V(is)|^{-2}\,ds\right)\\ &= \frac{1}{\pi}\int_0^\infty |G(is)|^2|V(is)|^{-2}\,ds.\end{aligned}$$

Lemma 2.3 is proved. □

Theorem 2.3. *Let conditions* (6) *and* (8) *hold for the coefficients of the stochastic differential-difference linear equation of neutral type and for roots of its characteristic quasi-polynomial* (7). *Then the condition*

$$B = \frac{1}{\pi}\int_0^\infty |G(is)|^2|V(is)|^{-2}\,ds < 1 \tag{28}$$

is necessary and sufficient for the exponential mean square stability of a solution of the problem (1), (2), *where* $G(z) \equiv f + \sum_{r=1}^q g_r e^{-z\theta_r}$.

Further, if $B > 1$, *then in every neighborhood of zero there exists an initial function* $\varphi(t)$ *such that*

$$\lim_{t\to\infty} M\left\{|x(t)|^2\right\} = \infty. \tag{29}$$

Proof. The sufficiency. Let condition (28) hold. Thus (20) holds, too. Conditions (6) and (8) imply that there exist constants $M_2 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} |Gy_t|^2 e^{tc_2} \le M_2\|\varphi\|^2, \qquad |y(t)|^2 e^{tc_2} \le M_2\|\varphi\|^2,\\ |X(t)|^2 e^{tc_2} \le M_2, \qquad |H(t)|^2 e^{tc_2} \le M_2\end{aligned} \tag{30}$$

for all $t > 0$ and $\varphi \in C([-h,0])$ (see [15]).

Note that the constant $c_2 > 0$ in the bounds (30) is the same as that in Theorem 2.2.

Inequalities (30) imply that

$$\begin{aligned}\Gamma(t,\varphi)e^{tc_2} &= |Gy_t|^2 e^{tc_2} + \int_0^t |H(t-s)|^2 e^{(t-s)c_2} e^{sc_2}\Gamma(s,\varphi)\,ds\\ &\le M_2\|\varphi\|^2 + M_2\int_0^t e^{sc_2}\Gamma(s,\varphi)\,ds \le M_2(1+c_1)\|\varphi\|^2\end{aligned} \tag{31}$$

for all $t > 0$ and $\varphi \in C([-h,0])$, where the constant c_1 is defined in the proof of Theorem 2.2.

Using inequalities (30) and (31) we obtain a bound for $M(t,\varphi)$ satisfying equation (17); namely,

$$M(t,\varphi) \le M_2\,(1+c_1)\,\|\varphi\|^2 e^{-tc_2}$$

for $t > 0$ and $\varphi \in C([-h,0])$.

This implies exponential stability (see condition (13) in Definition 2.1). The sufficiency is proved.

The necessity. Let conditions (6), (8) hold and let the solution $x(t) \equiv 0$ be exponentially stable, that is, $\mathsf{E}\,|x(t)|^2 \le Me^{-ct}\,\mathsf{E}\,\|\varphi\|^2$ for $t > 0$ and $\varphi \in C([-h,0])$, where $M > 0$

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and $c > 0$. Then

$$\Gamma(t,\varphi) = \mathsf{E}\left\{|Gx_t|^2\right\} \le (q+1)\left(|f|^2 + \sum_{r=1}^{q}|g_r|^2\right) \max_{-h\le\theta\le 0} \mathsf{E}\left\{|x(t+\theta)|^2\right\}$$
$$\le M(1+q)\left(|f|^2 + \sum_{r=1}^{q}|g_r|^2\right) e^{-ct}\|\varphi\|^2.$$

This means that there exists a constant $A_1 > 0$ such that

$$\int_0^\infty \Gamma(t,\varphi)e^{tc_2 dt} < c_1\|\varphi\|.$$

Taking into account equality (26) of Lemma 2.3, we derive inequality (28) from Theorem 2.2.

Further, equality (23) means that the poles of $1 - \int_0^\infty \Gamma(t,\varphi)e^{-\lambda t}\,dt$ coincide with the zeros of the function

$$1 - \int_0^\infty |H(t)|^2 e^{-\lambda t}\,dt,$$

and that $\int_0^\infty |H(t)|^2 e^{-\lambda t}\,dt$ is a continuous decreasing function of a real argument λ such that

$$\lim_{\lambda\to\infty}\int_0^\infty |H(t)|^2 e^{-\lambda t}\,dt = 0.$$

Note that there exists $\lambda_0 > 0$ such that

$$1 - \int_0^\infty |H(t)|^2 e^{-\lambda_0 t}\,dt = 0.$$

This means that the function $\int_0^\infty \Gamma(t,\varphi)e^{-\lambda t}\,dt$ has a pole at $\lambda = \lambda_0 > 0$. In its turn, the latter means that

$$\lim_{t\to\infty}\Gamma(t,\varphi) = \lim_{t\to\infty} e^{\lambda_0 t} = \infty,$$

whence (29) follows. Theorem 2.3 is proved. □

3. The critical case

Consider the behavior of a solution of the stochastic differential-difference linear equation of neutral type (1), (2) for $B = 1$.

Theorem 3.1. *Assume that*

1) *conditions* (6), (8) *hold;*
2)

$$B = \frac{1}{\pi}\int_0^\infty |G(is)|^2 |V(is)|^{-2}\,ds = 1; \tag{32}$$

3)

$$f + \sum_{k=1}^{q} g_k \ne 0.$$

Then there exists an initial function $\varphi \in C[-h,0]$ *such that*

$$0 < \lim_{t\to\infty} \mathsf{E}\,|x(t)|^2 < \infty. \tag{33}$$

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Proof. Let a strong solution of equation (1) be constructed for the initial condition

$$\varepsilon(t) = \begin{cases} 0, & -h \le t < 0, \\ \varepsilon, & t = 0. \end{cases} \tag{34}$$

Then $y(t) = \varepsilon X(t)$ is a solution of the deterministic equation (4), while equation (12) for $\Gamma(t,\varphi)$ becomes of the following form:

$$\begin{aligned} \int_0^\infty \Gamma(t,\varphi) e^{-zt}\,dt &= \varepsilon \int_0^\infty |GX_t|^2 e^{-zt}\,dt + \int_0^\infty |H(t)|^2 e^{-zt}\,ds \int_0^\infty \Gamma(t,\varphi) e^{-zt}\,dt \\ &= \int_0^\infty |H(t)|^2 e^{-zt}\,ds \left(\varepsilon + \int_0^\infty \Gamma(t,\varphi) e^{-zt}\,dt\right). \end{aligned} \tag{35}$$

Hence

$$\int_0^\infty \Gamma(t,\varphi) e^{-zt}\,dt = \frac{\varepsilon \int_0^\infty |H(t)|^2 e^{-zt}\,dt}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt}\,dt\right)}. \tag{36}$$

A relationship between the behavior of an original $\Gamma(t,\varphi)$ as $t \to \infty$ and that of its Laplace image as $z \to 0$ is well known, namely

$$\lim_{t\to\infty} \Gamma(t,\varphi) = \varepsilon \lim_{z\to 0} \frac{z \int_0^\infty |H(t)|^2 e^{-zt}\,dt}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt}\,dt\right)}; \tag{37}$$

see [6]. The limit as $z \to 0$ of the right hand side of equation (31) can be evaluated by using l'Hôpital's rule:

$$\begin{aligned} \lim_{z\to 0} \frac{\varepsilon z \int_0^\infty |H(t)|^2 e^{-zt}\,ds}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt}\,dt\right)} &= \lim_{z\to 0} \frac{\left(\varepsilon z \int_0^\infty |H(t)|^2 e^{-zt}\,ds\right)'}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt}\,dt\right)'} \\ &= \varepsilon \lim_{z\to 0} \frac{\int_0^\infty |H(t)|^2 e^{-zt}\,ds - z\int_0^\infty |H(t)|^2 t e^{-zt}\,ds}{\int_0^\infty |H(t)|^2 t e^{-zt}\,dt} \\ &= \varepsilon \lim_{z\to 0} \frac{\int_0^\infty |H(t)|^2 e^{-zt}\,ds}{\int_0^\infty |H(t)|^2 t e^{-zt}\,dt} = \frac{\varepsilon}{\int_0^\infty |H(t)|^2 t\,dt}. \end{aligned} \tag{38}$$

Conditions (8) imply that the integral in the denominator of the right hand side of (38) exists and

$$\varepsilon = \varepsilon \int_0^\infty |H(t)|^2\,dt \le \int_0^\infty |H(t)|^2 t\,dt \le K \int_0^\infty |H(t)|^2 e^{\rho t/2}\,dt = c < \infty, \tag{39}$$

where $K = \rho/2$. Thus

$$\frac{\varepsilon}{C} \le \lim_{t\to\infty} \Gamma(t,\varphi) \le 1. \tag{40}$$

Since $\Gamma(t,\varphi) = M\left\{Gx_t\right\}^2$,

$$\lim_{t\to\infty} \Gamma(t,\varphi) = \left(f + \sum_{k=1}^q g_k\right)^2 \lim_{t\to\infty} M\left\{x^2(t)\right\} \tag{41}$$

if the limit $\lim_{t\to\infty} \Gamma(t,\varphi)$ exists. Thus we derive from (40) and (41) that

$$0 < \frac{\varepsilon}{C}\left(f + \sum_{k=1}^q g_k\right)^{-2} \le \lim_{t\to\infty} \mathsf{E}\,|x(t)|^2 \le \left(f + \sum_{k=1}^q g_k\right)^{-2} < \infty.$$

Theorem 3.1 is proved. ∎

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Along with equation (1) we consider the perturbed equation

$$d\{Dx_t\} = \{Lx_t\}\,dt + (1+\alpha(t))\{Gx_t\}\,dw(t). \tag{42}$$

Assume that condition (32) holds. We study the asymptotic stability and instability of the problem (42), (2) (in other words, of the problem (1), (2) under permanent perturbations).

By $\Re_1$, we denote the space of continuous in $[0,\infty)$ functions ϕ such that

$$\exists\varepsilon\in(0,1),\quad \forall t>0\colon\qquad (1+\phi(t))^2 < 1-\varepsilon.$$

Let $\Re_2$ denote the space of continuous in $[0,\infty)$ functions ϕ such that

$$\exists\varepsilon>0,\quad \forall t>0\colon\qquad (1+\phi(t))^2 > 1+\varepsilon.$$

Theorem 3.2. *Let conditions* (6), (8) *hold and* $B=1$. *Then*

A) *the trivial solution of equation* (42) *is asymptotically stable in the mean square sense if* $\alpha\in\Re_1$;

B) *the trivial solution of equation* (42) *is unstable in the mean square sense if* $\alpha\in\Re_2$.

Proof. We rewrite equation (42) in the form of an integral equation, namely

$$x(t) = y(t) + \int_0^t (1+\alpha(s))X(t-s)Gx_s\,dw(s), \tag{43}$$

where $y(t)$ is a solution of the problem (4), (2), while $X(t)$ is its fundamental solution, $t\ge 0$.

Indeed, applying the linear operator D to (43) and then differentiating the result, we obtain

$$\begin{aligned}
dDx_t &= dDy_t + dD\left\{\int_0^t (1+\alpha(s))X(t-s)Gx_s\,dw(s)\right\}\\
&= Ly_t\,dt + Dd\left\{\int_0^t (1+\alpha(s))X(t-s)Gx_s\,dw(s)\right\}\\
&= Ly_t\,dt + \int_0^t (1+\alpha(s))\,dDX(t-s)Gx_s\,dw(s) + (1+\alpha(t))Gx_t\,dw(t)\\
&= Ly_t\,dt + \int_0^t (1+\alpha(s))LX(t-s)Gx_s\,dw(s)\,dt + (1+\alpha(t))Gx_t\,dw(t)\\
&= L\left\{y_t + \int_0^t (1+\alpha(s))X(t-s)Gx_s\,dw(s)\right\}dt + (1+\alpha(t))Gx_t\,dw(t)\\
&= Lx_t + (1+\alpha(t))Gx_t\,dw(t),
\end{aligned}$$

whence (43) follows.

Using integral equation (43) and equality (18) we get

$$\Gamma(t,\varphi) = |Gy_t|^2 + \int_0^t (1+\alpha(s))^2|H(t-s)|^2\Gamma(s,\varphi)\,ds. \tag{44}$$

A) Let $\alpha\in\Re_1$. Then equality (44) implies that

$$\Gamma(t,\varphi) < |Gy_t|^2 + (1-\varepsilon)\int_0^t |H(t-s)|^2\Gamma(s,\varphi)\,ds. \tag{45}$$

Applying the Laplace transform to both sides of (45), we prove an inequality for images, namely

$$\int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt < \int_0^\infty |Gy_t|^2e^{-zt}\,dt + (1-\varepsilon)\int_0^\infty |H(t)|^2e^{-zt}\,ds\int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt$$

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or

$$\int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt < \int_0^\infty |Gy_t|^2 e^{-zt}\,dt \left(1-(1-\varepsilon)\int_0^\infty |H(t)|^2 e^{-zt}\,dt\right)^{-1}. \tag{46}$$

If $B=1$, then the pole of the right hand side of inequality (46) with the largest real part coincides with λ_0 for which

$$\int_0^\infty |H(t)|^2 e^{-\lambda_0 t}\,dt = \frac{1}{1-\varepsilon} > 1. \tag{47}$$

Since the Laplace transform with a real argument ($\lambda \in \mathbf{R}$) is a decreasing continuous function and $B=1$, we get $\lambda_0 < 0$. This means that $\Gamma(t,\varphi)$ behaves at ∞ like a function whose absolute value does not exceed $Ne^{\lambda_0 t}$, $N>0$, that is,

$$\lim_{t\to\infty} \Gamma(t,\varphi) = 0. \tag{48}$$

This proves A).

B) Let $\alpha \in \Re_2$. Then equality (44) implies that

$$\Gamma(t,\varphi) > |Gy_t|^2 + (1+\varepsilon)\int_0^t |H(t-s)|^2\Gamma(s,\varphi)\,ds. \tag{49}$$

Applying the Laplace transform to both sides of (49), we prove the following inequality for the images:

$$\int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt > \int_0^\infty |Gy_t|^2 e^{-zt}\,dt + (1+\varepsilon)\int_0^\infty |H(t)|^2 e^{-zt}\,dt \int_0^\infty \Gamma(t,\varphi)e^{-zt}\,dt$$

or

$$\int_0^\infty \Gamma(t,\varphi)e^{-\lambda t}\,dt > \int_0^\infty |Gy_t|^2 e^{-zt}\,dt \left(1-(1+\varepsilon)\int_0^\infty |H(t)|^2 e^{-\lambda t}\,dt\right)^{-1}. \tag{50}$$

If $B=1$, then the pole of the right hand side of (46) with the largest real part coincides with the number $\lambda=\lambda_0$ for which

$$\int_0^\infty |H(t)|^2 e^{-\lambda t} ds = \frac{1}{1+\varepsilon} < 1. \tag{51}$$

Since the Laplace transform of a real argument ($\lambda \in R$) is a decreasing function and $B=1$, we get $\lambda_0 > 0$. This means that $\Gamma(t,\varphi)$ behaves at ∞ like a function whose absolute value is not less than $\delta e^{\lambda_0 t}$, that is,

$$\lim_{t\to\infty} \Gamma(t,\varphi) = \infty, \tag{52}$$

whence B) follows. Theorem 3.2 is proved. □

Remark 3.1. The spaces $\Re_1$ and $\Re_2$ in Theorem 4.2 can be changed for "wider spaces", namely for

$$\widetilde{\Re}_1 \equiv \left\{\phi \in C^0([0,\infty])\colon \exists\varepsilon>0, \forall t\in[0,h]\colon (1+\phi(t))^2 < 1-\varepsilon; \forall t>h\colon (1+\phi(t))^2 \le 1\right\},$$

$$\widetilde{\Re}_2 \equiv \left\{\phi \in C^0([0,\infty])\colon \exists\varepsilon>0, \forall t\in[0,h]\colon (1+\phi(t))^2 > 1+\varepsilon; \forall t>h\colon (1+\phi(t))^2 \ge 1\right\},$$

respectively.

Proof. We prove, for example, the part of Remark 3.1 related to the spaces $\Re_1$ and $\widetilde{\Re}_1$. Indeed, inequality (45) can be rewritten as follows:

$$\Gamma(t,\varphi) \le |Gy_t|^2 + (1-\varepsilon)\int_0^t |H(t-s)|^2\Gamma(s,\varphi)\,ds$$

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for $t \in [0, h]$ and

$$\Gamma(t,\varphi) \le |Gy_t|^2 + (1-\varepsilon)\int_0^h |H(t-s)|^2\Gamma(s,\varphi)\,ds + \int_h^t |H(t-s)|^2\Gamma(s,\varphi)\,ds$$

for $t > h$.

Then inequality (46) becomes of the form

$$\begin{aligned}&\int_0^\infty \Gamma(t,\varphi)e^{-\lambda t}\,dt \\ &\quad < \int_0^\infty |Gy_t|^2 e^{-zt}\,dt \left(1 - \int_0^\infty |H(t)|^2 e^{-\lambda t}\,dt + \varepsilon\int_0^h |H(t)|^2 e^{-\lambda t}\,dt\right)^{-1}.\end{aligned} \tag{53}$$

Since $B = 1$ and

$$\int_0^h |H(t)|^2 e^{-\lambda t}\,dt = K > 0,$$

all poles of the right hand side of (53) belong to the left semi-plane of the complex plane. This means that $\Gamma(t,\varphi)$ behaves at ∞ like an exponential function with a negative index, that is, a solution of the perturbed equation (42) is exponentially stable in the mean square sense. Remark 3.1 is proved. □

Bibliography

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Linear Functional-Differential Equations*, Nauka, Moscow, 1991; English transl., World Federation Publishers Company, Atlanta, GA, 1995. MR1144998 (92j:34123); MR1375462 (96m:34132)
2. E. A. Andreeva, V. B. Kolmanovskiĭ, and L. E. Shaĭkhet, *Control of Systems with Aftereffect*, Nauka, Moscow, 1992; English transl., V. B. Kolmanovskiĭ and L. E. Shaĭkhet, American Mathematical Society, Providence, RI, 1996. MR1185708 (93i:49001); MR1415834
3. R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskiĭ, *The theory of equations of neutral type*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Mathematical Analysis **19** (1982), no. 1, 55–126, 232. (Russian) MR657948 (83j:34078)
4. R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York–London, 1967. MR0147745 (26:5259)
5. Yu. V. Bereza and V. K. Yasyns'kyĭ, *The existence of solutions of stochastic differential functional equations of the neutral type with Poissonian switchings*, Visn. Kyiv. Univ., Ser. Fiz.-Mat. Nauky **5** (2002), no. 1, 19–27. (Ukrainian)
6. G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer-Verlag, New York–Heidelberg, 1974; Translated from the second German edition. MR0344810 (49:9549)
7. J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Nauka, Moscow, 1994; English transl. of the second edition, Springer-Verlag, Berlin, 2003. MR1943877 (2003j:60001)
8. V. B. Kolmanovskiĭ and V. R. Nosov, *Stability and Periodic Modes of Adaptable Systems with Aftereffect*, Nauka, Moscow, 1981. (Russian) MR641554 (83g:34001)
9. V. Yu. Slyusarchuk, *The Absolute Stability of Dynamical Systems with Aftereffect*, UDUVGP, Rivne, 2003. (Ukrainian)
10. I. Ya. Spektorskiĭ, *A generalization of the constant variation formula for a linear nonhomogeneous stochastic equation*, Problemy Upravlen. Inform. **1** (1998), no. 5, 107–112, 158. (Russian) MR1700664
11. R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. MR832183 (87e:15001)
12. D. Ya. Khusainov, *Estimates for the stability of solutions of systems of functional-differential equations of neutral type*, Ukrain. Mat. Zh. **1** (1991), no. 9, 1123–1135; English transl. in Ukrainian Math. J. **1** (1991), no. 9, 1053–1063. MR1149573 (92k:34102)
13. D. Ya. Khusainov and A. V. Shatyrko, *The method of Lyapunov functions in the studies of the stability of differential-functional systems*, Kiev University, Kiev, 1997. (Russian) MR1486825 (2000k:34124)
14. E. F. Tsar'kov, *Random Perturbations of Differential-Functional Equations*, Zinatne, Riga, 1989. (Russian) MR1036733 (90m:34164)

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15. J. Hale, *Theory of Functional Differential Equations*, Springer, Berlin, 1978. MR0508721 (58:22904)
16. I. I. Gihman and A. V. Skorohod, *Stochastic Differential Equations*, Springer, Berlin, 1972. MR0346904 (49:11625)

Department of Mathematical and Applied Statistics, Faculty for Applied Mathematics, Chernivtsi Fed'kovych National University, Kotsyubyns'kyi Street 2, Chernivtsi 58000, Ukraine
E-mail address: M_Ihor_V@rambler.ru

Department of Probability Theory and Mathematical Statistics, Riga Technical University, Riga, Latvia
E-mail address: carkovs@livas.lv

Department of Mathematical and Applied Statistics, Faculty for Applied Mathematics, Chernivtsi Fed'kovych National University, Kotsyubyns'kyi Street 2, Chernivtsi 58000, Ukraine
E-mail address: yasinsk@cv.ukrtel.net

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