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ASYMPTOTIC STABILITY OF THE MAXIMUM OF NORMAL STOCHASTIC PROCESSES

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ABSTRACT. Under quite general conditions, we prove that the maximum of a sequence of normal stochastic processes in the space $C_{[0,1]}$ is asymptotically stable almost surely.

The scheme of the maximum of independent random variables always attracts the attention of mathematicians because of various applications. Gnedenko [1] was the first to study degenerate limit laws for a sequence of extremes although the case of the limit normal distribution was known earlier.

Galambos [2] gives the main asymptotic results on the convergence to degenerate laws along with the classical theory of extremal values. He also provides several results on almost sure convergence.

Let (ξ_n) be a sequence of independent identically distributed random variables and let $z_n = \max_{1\le i\le n} \xi_i$. We say that a sequence (z_n) is relatively stable in probability if there exists a sequence of numbers (a_n) such that

$$\frac{z_n}{a_n} \xrightarrow{\mathsf{P}} 1 \tag{1}$$

as $n \to \infty$. We also say that (z_n) is stable in probability if

$$z_n - a_n \xrightarrow{\mathsf{P}} 0 \tag{2}$$

as $n \to \infty$.

Gnedenko [1] obtains the criteria for relations (1) and (2).

If the almost sure convergence is substituted for the convergence in probability in relations (1) and (2), then we say that a sequence (z_n) is relatively stable or stable almost surely. Some important results on the almost sure convergence are obtained by Barndorff-Nielsen [3].

Below is one of the classical results on almost sure convergence. Let $\gamma_1, \gamma_2, \dots$ be a sequence of independent normal random variables such that $\mathsf{E}\,\gamma_i = 0$ and $\mathsf{E}\,\gamma_i^2 = 1$. Put $z_n = \max_{1\le i\le n} \gamma_i$ and

$$b_n = \begin{cases} (2\ln n)^{1/2}, & n > 1, \\ 1, & n = 1. \end{cases} \tag{3}$$

Then

$$\lim_{n\to\infty} \frac{z_n}{b_n} = 1 \quad \text{a.s.,} \tag{4}$$

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Let B be a Banach lattice equipped with a norm $\|\cdot\|$ and modulus $|\cdot|$, let X be a normal random element defined on a probability space (Ω, A, P) and assuming values in B, and let (X_n) be independent copies of X, $Z_n := \max_{1\le k\le n} X_k$. The following is a natural generalization of equality (4) to the case of Banach lattices

$$\lim_{n\to\infty} \|Z_n/b_n - \mathfrak{S}X\| = 0 \quad \text{a.s.,} \tag{5}$$

where "a.s." abbreviates "almost surely", $\mathfrak{S}X$ is the mean square deviation of the random element X, and where the sequence (b_n) is defined by equality (3).

The relative almost sure stability (5) for the sequence (Z_n) is proved in [4] where the case of random elements (X_k) assuming values in the space $C_{[0,1]}$ is considered.

It turns out that the sequence (Z_n) is relatively stable for almost sure convergence in the main classical spaces. For example, relation (5) holds for Banach lattices that do not uniformly contain ℓ_∞^n (Plichko and Matsak [5]). The relative almost sure stability of (Z_n) is proved for the space c_0 in [5]. Relation (5) is established in [6] for an arbitrary separable σ-complete Banach lattice if a normal random element X belongs to some ideal B_u. A generalization of the result of [4] to the case of bounded normal stochastic processes is given in [7]. The question on whether or not relation (5) holds for a normal random element X in an arbitrary Banach lattice is still open.

It is known for the one-dimensional independent normal random variables that equality (5) can be extended up to almost sure stability; namely,

$$\lim_{n\to\infty} \|Z_n - b_n \cdot \mathfrak{S}X\| = 0 \quad \text{a.s.} \tag{6}$$

(see [2]).

A question arises on whether or not relative stability can be extended to equality (6) in the main Banach lattices?

The answer is positive for Banach lattices that do not uniformly contain the space l_∞^n. The case of lattices such as $C(Q)$ is more complicated. An example of a bounded normal stochastic process X is constructed in [7] showing that equality (5) holds but equality (6) does not. Therefore equality (5) holds for an arbitrary space $C(Q)$. On the other hand, there exists a compact Hausdorff space Q and a normal random element X assuming values in $C(Q)$ for which equality (6) does not hold.

The aim of this paper is to show that relation (6) holds in the space $C_{[0,1]}$ under rather wide assumptions imposed on a stochastic process.

In what follows we denote by $X = \{X(t), t \in [0,1]\}$ a normal stochastic process defined on $[0,1]$ and such that $\mathsf{E}\, X(t) = 0$. By $\mathfrak{S}X = \{\sigma(t), t \in [0,1]\}$ we denote the mean square deviation of the stochastic process X where $\sigma(t) = \left(\mathsf{E}\,|X(t)|^2\right)^{1/2}$. Also let $\|x\| = \sup_{t\in[0,1]} |x(t)|$ be the uniform norm (sup-norm).

Consider the stochastic process of extremal values

$$Z_n = \left\{ Z_n(t) = \max_{1\le k\le n} X_k(t),\ t \in [0,1] \right\},$$

where $X_n = \{X_n(t),\ t \in [0,1]\}$, $n \ge 1$, is a sequence of independent copies of X.

We assume that X is a separable stochastic process such that

$$d(t, t+h) \le \frac{C}{\left|\ln |h|\right|^a}, \qquad a > \frac{3}{2}, \tag{7}$$

where

$$d(t,s) = \left(\mathsf{E}\,|X(t) - X(s)|^2\right)^{1/2}. \tag{8}$$

It is well known (see, for example, [8]) that condition (7) implies the continuity of sample functions of the stochastic process X.

The following is the main result of the current paper.

Theorem 1. *Let $X = \{X(t),\ t \in [0,1]\}$ be a normal stochastic process satisfying condition* (7) *and let $\mathfrak{S}X$ denote the mean square deviation of X. Then the sequence (Z_n) is stable almost surely in the space $C_{[0,1]}$; that is, relation* (6) *holds.*

We need several auxiliary results to prove Theorem 1.

Put

$$\chi = \sup_{n\ge 1} \frac{|z_n - b_n|}{d_n},$$

where

$$d_n = \frac{\ln\ln n}{\sqrt{2\ln n}}$$

and z_n and b_n are defined in equalities (3) and (4), respectively.

Lemma 1. *There exists a number x_0 such that*

$$\mathsf{P}\,(\chi > x) \le C_1 \exp\left(-C_2 x\right) \tag{9}$$

for $x > x_0$.

Lemma 2. *Let $(c_k)_1^n$ and $(d_k)_1^n$ be two sequences of real numbers. Then*

$$\left| \max_{1\le k\le n} c_k - \max_{1\le k\le n} d_k \right| \le \max_{1\le k\le n} |c_k - d_k|.$$

Inequality (9) of Lemma 1 is proved in [9] (see (23) in the proof of Lemma 3 of [9]). Lemma 2 is elementary (see [4]).

Lemma 3. *Assume that $X = \{X(t),\ t \in [0,1]\}$ is a separable normal stochastic process, $d(t,s)$ is defined by equality* (8)*, and $\varphi(t)$ is an increasing function defined on $\mathbf{R}^+$ and such that*

$$d(t,s) \le \varphi(|t-s|)$$

for all $t, s \in [0,1]$. If the integral $\int^\infty \varphi(\exp(-x^2))\,dx$ converges, then there exists a random variable ζ such that

$$|X(t) - X(s)| \le \zeta \left[\varphi(|t-s|) \sqrt{\ln \frac{1}{t-s}} + \int_{\sqrt{\ln \frac{1}{t-s}}}^{\infty} \varphi\left(\exp\left(-x^2\right)\right)\,dx \right]$$

for all $t, s \in [0,1]$.

Lemma 4. *Let $X = \{X(t),\ t \in T\}$ be a normal stochastic process, where T is a parametric set. Assume that*

$$\sup_{t\in T} |X(t)| < \infty \quad a.s.$$

Then there exists $\alpha > 0$ such that

$$\mathsf{E} \exp\left(\alpha \sup_{t\in T} |X(t)| \right) < \infty.$$

Lemmas 3 and 4 follow from Corollary 6.2.5 and Theorems 1.3.2 and 1.3.3 of [10], respectively.

Proof of Theorem 1. It is clear that

$$\mathfrak{S}X = \{\sigma(t),\ t \in [0,1]\} \in C_{[0,1]}.$$

For $m \in N$, consider a partition (t_i) of the interval $[0,1]$, where $t_i = i/m$, $i = 0, 1, \dots, m$. We denote by $\omega(x, \delta)$ the modulus of continuity of the function $x(t)$ in the space $C_{[0,1]}$, namely

$$\omega(x,\delta) = \sup_{|t-s|\le\delta} |x(t) - x(s)|,$$
$$V_n = \max_{1\le i\le m} |Z_n(t_i) - b_n\sigma(t_i)|.$$

Then

$$\|Z_n(t) - b_n\sigma(t)\| \le V_n + b_n\omega(\sigma, 1/m) + \omega(Z_n, 1/m) \tag{10}$$

by the triangle inequality. We choose m depending on n as follows:

$$m = \left[\exp\left((\ln n)^\beta\right)\right], \qquad \beta = \frac{1-\varepsilon}{2}, \qquad 0 < \varepsilon < \min\left(\frac{1}{3}, \frac{2a-3}{2a-1}\right), \tag{11}$$

where ε is fixed. Now we show that all terms on the right hand side of inequality (10) approach zero as $n \to \infty$.

We start with $\omega(\sigma, 1/m)$. Since

$$\sigma(t) = \left(\mathsf{E}\,|X(t)|^2\right)^{1/2} = \sqrt{\pi/2}\,\mathsf{E}\,|X(t)|,$$

we have

$$|\sigma(t) - \sigma(s)| = \sqrt{\pi/2}\big|\mathsf{E}\,|\xi(t)| - \mathsf{E}\,|\xi(s)|\big| \le \sqrt{\pi/2}\,\mathsf{E}\,|\xi(t) - \xi(s)| \le \sqrt{\pi/2}\,d(t,s).$$

This together with relations (7) and (11) implies that

$$b_n\omega(\sigma, 1/m) \le b_n \frac{C}{|\ln m|^a} \le C\,|\ln n|^{(1-a(1-\varepsilon))/2} \to 0 \quad \text{as } n \to \infty. \tag{12}$$

Now we prove that

$$\mathsf{P}\left(\lim_{n\to\infty} V_n = 0\right) = 1. \tag{13}$$

Let $\chi, \chi_1, \chi_2, \dots$ be identically distributed random variables and let χ be defined as in Lemma 1. Then

$$\sum_{m>1} \mathsf{P}(\chi_m > C\ln m) \le C_1 \sum_{m>1} m^{-CC_2} < \infty$$

for $C > 1/C_2$ by Lemma 1. Thus (see [2])

$$\mathsf{P}\left(\max_{1\le i\le m} \chi_i > C\ln m \;\text{ i.o.}\right) = 0,$$

where "i.o." abbreviates "infinitely often". Hence

$$\sup_{m>1} \frac{\max_{1\le i\le m}\chi_i}{\ln m} = K(\omega) < \infty \quad \text{a.s.} \tag{14}$$

Put

$$\chi(t_i) = \sup_{n\ge1} \frac{|Z_n(t_i)/\sigma(t_i) - b_n|}{d_n}.$$

Without loss of generality one can assume that $\sigma(t_i) \ne 0$ (otherwise we set $\chi(t_i) = 0$). Then

$$|Z_n(t_i) - b_n\sigma(t_i)| \le d_n\sigma(t_i)\chi(t_i).$$

It is clear that the random variables $\chi(t_i)$ have the same distribution as χ, and thus one can apply bound (14):

$$\begin{aligned}V_n &\le \max_{0\le t\le 1}\sigma(t)\max_{0\le i\le m}\chi(t_i)d_n \le K(\omega)\max_{0\le t\le 1}\sigma(t)\ln m\, d_n\\ &= K(\omega)\max_{0\le t\le 1}\sigma(t)\left|\ln n\right|^{(1-\varepsilon)/2}\frac{\ln\ln n}{\sqrt{2\ln n}}\to 0\quad \text{as } n\to\infty,\end{aligned}$$

whence equality (13) follows.

To estimate from above the last term on the right hand side of inequality (10) we apply Lemma 2:

$$\begin{aligned}\omega\left(Z_n,\frac{1}{m}\right) &= \sup_{|t-s|\le\frac{1}{m}}\left|\max_{1\le k\le n}X_k(t)-\max_{1\le k\le n}X_k(s)\right|\\ &\le \sup_{|t-s|\le\frac{1}{m}}\max_{1\le k\le n}|X_k(t)-X_k(s)|\le\max_{1\le k\le n}\omega\left(X_k,\frac{1}{m}\right).\end{aligned}\tag{15}$$

By Lemma 3, the function

$$g(h)=\varphi(|h|)\sqrt{\ln\frac{1}{|h|}}+\int_{\sqrt{\ln\frac{1}{|h|}}}^{\infty}\varphi\left(\exp\left(-x^2\right)\right)\,dx$$

is the uniform sample modulus of the normal stochastic process $X(t)$. If condition (7) holds for $X(t)$, then we choose $\varphi(h)=C\left|\ln|h|\right|^{-a}$ and check that the function

$$g(h)=\left|\ln|h|\right|^{1/2-a}$$

is the uniform sample modulus for the stochastic process $X(t)$. Therefore

$$\zeta_k=\sup_{t,s\in[0,1]}\frac{|X_k(t)-X_k(s)|}{g|t-s|}<\infty\quad\text{a.s.}$$

and

$$\omega\left(X_k,\frac{1}{m}\right)\le g\left(\frac{1}{m}\right)\zeta_k=\frac{\zeta_k}{|\ln m|^{a-1/2}}.\tag{16}$$

Next we apply Lemma 4:

$$\begin{aligned}\sum_{k>1}\mathsf{P}\left(\zeta_k>C\sqrt{\ln k}\right) &\le \sum_{k>1}\mathsf{E}\exp\left(\alpha|\zeta_k|^2\right)\exp\left(-C^2\alpha\ln k\right)\\ &= \mathsf{E}\exp\left(\alpha|\zeta_1|^2\right)\sum_{k>1}k^{-C^2\alpha}<\infty\end{aligned}$$

for $C>1/\sqrt{\alpha}$.

Similarly to the proof of equality (14), this implies that

$$\sup_{n>1}\frac{\max_{1\le k\le n}\zeta_k}{\sqrt{\ln n}}=K_1(\omega)<\infty\quad\text{a.s.}\tag{17}$$

Combining relations (15)–(17) we get

$$\begin{aligned}\omega\left(Z_n,\frac{1}{m}\right) &\le g\left(\frac{1}{m}\right)\max_{1\le k\le n}\zeta_k\le K_1(\omega)\frac{1}{|\ln m|^{a-1/2}}\sqrt{\ln n}\\ &\le K_1(\omega)\frac{1}{|\ln n|^{(a-1/2)(1/2-\varepsilon/2)}}\sqrt{\ln n}\to 0\quad\text{as } n\to\infty,\end{aligned}$$

since $\left(a-\frac{1}{2}\right)\left(\frac{1}{2}-\frac{\varepsilon}{2}\right)>\frac{1}{2}$ by condition (11). $\square$

Examples. 1. Let $X(t) = W(t)$, $t \in [0,1]$, be the Brownian motion,

$$\mathsf{E}\,W(t) = 0, \qquad R(t,s) = \mathsf{E}\,W(t)W(s) = \min(t,s), \qquad \mathfrak{S}X = \big(\sqrt{t},\ t \in [0,1]\big).$$

Since $\mathsf{E}\,|W(t) - W(s)|^2 = |t-s|$, condition (7) holds and Theorem 1 implies that

$$\lim_{n\to\infty} \big\| Z_n - b_n \cdot \sqrt{t} \big\| = 0 \quad \text{a.s.}$$

2. Let $X(t) = W^o(t)$, $t \in [0,1]$, be the Brownian bridge, that is, a normal process such that $\mathsf{E}\,W^o(t) = 0$, $R(t,s) = s(1-t)$, $s \le t$, and $\mathfrak{S}X = \big(\sqrt{t(1-t)},\ t \in [0,1]\big)$. As in the preceding case, condition (7) holds, since

$$\mathsf{E}\,|W^o(t) - W^o(s)|^2 = (t-s)(1-(t-s)) \quad \text{for } s < t,$$

whence

$$\lim_{n\to\infty} \big\| Z_n - b_n \cdot \sqrt{t(1-t)} \big\| = 0 \quad \text{a.s.}$$

Remark 1. If condition (7) holds for $a > \frac{1}{2}$, then a separable normal stochastic process $X(t)$ has continuous sample functions (this follows, for example, from the Fernique bound of Lemma 3 above; also see [8, 10]). On the other hand, we do not know yet if relation (6) holds in this case.

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