

**ASYMPTOTIC BEHAVIOR OF THE SOLUTION  
OF A LINEAR STOCHASTIC DIFFERENTIAL-DIFFERENCE  
EQUATION OF NEUTRAL TYPE**

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ABSTRACT. Necessary and sufficient conditions are found for the exponential mean square stability of a stochastic differential-difference linear equation of neutral type in the scalar case.

1. INTRODUCTION

The Lyapunov stability of solutions of deterministic differential-functional equations of neutral type is studied in many papers. A special place among these papers is occupied by Hale [15], Azbelev et al. [1], Kamenskiĭ et al. [3], Khusainov and Shatyrko [13, 12], Slyusarchuk [9].

Kolmanovskii and Nosov [8], Tsar'kov [14], and Khusainov [13] studied the behavior of solutions of stochastic differential-functional equations. The existence of a strong solution for a stochastic differential-functional equation of neutral type and generalizations of the second Lyapunov method are discussed by Kolmanovskii and Shaikhmet in [2], Bereza and Yasyns'kyĭ [5], and others.

In this paper, we obtain necessary and sufficient conditions for the exponential mean square stability of a stochastic differential-difference linear equation of neutral type and study the nonstable as well as the critical case.

2. MAIN PART

Let  $(\Omega, F, P, \mathfrak{F})$  be a probability basis [7] where  $\mathfrak{F} \equiv \{F_t, t \geq 0\}$  is a filtration. Let a strong solution [2, 16]  $x(t) = x(t, \omega) \in \mathbf{R}^1$  of the stochastic differential-difference linear equation of neutral type

$$(1) \quad d\{Dx_t\} = \{Lx_t\} dt + \{Gx_t\} dw(t)$$

with the initial condition

$$(2) \quad x_0 = \varphi$$

be defined on the probability basis  $(\Omega, F, P, \mathfrak{F})$ . Here

$$x_t \equiv \{x(t+s), -h \leq s \leq 0\} \in C([-h, 0]),$$

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$\varphi \in C([-h, 0])$  is an  $F_0$ -measurable stochastic process,  $w(t) = w(t, \omega)$  is a one-dimensional Wiener process adapted to  $\mathfrak{S} = \{F_t, t \geq 0\}$ , and where  $D$ ,  $L$ , and  $G$  are difference operators defined for  $\psi \in C([-h, 0])$  by the following relations:

$$(3) \quad \begin{aligned} D\psi &\equiv \psi(0) + \sum_{k=1}^n \delta_k \psi(-\tau_k), & 0 < \tau_1 < \tau_2 < \dots < \tau_n \leq h, \\ L\psi &\equiv \alpha\psi(0) + \sum_{k=1}^m b_k \psi(-\lambda_k), & 0 < \lambda_1 < \lambda_2 < \dots < \lambda_m \leq h, \\ G\psi &\equiv f\psi(0) + \sum_{k=1}^q g_k \psi(-\theta_k), & 0 < \theta_1 < \theta_2 < \dots < \theta_q \leq h. \end{aligned}$$

For the stochastic differential-difference linear equation of neutral type (1), (2), the existence and uniqueness (up to stochastic equivalence) hold in the case of a strong solution  $x(t) \in \mathbf{R}^1$  such that  $\mathbf{E} \{x^2(t)\} < \infty$  (see [5]). Along with equation (1) consider the deterministic differential-difference equation of neutral type [1, 15]

$$(4) \quad d\{Dy_t\} = \{Ly_t\} dt$$

with the initial condition

$$(5) \quad y(t) = \varphi(t), \quad -h \leq t \leq 0.$$

First we mention some auxiliary results.

**Lemma 2.1.** *If*

$$(6) \quad \sum_{k=1}^n |\delta_k| < 1,$$

*then the solution  $y(t) \equiv 0$  of the deterministic differential-difference equation of neutral type (4), (5) is exponentially stable if and only if the roots of the characteristic quasi-polynomial*

$$(7) \quad V(z) \equiv z \left( 1 + \sum_{k=1}^n e^{-z\tau_k} \delta_k \right) - a - \sum_{l=1}^m e^{-z\lambda_l} b_l$$

*belong to the left semi-plane of the complex plane  $\mathbf{C}$ , that is,*

$$(8) \quad \exists \rho > 0, \quad \forall z \in \mathbf{C}: \quad V(z) = 0 \Rightarrow \operatorname{Re} z < -\rho.$$

Consider the Cauchy function  $X(t)$  [15] as a solution of equation (4) satisfying the initial condition

$$(9) \quad X(t) \equiv \mathbf{1}(t) = \begin{cases} 0, & -h \leq t < 0; \\ 1, & t = 0. \end{cases}$$

Then the Cauchy function  $X(t)$  can be represented in terms of the characteristic quasi-polynomial [4, 15].

**Lemma 2.2.** *The Cauchy function  $X(t)$  is of the form*

$$(10) \quad X(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \mu} e^{zt} V^{-1}(z) dz,$$

*where  $\mu > -\rho$ .*

The following assertion holds.

**Theorem 2.1.** *A solution of the stochastic differential-difference linear equation of neutral type (1), (2) satisfies the stochastic integral equation*

$$(11) \quad x(t) = y(t) + \int_0^t X(t-s)Gx_s dw(s),$$

where  $y(t)$  is a solution of (4), (2).

*Proof.* Since the neutral difference operator  $D$  is linear, we get

$$(12) \quad Dx_t = Dy_t + \int_0^t DX_{t-s}Gx_s dw(s).$$

Substitute (12) into equation (1) and take into account the linearity of the difference operator  $L$ . Then we obtain

$$\begin{aligned} dDx_t &= dDy_t + d \int_0^t DX_{t-s}Gx_s dw(s) \\ &= Ly_t dt + DX_0Gx_t dw(t) + \int_0^t d_t DX_{t-s}Gx_s dw(s) \\ &= Ly_t dt + Gx_t dw(t) + \int_0^t (LX_{t-s} dt) Gx_s dw(s) \\ &= Ly_t dt + Gx_t dw(t) + L \left\{ \int_0^t X(t-s) dt Gx_s dw(s) \right\} dt \\ &= L \left\{ y_t + \int_0^t X(t-s) dt Gx_s dw(s) \right\} dt + Gx_t dw(t) = Lx_t dt + Gx_t dw(t). \end{aligned}$$

The actions of the difference operator  $L$  and the integration can be interchanged indeed, since  $L$  is linear. Thus the stochastic process defined by (11) satisfies a stochastic differential-difference linear equation of neutral type. Theorem 2.1 is proved.  $\square$

Now we define the notion of exponential mean square stability of a solution of a stochastic differential-difference linear equation of neutral type (1), (2).

**Definition 2.1.** The trivial solution  $x(t) \equiv 0$  of the problem (1), (2) is called exponentially mean square stable if there are constants  $M > 0$  and  $c > 0$  such that

$$(13) \quad \mathbf{E} |x(t)|^2 \leq M e^{-ct} \mathbf{E} \|\varphi\|^2$$

for all  $t \geq 0$  and  $\varphi \in C([-h, 0])$ , where  $\|\varphi\| \equiv \max_{-h \leq t \leq 0} |\varphi(t)|$  and  $\mathbf{E}\{\cdot\}$  means the mathematical expectation.

We need the following notation and related results to find necessary and sufficient conditions for the exponential stability of the trivial solution  $x(t) \equiv 0$  of a stochastic differential-difference linear equation of neutral type (1), (2).

Using the integral representation (11) of the strong solution of equation (1), and the representation (10) of the Cauchy function  $X(t)$ , we write an integral Volterra equation of neutral type for the stochastic process  $\gamma(t) \equiv Gx_t$ :

$$(14) \quad \gamma(t) \equiv Gy_t + \int_0^t H(t-s)\gamma(s) dw(s),$$

where

$$(15) \quad H(t) \equiv GX_t = \frac{1}{2\pi i} \int_{\text{Re } z = \mu} e^{zt} G_1(z) V^{-1}(z) dz$$

and

$$(16) \quad G_1(z) \equiv f + \sum_{i=1}^q g_i e^{-z\theta_i}.$$

To derive the latter equation, we apply the difference operator  $G$  to both sides of equation (10) and obtain

$$\begin{aligned} GX_t &= G \left\{ \frac{1}{2\pi i} \int_{\operatorname{Re} z = \mu} e^{zt} V^{-1}(z) dz \right\} = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \mu} \left\{ f e^{zt} + \sum_{k=1}^q g_k e^{z(t-\theta_k)} \right\} V^{-1}(z) dz \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re} z = \mu} \left\{ f + \sum_{k=1}^q g_k e^{-z\theta_k} \right\} e^{zt} V^{-1}(z) dz = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \mu} G_1(z) e^{zt} V^{-1}(z) dz, \end{aligned}$$

whence (14) follows.

Introduce the following notation:

$$\begin{aligned} M(t, \varphi) &\equiv \mathbf{E} \{ |x(t)|^2 / F^0 \} = \mathbf{E} \{ |x(t)|^2 \}, \\ \Gamma(t, \varphi) &\equiv \mathbf{E} \{ |\gamma(t)|^2 / F^0 \} = \mathbf{E} \{ |\gamma(t)|^2 \}. \end{aligned}$$

The conditional mathematical expectations above are equal to the corresponding unconditional mathematical expectations, since the stochastic processes  $x(t)$  and  $\gamma(t)$  are independent of the initial  $\sigma$ -algebra  $F_0$  [7].

Further we square the integral representation (11) of a strong solution of the stochastic differential-difference linear equation of neutral type and evaluate the mathematical expectations to obtain the following equation for the second moment  $M(t, \varphi)$ :

$$(17) \quad M(t, \varphi) = |y(t)|^2 + \int_0^t |X(t-s)|^2 \Gamma(s, \varphi) ds.$$

In deriving this equality we take into account that the mathematical expectation of a Wiener-Itô integral is zero and that the mathematical expectation of the square of a Wiener-Itô integral is equal to the usual Riemann integral [7].

Analogously we get

$$(18) \quad \Gamma(t, \varphi) = |Gy_t|^2 + \int_0^t |H(t-s)|^2 \Gamma(s, \varphi) ds.$$

Next we prove the following auxiliary result.

**Theorem 2.2.** *Let conditions (6) and (8) hold. Then a solution of the integral equation (18) is such that*

$$(19) \quad \exists C_1 > 0, \quad \exists C_2 > 0, \quad \forall \varphi \in C([-h, 0]): \quad \int_0^\infty \Gamma(t, \varphi) e^{tC_2} dt < C_1 \|\varphi\|^2$$

if and only if

$$(20) \quad B \equiv \int_0^\infty |H(t)|^2 dt < 1.$$

*Proof.* Conditions (6) and (8) imply that there exists a constant  $M > 0$  such that

$$(21) \quad |H(t)| \leq M e^{-\rho t}, \quad |Gy_t| \leq M e^{-\rho t} \|\varphi\|$$

for  $\varphi \in C([-h, 0])$ . This means that one can apply the Laplace transform to (18) and obtain the following equation:

$$(22) \quad \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt = \int_0^\infty |Gy_t|^2 e^{-zt} dt + \int_0^\infty |H(t)|^2 e^{-zt} dt \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt,$$

which is equivalent to the equation

$$(23) \quad \int_0^\infty \Gamma(t, \varphi) e^{-\lambda t} dt = \int_0^\infty |Gy_t|^2 e^{-z t} dt \left( 1 - \int_0^\infty |H(t)|^2 e^{-\lambda t} dt \right)^{-1}$$

if  $\operatorname{Re} z$  is sufficiently large. Without loss of generality we assume that  $\operatorname{Im} z = 0$ . Note that the function  $R(z) = \int_0^\infty |H(t)|^2 e^{-z t} ds$ , as a function of  $z \in \mathbf{R}$ , is continuous and that  $R(z)$  is a decreasing function in the interval  $(-\rho, \infty)$  if  $\operatorname{Im} z = 0$ .

*The sufficiency.* Let relation (20) hold. Then there exists a number  $\varepsilon > 0$  such that

$$\int_0^\infty |H(t)|^2 e^{-z t} dt < 1$$

for  $z \in [-\varepsilon, \infty)$ . Then

$$(24) \quad \begin{aligned} \int_0^\infty \Gamma(t, \varphi) e^{-z t} dt &< \int_0^\infty |Gy_t|^2 e^{-z t} dt \left( 1 - \int_0^\infty |H(t)|^2 e^\varepsilon dt \right)^{-1} \\ &\leq \int_0^\infty |Gy_t|^2 e^\varepsilon dt \left( 1 - \int_0^\infty |H(t)|^2 e^\varepsilon dt \right)^{-1} \\ &\leq M^2 \|\varphi\|^2 (\rho - \varepsilon)^{-1} \left( 1 - \int_0^\infty |H(t)|^2 e^\varepsilon dt \right)^{-1} \end{aligned}$$

for  $z \in [-\varepsilon, \infty)$ . Inequality (24) implies condition (19) if we let  $C_2 = \varepsilon$  and

$$C_1 = M^2 (\rho - \varepsilon)^{-1} \left( 1 - \int_0^\infty |H(t)|^2 e^\varepsilon dt \right)^{-1}.$$

*The necessity.* Let condition (19) hold and  $\operatorname{Im} z = 0$ . We prove (20) by contradiction. Assume that (20) does not hold, that is,

$$\int_0^\infty |H(t)|^2 dt \geq 1.$$

Then  $\int_0^\infty |H(t)|^2 e^{-z t} dt > 1$  for  $z < 0$ . This means that

$$(25) \quad \int_0^\infty \Gamma(t, \varphi) e^{-z t} dt = \int_0^\infty |Gy_t|^2 e^{-z t} dt \left( 1 - \int_0^\infty |H(t)|^2 e^{-z t} dt \right)^{-1} < 0,$$

whence  $\Gamma(t, \varphi) < 0$  for  $\operatorname{Re} z < 0$ . This is a contradiction, since  $\int_0^\infty \Gamma(t, \varphi) e^{-\lambda t} dt > 0$  by definition. Theorem 2.2 is proved.  $\square$

**Lemma 2.3.** *The integral on the left hand side of condition (20) is equal to*

$$B \equiv \int_0^\infty |H(t)|^2 dt = \frac{1}{\pi} \int_0^\infty |G(is)|^2 |V(is)|^{-2} ds.$$

*Proof.* Using integral representation (15) in place of  $H(t)$  we have

$$(26) \quad H(t) = \frac{1}{2\pi i} \int_{-\infty}^\infty G_1(is) e^{z t} V^{-1}(is) ds.$$

Note that  $G_1(is) e^{z t} V^{-1}(is)$  is the image of  $H(t)$  under the Laplace transform, that is,

$$(27) \quad G_1(is) e^{z t} V^{-1}(is) = \int_0^\infty e^{-i s t} H(t) dt.$$

Then the Plancherel–Parseval equality [4] implies that

$$\begin{aligned}
\int_0^\infty |H(t)|^2 dt &= \frac{1}{2\pi} \int_{-\infty}^\infty |G(is)|^2 |V(is)|^{-2} ds \\
&= \frac{1}{2\pi} \left( \int_{-\infty}^0 |G(is)|^2 |V(is)|^{-2} ds + \int_0^\infty |G(is)|^2 |V(is)|^{-2} ds \right) \\
&= \frac{1}{2\pi} \left( \int_0^\infty |G(-is)|^2 |V(-is)|^{-2} ds + \int_0^\infty |G(is)|^2 |V(is)|^{-2} ds \right) \\
&= \frac{1}{\pi} \int_0^\infty |G(is)|^2 |V(is)|^{-2} ds.
\end{aligned}$$

Lemma 2.3 is proved.  $\square$

**Theorem 2.3.** *Let conditions (6) and (8) hold for the coefficients of the stochastic differential-difference linear equation of neutral type and for roots of its characteristic quasi-polynomial (7). Then the condition*

$$(28) \quad B = \frac{1}{\pi} \int_0^\infty |G(is)|^2 |V(is)|^{-2} ds < 1$$

*is necessary and sufficient for the exponential mean square stability of a solution of the problem (1), (2), where  $G(z) \equiv f + \sum_{r=1}^q g_r e^{-z\theta_r}$ .*

*Further, if  $B > 1$ , then in every neighborhood of zero there exists an initial function  $\varphi(t)$  such that*

$$(29) \quad \lim_{t \rightarrow \infty} M \{ |x(t)|^2 \} = \infty.$$

*Proof. The sufficiency.* Let condition (28) hold. Thus (20) holds, too. Conditions (6) and (8) imply that there exist constants  $M_2 > 0$  and  $c_2 > 0$  such that

$$(30) \quad \begin{aligned} |Gy_t|^2 e^{tc_2} &\leq M_2 \|\varphi\|^2, & |y(t)|^2 e^{tc_2} &\leq M_2 \|\varphi\|^2, \\ |X(t)|^2 e^{tc_2} &\leq M_2, & |H(t)|^2 e^{tc_2} &\leq M_2 \end{aligned}$$

for all  $t > 0$  and  $\varphi \in C([-h, 0])$  (see [15]).

Note that the constant  $c_2 > 0$  in the bounds (30) is the same as that in Theorem 2.2.

Inequalities (30) imply that

$$(31) \quad \begin{aligned} \Gamma(t, \varphi) e^{tc_2} &= |Gy_t|^2 e^{tc_2} + \int_0^t |H(t-s)|^2 e^{(t-s)c_2} e^{sc_2} \Gamma(s, \varphi) ds \\ &\leq M_2 \|\varphi\|^2 + M_2 \int_0^t e^{sc_2} \Gamma(s, \varphi) ds \leq M_2 (1 + c_1) \|\varphi\|^2 \end{aligned}$$

for all  $t > 0$  and  $\varphi \in C([-h, 0])$ , where the constant  $c_1$  is defined in the proof of Theorem 2.2.

Using inequalities (30) and (31) we obtain a bound for  $M(t, \varphi)$  satisfying equation (17); namely,

$$M(t, \varphi) \leq M_2 (1 + c_1) \|\varphi\|^2 e^{-tc_2}$$

for  $t > 0$  and  $\varphi \in C([-h, 0])$ .

This implies exponential stability (see condition (13) in Definition 2.1). The sufficiency is proved.

*The necessity.* Let conditions (6), (8) hold and let the solution  $x(t) \equiv 0$  be exponentially stable, that is,  $\mathbf{E} |x(t)|^2 \leq M e^{-ct} \mathbf{E} \|\varphi\|^2$  for  $t > 0$  and  $\varphi \in C([-h, 0])$ , where  $M > 0$

and  $c > 0$ . Then

$$\begin{aligned}\Gamma(t, \varphi) &= \mathbb{E} \{|Gx_t|^2\} \leq (q+1) \left( |f|^2 + \sum_{r=1}^q |g_r|^2 \right) \max_{-h \leq \theta \leq 0} \mathbb{E} \{|x(t+\theta)|^2\} \\ &\leq M(1+q) \left( |f|^2 + \sum_{r=1}^q |g_r|^2 \right) e^{-ct} \|\varphi\|^2.\end{aligned}$$

This means that there exists a constant  $A_1 > 0$  such that

$$\int_0^\infty \Gamma(t, \varphi) e^{tc_2} dt < c_1 \|\varphi\|.$$

Taking into account equality (26) of Lemma 2.3, we derive inequality (28) from Theorem 2.2.

Further, equality (23) means that the poles of  $1 - \int_0^\infty \Gamma(t, \varphi) e^{-\lambda t} dt$  coincide with the zeros of the function

$$1 - \int_0^\infty |H(t)|^2 e^{-\lambda t} dt,$$

and that  $\int_0^\infty |H(t)|^2 e^{-\lambda t} dt$  is a continuous decreasing function of a real argument  $\lambda$  such that

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty |H(t)|^2 e^{-\lambda t} dt = 0.$$

Note that there exists  $\lambda_0 > 0$  such that

$$1 - \int_0^\infty |H(t)|^2 e^{-\lambda_0 t} dt = 0.$$

This means that the function  $\int_0^\infty \Gamma(t, \varphi) e^{-\lambda t} dt$  has a pole at  $\lambda = \lambda_0 > 0$ . In its turn, the latter means that

$$\lim_{t \rightarrow \infty} \Gamma(t, \varphi) = \lim_{t \rightarrow \infty} e^{\lambda_0 t} = \infty,$$

whence (29) follows. Theorem 2.3 is proved.  $\square$

### 3. THE CRITICAL CASE

Consider the behavior of a solution of the stochastic differential-difference linear equation of neutral type (1), (2) for  $B = 1$ .

**Theorem 3.1.** *Assume that*

- 1) conditions (6), (8) hold;
- 2)

$$(32) \quad B = \frac{1}{\pi} \int_0^\infty |G(is)|^2 |V(is)|^{-2} ds = 1;$$

- 3)

$$f + \sum_{k=1}^q g_k \neq 0.$$

*Then there exists an initial function  $\varphi \in C[-h, 0]$  such that*

$$(33) \quad 0 < \lim_{t \rightarrow \infty} \mathbb{E} |x(t)|^2 < \infty.$$

*Proof.* Let a strong solution of equation (1) be constructed for the initial condition

$$(34) \quad \varepsilon(t) = \begin{cases} 0, & -h \leq t < 0, \\ \varepsilon, & t = 0. \end{cases}$$

Then  $y(t) = \varepsilon X(t)$  is a solution of the deterministic equation (4), while equation (12) for  $\Gamma(t, \varphi)$  becomes of the following form:

$$(35) \quad \begin{aligned} \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt &= \varepsilon \int_0^\infty |GX_t|^2 e^{-zt} dt + \int_0^\infty |H(t)|^2 e^{-zt} ds \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt \\ &= \int_0^\infty |H(t)|^2 e^{-zt} ds \left( \varepsilon + \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt \right). \end{aligned}$$

Hence

$$(36) \quad \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt = \frac{\varepsilon \int_0^\infty |H(t)|^2 e^{-zt} dt}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt} dt\right)}.$$

A relationship between the behavior of an original  $\Gamma(t, \varphi)$  as  $t \rightarrow \infty$  and that of its Laplace image as  $z \rightarrow 0$  is well known, namely

$$(37) \quad \lim_{t \rightarrow \infty} \Gamma(t, \varphi) = \varepsilon \lim_{z \rightarrow 0} \frac{z \int_0^\infty |H(t)|^2 e^{-zt} dt}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt} dt\right)};$$

see [6]. The limit as  $z \rightarrow 0$  of the right hand side of equation (31) can be evaluated by using l'Hôpital's rule:

$$(38) \quad \begin{aligned} \lim_{z \rightarrow 0} \frac{\varepsilon z \int_0^\infty |H(t)|^2 e^{-zt} ds}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt} dt\right)} &= \lim_{z \rightarrow 0} \frac{(\varepsilon z \int_0^\infty |H(t)|^2 e^{-zt} ds)'}{\left(1 - \int_0^\infty |H(t)|^2 e^{-zt} dt\right)'} \\ &= \varepsilon \lim_{z \rightarrow 0} \frac{\int_0^\infty |H(t)|^2 e^{-zt} ds - z \int_0^\infty |H(t)|^2 t e^{-zt} ds}{\int_0^\infty |H(t)|^2 t e^{-zt} dt} \\ &= \varepsilon \lim_{z \rightarrow 0} \frac{\int_0^\infty |H(t)|^2 e^{-zt} ds}{\int_0^\infty |H(t)|^2 t e^{-zt} dt} = \frac{\varepsilon}{\int_0^\infty |H(t)|^2 t dt}. \end{aligned}$$

Conditions (8) imply that the integral in the denominator of the right hand side of (38) exists and

$$(39) \quad \varepsilon = \varepsilon \int_0^\infty |H(t)|^2 dt \leq \int_0^\infty |H(t)|^2 t dt \leq K \int_0^\infty |H(t)|^2 e^{\rho t/2} dt = c < \infty,$$

where  $K = \rho/2$ . Thus

$$(40) \quad \frac{\varepsilon}{C} \leq \lim_{t \rightarrow \infty} \Gamma(t, \varphi) \leq 1.$$

Since  $\Gamma(t, \varphi) = M \{Gx_t\}^2$ ,

$$(41) \quad \lim_{t \rightarrow \infty} \Gamma(t, \varphi) = \left( f + \sum_{k=1}^q g_k \right)^2 \lim_{t \rightarrow \infty} M \{x^2(t)\}$$

if the limit  $\lim_{t \rightarrow \infty} \Gamma(t, \varphi)$  exists. Thus we derive from (40) and (41) that

$$0 < \frac{\varepsilon}{C} \left( f + \sum_{k=1}^q g_k \right)^{-2} \leq \lim_{t \rightarrow \infty} \mathbb{E} |x(t)|^2 \leq \left( f + \sum_{k=1}^q g_k \right)^{-2} < \infty.$$

Theorem 3.1 is proved.  $\square$



Along with equation (1) we consider the perturbed equation

$$(42) \quad d\{Dx_t\} = \{Lx_t\} dt + (1 + \alpha(t))\{Gx_t\} dw(t).$$

Assume that condition (32) holds. We study the asymptotic stability and instability of the problem (42), (2) (in other words, of the problem (1), (2) under permanent perturbations).

By  $\mathfrak{R}_1$ , we denote the space of continuous in  $[0, \infty)$  functions  $\phi$  such that

$$\exists \varepsilon \in (0, 1), \quad \forall t > 0: \quad (1 + \phi(t))^2 < 1 - \varepsilon.$$

Let  $\mathfrak{R}_2$  denote the space of continuous in  $[0, \infty)$  functions  $\phi$  such that

$$\exists \varepsilon > 0, \quad \forall t > 0: \quad (1 + \phi(t))^2 > 1 + \varepsilon.$$

**Theorem 3.2.** *Let conditions (6), (8) hold and  $B = 1$ . Then*

- A) *the trivial solution of equation (42) is asymptotically stable in the mean square sense if  $\alpha \in \mathfrak{R}_1$ ;*
- B) *the trivial solution of equation (42) is unstable in the mean square sense if  $\alpha \in \mathfrak{R}_2$ .*

*Proof.* We rewrite equation (42) in the form of an integral equation, namely

$$(43) \quad x(t) = y(t) + \int_0^t (1 + \alpha(s))X(t-s)Gx_s dw(s),$$

where  $y(t)$  is a solution of the problem (4), (2), while  $X(t)$  is its fundamental solution,  $t \geq 0$ .

Indeed, applying the linear operator  $D$  to (43) and then differentiating the result, we obtain

$$\begin{aligned} dDx_t &= dDy_t + dD \left\{ \int_0^t (1 + \alpha(s))X(t-s)Gx_s dw(s) \right\} \\ &= Ly_t dt + Dd \left\{ \int_0^t (1 + \alpha(s))X(t-s)Gx_s dw(s) \right\} \\ &= Ly_t dt + \int_0^t (1 + \alpha(s)) dDX(t-s)Gx_s dw(s) + (1 + \alpha(t))Gx_t dw(t) \\ &= Ly_t dt + \int_0^t (1 + \alpha(s))LX(t-s)Gx_s dw(s) dt + (1 + \alpha(t))Gx_t dw(t) \\ &= L \left\{ y_t + \int_0^t (1 + \alpha(s))X(t-s)Gx_s dw(s) \right\} dt + (1 + \alpha(t))Gx_t dw(t) \\ &= Lx_t + (1 + \alpha(t))Gx_t dw(t), \end{aligned}$$

whence (43) follows.

Using integral equation (43) and equality (18) we get

$$(44) \quad \Gamma(t, \varphi) = |Gy_t|^2 + \int_0^t (1 + \alpha(s))^2 |H(t-s)|^2 \Gamma(s, \varphi) ds.$$

A) Let  $\alpha \in \mathfrak{R}_1$ . Then equality (44) implies that

$$(45) \quad \Gamma(t, \varphi) < |Gy_t|^2 + (1 - \varepsilon) \int_0^t |H(t-s)|^2 \Gamma(s, \varphi) ds.$$

Applying the Laplace transform to both sides of (45), we prove an inequality for images, namely

$$\int_0^\infty \Gamma(t, \varphi) e^{-zt} dt < \int_0^\infty |Gy_t|^2 e^{-zt} dt + (1 - \varepsilon) \int_0^\infty |H(t)|^2 e^{-zt} ds \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt$$

or

$$(46) \quad \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt < \int_0^\infty |Gy_t|^2 e^{-zt} dt \left( 1 - (1 - \varepsilon) \int_0^\infty |H(t)|^2 e^{-zt} dt \right)^{-1}.$$

If  $B = 1$ , then the pole of the right hand side of inequality (46) with the largest real part coincides with  $\lambda_0$  for which

$$(47) \quad \int_0^\infty |H(t)|^2 e^{-\lambda_0 t} dt = \frac{1}{1 - \varepsilon} > 1.$$

Since the Laplace transform with a real argument ( $\lambda \in \mathbf{R}$ ) is a decreasing continuous function and  $B = 1$ , we get  $\lambda_0 < 0$ . This means that  $\Gamma(t, \varphi)$  behaves at  $\infty$  like a function whose absolute value does not exceed  $N e^{\lambda_0 t}$ ,  $N > 0$ , that is,

$$(48) \quad \lim_{t \rightarrow \infty} \Gamma(t, \varphi) = 0.$$

This proves A).

B) Let  $\alpha \in \mathfrak{R}_2$ . Then equality (44) implies that

$$(49) \quad \Gamma(t, \varphi) > |Gy_t|^2 + (1 + \varepsilon) \int_0^t |H(t-s)|^2 \Gamma(s, \varphi) ds.$$

Applying the Laplace transform to both sides of (49), we prove the following inequality for the images:

$$\int_0^\infty \Gamma(t, \varphi) e^{-zt} dt > \int_0^\infty |Gy_t|^2 e^{-zt} dt + (1 + \varepsilon) \int_0^\infty |H(t)|^2 e^{-zt} dt \int_0^\infty \Gamma(t, \varphi) e^{-zt} dt$$

or

$$(50) \quad \int_0^\infty \Gamma(t, \varphi) e^{-\lambda t} dt > \int_0^\infty |Gy_t|^2 e^{-\lambda t} dt \left( 1 - (1 + \varepsilon) \int_0^\infty |H(t)|^2 e^{-\lambda t} dt \right)^{-1}.$$

If  $B = 1$ , then the pole of the right hand side of (46) with the largest real part coincides with the number  $\lambda = \lambda_0$  for which

$$(51) \quad \int_0^\infty |H(t)|^2 e^{-\lambda t} ds = \frac{1}{1 + \varepsilon} < 1.$$

Since the Laplace transform of a real argument ( $\lambda \in R$ ) is a decreasing function and  $B = 1$ , we get  $\lambda_0 > 0$ . This means that  $\Gamma(t, \varphi)$  behaves at  $\infty$  like a function whose absolute value is not less than  $\delta e^{\lambda_0 t}$ , that is,

$$(52) \quad \lim_{t \rightarrow \infty} \Gamma(t, \varphi) = \infty,$$

whence B) follows. Theorem 3.2 is proved.  $\square$

*Remark 3.1.* The spaces  $\mathfrak{R}_1$  and  $\mathfrak{R}_2$  in Theorem 4.2 can be changed for “wider spaces”, namely for

$$\tilde{\mathfrak{R}}_1 \equiv \{ \phi \in C^0([0, \infty]): \exists \varepsilon > 0, \forall t \in [0, h]: (1 + \phi(t))^2 < 1 - \varepsilon; \forall t > h: (1 + \phi(t))^2 \leq 1 \},$$

$$\tilde{\mathfrak{R}}_2 \equiv \{ \phi \in C^0([0, \infty]): \exists \varepsilon > 0, \forall t \in [0, h]: (1 + \phi(t))^2 > 1 + \varepsilon; \forall t > h: (1 + \phi(t))^2 \geq 1 \},$$

respectively.

*Proof.* We prove, for example, the part of Remark 3.1 related to the spaces  $\mathfrak{R}_1$  and  $\tilde{\mathfrak{R}}_1$ . Indeed, inequality (45) can be rewritten as follows:

$$\Gamma(t, \varphi) \leq |Gy_t|^2 + (1 - \varepsilon) \int_0^t |H(t-s)|^2 \Gamma(s, \varphi) ds$$

for  $t \in [0, h]$  and

$$\Gamma(t, \varphi) \leq |Gy_t|^2 + (1 - \varepsilon) \int_0^h |H(t-s)|^2 \Gamma(s, \varphi) ds + \int_h^t |H(t-s)|^2 \Gamma(s, \varphi) ds$$

for  $t > h$ .

Then inequality (46) becomes of the form

$$(53) \quad \int_0^\infty \Gamma(t, \varphi) e^{-\lambda t} dt < \int_0^\infty |Gy_t|^2 e^{-zt} dt \left( 1 - \int_0^\infty |H(t)|^2 e^{-\lambda t} dt + \varepsilon \int_0^h |H(t)|^2 e^{-\lambda t} dt \right)^{-1}.$$

Since  $B = 1$  and

$$\int_0^h |H(t)|^2 e^{-\lambda t} dt = K > 0,$$

all poles of the right hand side of (53) belong to the left semi-plane of the complex plane. This means that  $\Gamma(t, \varphi)$  behaves at  $\infty$  like an exponential function with a negative index, that is, a solution of the perturbed equation (42) is exponentially stable in the mean square sense. Remark 3.1 is proved.  $\square$

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