

## ASYMPTOTIC RESULTS FOR THE ABSORPTION TIMES OF RANDOM WALKS WITH A BARRIER

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ABSTRACT. A sequence  $R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)} + \xi_k < n\}}$ ,  $k \in \mathbf{N}$ ,  $R_0^{(n)} := 0$ , is called a random walk with a barrier  $n \in \mathbf{N}$ , where the  $\xi_k$  are independent copies of a random variable  $\xi$  assuming positive integer values. The asymptotic behavior of the absorption times is studied in the paper for a random walk with a barrier. This behavior depends on the properties of the tail of the distribution of the random variable  $\xi$ .

### 1. INTRODUCTION AND MAIN RESULTS

Let  $\xi$  be a random variable with a proper distribution

$$p_k := \mathbf{P}\{\xi = k\}, \quad k \in \mathbf{N}.$$

For simplicity, we assume that  $p_1 > 0$  in what follows.

For every  $n \in \mathbf{N}$ , we define the random walk with the barrier  $n$  to be

$$R_0^{(n)} := 0, \quad R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)} + \xi_k < n\}}, \quad k \in \mathbf{N},$$

where  $\{\xi_k : k \in \mathbf{N}\}$  are independent copies of the random variables  $\xi$ . Let

$$T_n := \inf \left\{ k \in \mathbf{N} : R_k^{(n)} = n - 1 \right\} = \sum_{l=1}^{\infty} 1_{\{R_l^{(n)} < n-1\}} + 1$$

be the absorption time. Note that  $T_1 = 1$  almost surely, while  $T_n < \infty$  almost surely for  $n \geq 2$  (see Lemma 2.1).

Below we provide some examples of applications of random walks  $\{R_k^{(n)} : k \in \mathbf{N}\}$  with a barrier.

**Example 1.** Let  $\{Z_k : k \in \mathbf{N}\}$  be a death chain with the space of states  $\mathbf{N}$ . Let the transition probabilities be such that  $\pi_{ij} > 0$  for  $j < i$  and  $\pi_{ij} = 0$  otherwise. Define the random variables

$$X_n := \inf \{k \geq 1 : Z_k = 1 \text{ given } Z_0 = n\},$$

which are called the absorption times of the death chain. It is shown in the paper [8] that if

$$\pi_{n,n-k} = \frac{\mathbf{P}\{\xi = k\}}{\mathbf{P}\{\xi < n-1\}},$$

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then the distribution of  $X_n$  for natural numbers  $n \geq 1 + \sup\{k \in \mathbf{N}: \mathbf{P}\{\xi \geq k\} = 1\}$  coincides with the distribution of the random variable

$$M_n := \#\{i \in \mathbf{N}: R_{i-1}^{(n)} \neq R_i^{(n)}\} = \sum_{l=0}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}}.$$

By  $Y_n$ , we denote the number of deleted edges in the procedure of separating the root of a random recursive tree with  $n$  vertices (this procedure is proposed in [10]). It is shown in the paper [9] that if

$$\mathbf{P}\{\xi = k\} := \frac{1}{k(k+1)}, \quad k \in \mathbf{N},$$

then

$$Y_n \xrightarrow{d} X_n \xrightarrow{d} M_n, \quad n \in \mathbf{N}.$$

**Example 2.** Assume that the wage fund of an employer is 50,000 units of a currency per month. We also assume that the salary demands of job applicants are independent copies of a random variable  $\xi$  assuming values in multiples of 500 units. Then the random variable  $M_{50\,001}$  means the number of employees, while the random variable  $T_{50\,001}$  is the total number of seekers trying to get a job at this business.

Some further applications of random walks with a barrier are also considered in the paper [6].

In what follows, a law  $\mu_\alpha$ ,  $\alpha \in [1, 2)$ , is called stable if its characteristic function is of the following form:

$$\begin{cases} \exp\{-|t|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i \sin(\pi\alpha/2) \operatorname{sgn}(t))\}, & 1 < \alpha < 2, \\ \exp\{-|t|(\pi/2 - i \ln |t| \operatorname{sgn}(t))\}, & \alpha = 1. \end{cases}$$

Now we provide the main results of the paper concerning the asymptotic behavior of the sequence  $\{T_n: n \in \mathbf{N}\}$  as  $n \rightarrow \infty$ .

**Theorem 1.1.** *If  $m := \mathbf{E} \xi < \infty$ , then the following statements are equivalent.*

- (i) *There are sequences  $\{a(n), b(n): n \in \mathbf{N}\}$ ,  $a(n) > 0$  and  $b(n) \in \mathbf{R}$  such that  $(T_n - b(n))/a(n)$  weakly converges as  $n \rightarrow \infty$  to a nondegenerate and proper probability distribution.*
- (ii) *For some  $\alpha \in [1, 2]$  and for some slowly varying at  $\infty$  function  $L$ ,*

$$(1) \quad \sum_{k=1}^n k^2 p_k \sim n^{2-\alpha} L(n), \quad n \rightarrow \infty.$$

*If  $\sigma^2 := \operatorname{Var} \xi < \infty$ , then one can choose  $b(n) := n/m$  and  $a(n) := (m^{-3}\sigma^2 n)^{1/2}$ ; the limit distribution in this case is normal with zero mean and variance 1.*

*If  $\sigma^2 = \infty$  and (1) holds for  $\alpha = 2$ , then one can choose  $b(n) := n/m$  and  $a(n) := c(n)m^{-3/2}$ , where the nonnegative sequence  $c(n)$  is such that  $\lim_{n \rightarrow \infty} nL(c(n))/c^2(n) = 1$ . The limit distribution in this case is normal with zero mean and variance 1.*

*If (1) holds for  $\alpha \in [1, 2)$ , then one can choose  $b(n) := n/m$  and  $a(n) := c(n)/m^{(\alpha+1)/\alpha}$ , where  $c(n)$  is a nonnegative sequence such that*

$$\lim_{n \rightarrow \infty} nL(c(n))/c^\alpha(n) = \frac{2-\alpha}{\alpha}.$$

*The limit distribution in this case is  $\mu_\alpha$ .*

**Theorem 1.2.** Let, for  $\alpha \in (0, 1)$ ,

$$(2) \quad \sum_{k=n}^{\infty} p_k \sim n^{-\alpha} L(n), \quad n \rightarrow \infty,$$

where  $L$  is a slowly varying at  $\infty$  function. Then

$$\frac{T_n}{a(n)} \xrightarrow{d} \int_0^{\infty} e^{-U(t)} dt, \quad n \rightarrow \infty,$$

where  $a(n) := n^\alpha L^{-1}(n)$  and where  $\{U(t) : t \geq 0\}$  is the subordinator with zero shift and Lévy measure

$$\nu(dt) = \frac{e^{-t/\alpha}}{(1 - e^{-t/\alpha})^{\alpha+1}} dt, \quad t > 0.$$

**Theorem 1.3.** Let  $E\xi = \infty$  and

$$(3) \quad \sum_{k=n}^{\infty} p_k \sim L(n)/n, \quad n \rightarrow \infty,$$

for some slowly varying at  $\infty$  function  $L$ . Let  $c(t)$  be an arbitrary positive function such that  $\lim_{t \rightarrow \infty} tL(c(t))/c(t) = 1$ . Put  $\psi(t) := t \int_0^{c(t)} \mathbb{P}\{\xi > y\} dy$ . Let  $b(t)$  be another arbitrary positive function such that

$$b(\psi(t)) \sim \psi(b(t)) \sim t, \quad t \rightarrow \infty.$$

Put  $a(t) := t^{-1}b(t)c(b(t))$ . Then  $(T_n - b(n))/a(n)$  weakly converges to the stable law  $\mu_1$ .

It is shown in the paper [8] that Theorems 1.1, 1.2, and 1.3 hold if  $T_n$  is changed for  $M_n$ . The results of our paper do not follow from results of [8]; however, our proofs are similar to those in [8].

## 2. AUXILIARY RESULTS

For all  $m \in \mathbf{N}$  and all  $i \in \mathbf{N}$ , let  $\widehat{R}_0^{(m)}(i) := 0$ ,

$$\widehat{R}_k^{(m)}(i) := \widehat{R}_{k-1}^{(m)}(i) + \xi_{i+k} 1_{\{\widehat{R}_{k-1}^{(m)}(i) + \xi_{i+k} < m\}}, \quad k \in \mathbf{N},$$

and

$$\widehat{T}_m(i) := \sum_{l=1}^{\infty} 1_{\{\widehat{R}_l^{(m)}(i) < m-1\}} + 1.$$

Note that the distributions of the random variables  $\widehat{T}_m(i)$  and  $T_m$  are identical. Put  $\widehat{T} := \widehat{T}(1)$ .

**Lemma 2.1.** The random variable  $T_n$  is finite almost surely for all finite  $n \in \mathbf{N}$ .

*Proof.* We use the induction in  $n$ . It is clear that  $T_1 = 1$  almost surely. Let  $T_k < \infty$  almost surely for  $k = 1, 2, \dots, n-1$ . We show that  $T_n < \infty$  almost surely, too.

The sequence  $\{\widehat{R}_k^{(n)}(1) \in \mathbf{N}\}$  as well as the sequence  $\widehat{T}_n$  does not depend on  $\xi_1$ . Moreover,

$$T_n = (1 + \widehat{T}_{n-\xi_1}(1)) 1_{\{\xi_1 \leq n-2\}} + 1_{\{\xi_1 = n-1\}} + (1 + \widehat{T}_n(1)) 1_{\{\xi_1 \geq n\}}.$$

By the induction assumption,  $\widehat{T}_k < \infty$ ,  $k = 1, 2, \dots, n-1$ . Thus

$$\widehat{T}_{n-\xi_1} 1_{\{\xi_1 \leq n-2\}} = \sum_{k=1}^{n-2} \widehat{T}_{n-k} 1_{\{\xi_1 = k\}} < \infty.$$

Therefore

$$\mathbb{P}\{T_n = \infty\} = \mathbb{P}\left\{(1 + \widehat{T}_n)1_{\{\xi_1 \geq n\}} = \infty\right\} = \mathbb{P}\{T_n = \infty\} \mathbb{P}\{\xi_1 \geq n\},$$

whence

$$(4) \quad \mathbb{P}\{T_n = \infty\} = 0,$$

since  $\mathbb{P}\{\xi_1 \geq n\} < 1$ . The lemma is proved.  $\square$

Put  $S_0 := 0$  and  $S_n := \xi_1 + \dots + \xi_n$  for  $n \in \mathbf{N}$ . Let

$$N_n := \inf\{k \geq 1: S_k \geq n\}.$$

We show that the sequences  $T_n$ ,  $n \in \mathbf{N}$ , and  $\mathbb{E}T_n$ ,  $n \in \mathbf{N}$ , satisfy certain recurrence relations.

**Proposition 2.1.** *For all  $n \in \mathbf{N}$ ,*

$$(5) \quad T_n = N_n + \widehat{T}_{Y_n}(N_n) - 2 \cdot 1_{\{Y_n=1\}}$$

with probability one, where  $Y_n := n - S_{N_n-1}$ .

*Proof.* We have

$$\begin{aligned} T_n &= T_n (1_{\{Y_n=1\}} + 1_{\{Y_n \neq 1\}}) = (N_n - 1)1_{\{Y_n=1\}} + T_n 1_{\{Y_n \neq 1\}} \\ &= (N_n - 1)1_{\{Y_n=1\}} + \sum_{k=1}^{N_n-1} 1_{\{R_k^{(n)} < n-1\}} 1_{\{Y_n \neq 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)} < n-1\}} 1_{\{Y_n \neq 1\}} \\ &\quad + 2 \cdot 1_{\{Y_n \neq 1\}} \\ &= N_n - 1 + 2 \cdot 1_{\{Y_n \neq 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)} < n-1\}} 1_{\{Y_n \neq 1\}}. \end{aligned}$$

If the random event  $\{Y_n = 1, k \geq N_n + 1\}$  occurs, then  $R_k^{(n)} = n - 1$ . Thus

$$\sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)} < n-1\}} 1_{\{Y_n \neq 1\}} = \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)} < n-1\}}.$$

Therefore,

$$\begin{aligned} T_n &= N_n - 1 + 2 \cdot 1_{\{Y_n \neq 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)} < n-1\}} \\ &= N_n - 1 + 2 \cdot 1_{\{Y_n \neq 1\}} + \sum_{k=1}^{\infty} 1_{\{\widehat{R}_k^{(n-S_{N_n-1})}(N_n) < n-S_{N_n-1}-1\}} \\ &= N_n + \widehat{T}_{n-S_{N_n-1}}(N_n) - 2 \cdot 1_{\{Y_n=1\}}, \end{aligned}$$

and this completes the proof of Proposition 2.1.  $\square$

**Proposition 2.2.** *For all  $n \in \mathbf{N}$  and  $k \in \mathbf{N}$ ,*

$$(6) \quad \mathbb{E}T_n^k = r_n \sum_{l=1}^{n-1} \mathbb{E}T_{n-l}^k p_l + k \cdot r_n \mathbb{E}T_n^{k-1} + D_k(\mathbb{E}T_n, \mathbb{E}T_n^2, \dots, \mathbb{E}T_n^{k-2}),$$

where

$$r_n := \frac{1}{p_1 + p_2 + \dots + p_{n-1}}, \quad D_k(x_1, x_2, \dots, x_n) := v^{(0)} + \sum_{k=1}^n v^{(k)} x_k,$$

and  $\{v^{(k)}, k = 0, 1, 2, \dots, n\}$  are some real numbers.

In particular, if  $k = 1$ , then

$$(7) \quad \mathbf{E} T_n = r_n \mathbf{P}\{\xi_1 \neq n-1\} + r_n \sum_{k=1}^{n-1} \mathbf{E} T_{n-k} p_k.$$

*Proof.* We have

$$\begin{aligned} \mathbf{P}\{T_n = i\} &= \sum_{l=1}^{\infty} \mathbf{P}\{T_n = i, \xi_1 = l\} \\ &= \sum_{l=1}^{n-2} \mathbf{P}\{T_n = i, \xi_1 = l\} + \mathbf{P}\{T_n = i, \xi_1 = n-1\} + \mathbf{P}\{T_n = i, \xi_1 \geq n\} \\ &= \sum_{l=1}^{n-1} \mathbf{P}\{T_{n-l} = i-1\} \mathbf{P}\{\xi_1 = l\} - 1_{\{i=2\}} \mathbf{P}\{\xi_1 = n-1\} \\ &\quad + 1_{\{i=1\}} \mathbf{P}\{\xi_1 = n-1\} + \mathbf{P}\{T_n = i-1\} \mathbf{P}\{\xi_1 \geq n\}. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{E} T_n^k &= \sum_{i=0}^{\infty} i^k \mathbf{P}\{T_n = i\} \\ &= \sum_{i=0}^{\infty} i^k \left[ \sum_{l=1}^{n-1} \mathbf{P}\{T_{n-l} = i-1\} \mathbf{P}\{\xi_1 = l\} - 1_{\{i=2\}} \mathbf{P}\{\xi_1 = n-1\} \right. \\ &\quad \left. + 1_{\{i=1\}} \mathbf{P}\{\xi_1 = n-1\} + \mathbf{P}\{T_n = i-1\} \mathbf{P}\{\xi_1 \geq n\} \right]. \end{aligned}$$

Further

$$\begin{aligned} \sum_{i=0}^{\infty} (i-1+1)^k \sum_{l=1}^{n-1} \mathbf{P}\{T_{n-l} = i-1\} \mathbf{P}\{\xi_1 = l\} &= \sum_{l=1}^{n-1} \mathbf{E}(T_{n-l})^k p_l + \sum_{j=1}^{n-1} \binom{k}{j} \mathbf{E} T_{n-l}^j + 1, \\ \sum_{i=0}^{\infty} i^k \mathbf{P}\{T_n = k-1\} \mathbf{P}\{\xi_1 \geq n\} &= \left( \mathbf{E}(T_n)^k + \sum_{j=1}^{n-1} \binom{k}{j} \mathbf{E} T_n^j + 1 \right) \sum_{l=n}^{\infty} p_l, \end{aligned}$$

whence

$$\mathbf{E} T_n^k = r_n \sum_{l=1}^{n-1} \mathbf{E} T_{n-l}^k p_l + k r_n \left[ \sum_{l=1}^{n-1} \mathbf{E} T_{n-l}^{k-1} p_l + \mathbf{E} T_n^{k-1} \sum_{l=n}^{\infty} p_l \right] + D_{k_1} (\mathbf{E} T_n, \mathbf{E} T_n^2, \dots, \mathbf{E} T_n^{k-2}).$$

Using the latter relation for the moment  $\mathbf{E} T_n^{k-1}$ , we complete the proof of (6).  $\square$

**Lemma 2.2.** *Let condition (2) hold for  $\alpha \in (0, 1)$ . Then*

$$\lim_{n \rightarrow \infty} \frac{L(n)}{n^\alpha} \mathbf{E} T_n = \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{\Gamma(1-\alpha)\Gamma(1+\alpha) - 1}.$$

*Proof.* It is known that

$$\lim_{n \rightarrow \infty} \frac{L(n)}{n^\alpha} \mathbf{E} N_n = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}$$

(see, for example, [8]). Put

$$b := \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{\Gamma(1-\alpha)\Gamma(1+\alpha) - 1}.$$

We show that

$$(8) \quad \overline{\lim}_{n \rightarrow \infty} \frac{\mathbf{E} T_n}{\mathbf{E} N_n} \leq b.$$

Assume the converse. Then, for an arbitrary  $\varepsilon > 0$ , the inequality  $\mathbf{E} T_n > (b + \varepsilon) \mathbf{E} N_n$  holds for infinitely many numbers  $n$ . Thus one can choose a positive number  $\varepsilon$  such that  $\mathbf{E} T_n > (b + \varepsilon) \mathbf{E} N_n + c$  infinitely often for any fixed  $c > 0$ . Let

$$n_c := \inf\{n \geq 1: \mathbf{E} T_n > (b + \varepsilon) \mathbf{E} N_n + c\}.$$

Note that  $\lim_{c \rightarrow \infty} n_c = \infty$ . Then

$$\mathbf{E} T_n \leq (b + \varepsilon) \mathbf{E} N_n + c, \quad n \in \{1, 2, \dots, n_c - 1\}.$$

Therefore

$$\begin{aligned} (b + \varepsilon) \mathbf{E} N_{n_c} + c &< \mathbf{E} T_{n_c} \stackrel{(7)}{=} r_{n_c} \mathbf{P}\{\xi_1 \neq n_c - 1\} + r_{n_c} \sum_{k=1}^{n_c-1} \mathbf{E} T_{n_c-k} p_k \\ &\leq r_{n_c} \mathbf{P}\{\xi_1 \neq n_c - 1\} + c + r_{n_c} (b + \varepsilon) \sum_{k=1}^{n_c-1} \mathbf{E} N_{n_c-k} p_k. \end{aligned}$$

Lemma 3.4 of [8] implies that  $\mathbf{E} N_n$  is such that

$$\mathbf{E} N_n = 1 + \sum_{k=1}^{n-1} \mathbf{E} N_{n-k} p_k.$$

Hence

$$(b + \varepsilon) \mathbf{E} N_{n_c} + c < r_{n_c} \mathbf{P}\{\xi_1 \neq n_c - 1\} + c + r_{n_c} (b + \varepsilon) (\mathbf{E} N_{n_c} - 1)$$

or, equivalently,

$$0 < r_{n_c} \mathbf{P}\{\xi_1 \neq n_c - 1\} - r_{n_c} (b + \varepsilon) + \mathbf{E} N_{n_c} (r_{n_c} - 1) (b + \varepsilon).$$

Since  $r_n - 1 \sim n^{-\alpha} L(n)$ ,  $n \rightarrow \infty$ , we pass to the limit in the latter inequality as  $c \rightarrow \infty$  and obtain

$$\frac{\varepsilon}{b} < b - 1 + 1 - b = 0.$$

This contradiction proves (8). Similar reasoning shows that the converse inequality holds for the lower limit as well. This proves the lemma.  $\square$

**Theorem 2.3.** *If*

$$(9) \quad \sum_{l=1}^n \sum_{k=l}^{\infty} p_k \sim L(n), \quad n \rightarrow \infty,$$

for some slowly varying at  $\infty$  function  $L$ , then

$$(10) \quad \frac{T_n}{\mathbf{E} T_n} \xrightarrow{\mathbf{P}} 1.$$

Moreover  $\mathbf{E} T_n \sim n/L(n)$  as  $n \rightarrow \infty$ .

*Proof.* To prove the theorem, it is sufficient to show that

$$(11) \quad \lim_{n \rightarrow \infty} \frac{\mathbf{E} T_n^k}{\mathbf{E} N_n^k} = 1$$

for all  $k \in \mathbf{N}$ , since  $\mathbf{E} N_n^k \sim n^k/L^k(n)$  according to Remark 3.5 in [8].

We prove relation (11) by induction in  $k$ . Assume that it holds for  $k \in \{1, 2, \dots, m-1\}$ , but, at the same time,

$$\overline{\lim}_{n \rightarrow \infty} \frac{\mathbf{E} T_n^k}{\mathbf{E} N_n^k} > 1.$$

Then, for an arbitrary  $\varepsilon > 0$ , the inequality  $\mathbf{E} T_n^k > (\varepsilon + 1) \mathbf{E} N_n^k$  holds for infinitely many numbers  $n$ . Then one can choose  $\varepsilon$  such that  $\mathbf{E} T_n^k > (\varepsilon + 1) \mathbf{E} N_n^k + c$  infinitely often for any fixed  $c > 0$ . Put

$$n_c := \inf \{n \geq 1: \mathbf{E} T_n^k > (\varepsilon + 1) \mathbf{E} N_n^k + c\}.$$

Note that  $\lim_{c \rightarrow \infty} n_c = \infty$ . Then

$$(12) \quad \mathbf{E} T_n^k \leq (1 + \varepsilon) \mathbf{E} N_n^k + c, \quad n \in \{1, 2, \dots, n_c - 1\}.$$

According to Lemma 3.4 of [8], the absorption times of the random variable  $N_n$  satisfy the relation

$$\mathbf{E} N_n^k = D_k(\mathbf{E} N_n, \dots, \mathbf{E} N_n^{k-2}) + k \mathbf{E} N_n^{k-1} + \sum_{i=1}^{n-1} \mathbf{E} N_{n-i}^k p_i.$$

Thus

$$\begin{aligned} (1 + \varepsilon) \mathbf{E} N_{n_c}^k + c &< \mathbf{E} T_{n_c}^k \stackrel{(6)}{=} r_n \sum_{l=1}^{n_c-1} \mathbf{E} T_{n_c-l}^k p_l + k \cdot r_n \mathbf{E} T_{n_c}^{k-1} + D_k(\mathbf{E} T_{n_c}, \dots, \mathbf{E} T_{n_c}^{k-2}) \\ &\stackrel{(12)}{\leq} r_n (1 + \varepsilon) \sum_{l=1}^{n_c-1} \mathbf{E} N_{n_c-l}^k p_l + k \cdot r_n \mathbf{E} T_{n_c}^{k-1} + D_k(\mathbf{E} T_{n_c}, \dots, \mathbf{E} T_{n_c}^{k-2}) + c \\ &= (1 + \varepsilon)(r_{n_c} - 1) (\mathbf{E} N_{n_c}^k - D_k(\mathbf{E} N_{n_c}) - k \mathbf{E} N_{n_c}^{k-1}) + (1 + \varepsilon) \mathbf{E} N_{n_c}^k \\ &\quad - (1 + \varepsilon)(D_k(\mathbf{E} N_{n_c}) + k \mathbf{E} N_{n_c}^{k-1}) + k \cdot r_n \mathbf{E} T_{n_c}^{k-1} + D_k(\mathbf{E} T_{n_c}) + c, \end{aligned}$$

where  $D_k(\mathbf{E} X) := D_k(\mathbf{E} X, \mathbf{E} X^2, \dots, \mathbf{E} X^{k-2})$ . Therefore

$$\begin{aligned} 0 &< (1 + \varepsilon)(r_{n_c} - 1) (\mathbf{E} N_{n_c}^k - D_k(\mathbf{E} N_{n_c}) - k \mathbf{E} N_{n_c}^{k-1}) - (1 + \varepsilon) (D_k(\mathbf{E} N_{n_c}) + k \mathbf{E} N_{n_c}^{k-1}) \\ &\quad + k r_n \mathbf{E} T_{n_c}^{k-1} + D_k(\mathbf{E} T_{n_c}). \end{aligned}$$

We divide the latter inequality by  $\mathbf{E} N_{n_c}^{k-1}$  and pass to the limit as  $c \rightarrow \infty$ . In view of Remark 3.5 of [8], we have  $\mathbf{E} N_n^{k-1} \sim n^{k-1}/L^{k-1}(n)$ . Condition (9) and Theorem 1.7.2 of [2] imply that  $\lim_{n \rightarrow \infty} (r_n - 1)n/L(n) = 0$ . By the assumption of the induction,

$$\lim_{c \rightarrow \infty} \frac{D_k(\mathbf{E} T_{n_c}, \dots, \mathbf{E} T_{n_c}^{k-2})}{\mathbf{E} N_{n_c}^{k-1}} = 0 \quad \text{and} \quad \lim_{c \rightarrow \infty} \frac{\mathbf{E} T_{n_c}^{k-1}}{\mathbf{E} N_{n_c}^{k-1}} = 1,$$

whence

$$0 < k - (1 + \varepsilon)k = -\varepsilon k.$$

This contradiction proves that

$$\overline{\lim}_{n \rightarrow \infty} \frac{\mathbf{E} T_n^k}{\mathbf{E} N_n^k} \leq 1.$$

The inequality for the lower limit is proved similarly. Thus (11) holds for all  $k \in \mathbf{N}$ . The theorem is proved.  $\square$

*Remark 2.1.* One can see from the proof of Theorem 2.3 that

$$(13) \quad \frac{T_n}{N_n} \xrightarrow{\mathbf{P}} 1, \quad n \rightarrow \infty,$$

provided condition (9) holds.

## 3. PROOF OF THEOREM 1.1

It is known that

$$Y_n \xrightarrow{d} Y \quad \text{as } n \rightarrow \infty$$

if  $m := E\xi < \infty$  (see, for example, [8]), where the distribution of the random variable  $Y$  is  $P\{Y = k\} = m^{-1}P\{\xi \geq k\}$ . Thus we obtain from (5) and (15) that

$$T_n - N_n \xrightarrow{d} T'_Y - 2 \cdot 1_{\{Y=1\}}, \quad n \rightarrow \infty.$$

Considering an arbitrary sequence  $d(n)$  such that  $\lim_{n \rightarrow \infty} d(n) = \infty$ , we get

$$\frac{T_n - N_n}{d(n)} \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

Assume that the distribution of  $\xi$  does not belong to the domain of attraction of any stable law with the index  $\alpha \in [1, 2]$ . Then there are no subsequences  $x(n) \in \mathbf{R}$  and  $y(n) > 0$  such that  $(S_n - x(n))/y(n)$  converges weakly to a nondegenerate proper probability law. Since

$$P\{N_n > m\} = P\{S_m \leq n - 1\},$$

the same property holds for  $N_n$ , too.

On the other hand, assume that (1) holds and let  $\alpha \in [1, 2)$ . Then

$$\sum_{k=n}^{\infty} p_k \sim \frac{\alpha}{2-\alpha} n^{-\alpha} L(n).$$

According to a theorem in the paper [4],

$$\frac{N_n - b(n)}{a(n)} \Rightarrow \mu_\alpha, \quad n \rightarrow \infty,$$

where  $a(n)$  and  $b(n)$  are as defined in [4].

If  $\alpha = 2$ , then there exists a sequence of nonrandom numbers  $a_n$  such that

$$na_n^{-2}L(a_n) \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

and

$$\lim_{n \rightarrow \infty} P \left\{ \frac{S_n - nm}{a_n} \leq x \right\} = \mu_2(-\infty, x],$$

where  $\mu_2$  is the standard normal distribution (see, for example, [12]). Then we derive from Theorem 2 of [5] that

$$\frac{N_n - b(n)}{a(n)} \Rightarrow \mu_2.$$

## 4. PROOF OF THEOREM 1.2

Repeating the reasoning of the proof of Theorem 1.4 in [8], one can show that the sequence  $T_n/a(n)$  converges in distribution to some random variable  $T$ . Put

$$Y_n := n - S_{N_n-1}.$$

Then

$$(14) \quad \frac{T_n}{a(n)} = \frac{\widehat{T}_{Y_n}(N_n)}{a(Y_n)} \frac{a(Y_n)}{a(n)} + \frac{N_n - 2 \cdot 1_{\{Y_n=1\}}}{a(n)}.$$



Let  $T'$  be the absorption time for the sequence  $\{(R_l^{(\cdot)})' : l \in \mathbf{N}_0\}$ , which is an independent copy of  $\{R_l^{(\cdot)}, l \in \mathbf{N}_0\}$  and which is independent of  $(N_n, Y_n)$ . For every fixed number  $m \in \mathbf{N}_0$ ,

$$\begin{aligned} \mathbb{P}\{\widehat{T}_{Y_n}(N_n) = m\} &= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\{\widehat{T}_{n-j}(i) = m, N_n = i, S_{N_n-1} = j\} \\ &= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\left\{\sum_{l=0}^{\infty} \mathbb{1}_{\{\widehat{R}_l^{(n-j)}(i) < n-j-1\}} = m-1, N_n = i, S_{N_n-1} = j\right\} \\ &= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\left\{\sum_{l=0}^{\infty} \mathbb{1}_{\{\widehat{R}_l^{(n-j)}(i) < n-j-1\}} = m-1\right\} \mathbb{P}\{N_n = i, S_{N_n-1} = j\} \\ &= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\left\{\sum_{l=0}^{\infty} \mathbb{1}_{\{(R_l^{(n-j)})' < n-j-1\}} = m-1\right\} \mathbb{P}\{N_n = i, S_{N_n-1} = j\} \\ &= \sum_{i=1}^n \sum_{j=0}^{n-1} \mathbb{P}\{T'_{n-j} = m\} \mathbb{P}\{N_n = i, S_{N_n-1} = j\} = \mathbb{P}\{T'_{Y_n} = m\}. \end{aligned}$$

This proves the identity of the distributions, namely

$$(15) \quad \widehat{T}_{Y_n}(N_n) \stackrel{d}{=} T'_{Y_n},$$

where the sequence  $\{T'_n : n \in \mathbf{N}\}$  does not depend on  $(N_n, n - S_{N_n-1})$  and has the same distribution as

$$\{T_n : n \in \mathbf{N}\}.$$

Since  $Y_n \xrightarrow{\mathbb{P}} \infty$ , we have

$$(16) \quad \frac{T'_{Y_n}}{a(Y_n)} \xrightarrow{\mathbb{P}} T',$$

where  $T'$  has the same distribution as  $T$ . Proposition 3.9 of [8] implies that

$$(17) \quad \left(\frac{a(Y_n)}{a(n)}, \frac{N_n}{a(n)}\right) \xrightarrow{d} \left(e^{-U(V)}, \int_0^V e^{-U(t)} dt\right), \quad n \rightarrow \infty,$$

where  $V$  is a random variable with the density  $e^{-x} \mathbb{1}_{\{x>0\}}$  that does not depend on the subordinator  $U(t)$ . Put  $X := \exp\{-U(V)\}$  and  $Y := \int_0^V e^{-U(t)} dt$ . Since the left hand side of (14) weakly converges, the random variables

$$\xi_n := \left(\frac{T'_{Y_n}}{a(Y_n)}, \frac{a(Y_n)}{a(n)}, \frac{N_n - 2 \cdot \mathbb{1}_{\{Y_n=1\}}}{a(n)}\right)$$

weakly converge, too. Taking into account (14) and (17) we prove that

$$\xi_n \xrightarrow{d} (T', X, Y).$$

Passing to the limit as  $n \rightarrow \infty$  we prove that

$$(18) \quad T \xrightarrow{d} XT' + Y,$$

where  $T'$  does not depend on the pair  $(X, Y)$ . To prove this relation, it suffices to show that

$$\lim_{n \rightarrow \infty} \left| \mathbb{E} \exp \left\{ i \left( t \frac{T'_{Y_n}}{a(Y_n)} + v \frac{a(Y_n)}{a(n)} + u \frac{N_n - 2 \cdot \mathbb{1}_{\{Y_n=1\}}}{a(n)} \right) \right\} - \mathbb{E} e^{itT'} \mathbb{E} e^{i(vX+uY)} \right| = 0$$

for all real  $t, u$ , and  $v$ .

In view of (16) and (17) and since  $Y_n \rightarrow \infty$  as  $n \rightarrow \infty$ , we obtain that, for all  $\varepsilon > 0$ , there are  $N_i = N_i(\varepsilon)$ ,  $i = 1, 2, 3$ , such that

$$(19) \quad \left| \mathbb{E} \exp \left\{ it \frac{T'_{Y_n}}{a(Y_n)} \right\} - \mathbb{E} e^{itT'} \right| < \varepsilon, \quad n > N_1,$$

$$(20) \quad \left| \mathbb{E} \exp \left\{ i \left( v \frac{a_{Y_n}}{a(n)} + u \frac{N_n - 2 \cdot 1_{\{Y_n=1\}}}{a(n)} \right) \right\} - \mathbb{E} e^{i(vX+uY)} \right| < \varepsilon, \quad n > N_2,$$

$$\mathbb{P}\{Y_n \leq \max\{N_1, N_2\}\} < \varepsilon, \quad n > N_3.$$

Put  $N := \max\{N_1, N_2\}$ . Then

$$\begin{aligned} & \left| \mathbb{E} \exp \left\{ i \left( t \frac{T'_{Y_n}}{a(Y_n)} + v \frac{a_{Y_n}}{a(n)} + u \frac{N_n - 2 \cdot 1_{\{Y_n=1\}}}{a(n)} \right) \right\} - \mathbb{E} e^{itT'} \mathbb{E} e^{i(vX+uY)} \right| \\ &= \left| \sum_{s=0}^{n-1} \sum_{l=1}^{s+1} \left( \mathbb{E} \exp \left\{ i \left( t \frac{T'_{n-s}}{a(n-s)} + v \frac{a_{n-s}}{a(n)} + u \frac{l - 2 \cdot 1_{\{n-s=1\}}}{a(n)} \right) \right\} \right. \right. \\ & \quad \left. \left. - \mathbb{E} e^{itT'} \mathbb{E} e^{i(vX+uY)} \right) \mathbb{P}\{S_{l-1} = s, N_n = l\} \right| \\ &\leq \left| \sum_{s=0}^{n-N} \sum_{l=1}^{s+1} \dots \right| + \left| \sum_{s=n-N+1}^{n-1} \sum_{l=1}^{s+1} \dots \right|. \end{aligned}$$

For all  $n > \max(N, N_3)$ , the second term is estimated as follows:

$$\begin{aligned} & \left| \sum_{s=n-N+1}^{n-1} \sum_{l=1}^{s+1} \left( \mathbb{E} \exp \left\{ i \left( t \frac{T'_{n-s}}{a(n-s)} + v \frac{a_{n-s}}{a(n)} + u \frac{l - 2 \cdot 1_{\{n-s=1\}}}{a(n)} \right) \right\} \right. \right. \\ & \quad \left. \left. - \mathbb{E} e^{itT'} \mathbb{E} e^{i(vX+uY)} \right) \mathbb{P}\{S_{l-1} = s, N_n = l\} \right| \\ &\leq 2 |\mathbb{P}\{Y_n \leq N\}| < 2\varepsilon. \end{aligned}$$

For the first term,

$$\begin{aligned} & \left| \sum_{s=0}^{n-N} \sum_{l=1}^{s+1} \left( \mathbb{E} \exp \left\{ i \left( t \frac{T'_{n-s}}{a(n-s)} + v \frac{a_{n-s}}{a(n)} + u \frac{l - 2 \cdot 1_{\{n-s=1\}}}{a(n)} \right) \right\} \right. \right. \\ & \quad \left. \left. - \mathbb{E} e^{itT'} \mathbb{E} e^{i(vX+uY)} \right) \mathbb{P}\{S_{l-1} = s, N_n = l\} \right| \\ &\leq \sum_{s=0}^{n-N} \sum_{l=1}^{s+1} \left| \left( \mathbb{E} \exp \left\{ i \left( t \frac{T'_{n-s}}{a(n-s)} \right) \right\} - \mathbb{E} e^{itT'} \right) \mathbb{E} e^{i(vX+uY)} \right| \mathbb{P}\{S_{l-1} = s, N_n = l\} \\ & \quad + \sum_{s=0}^{n-N} \sum_{l=1}^{s+1} \left| \mathbb{E} \exp \left\{ i \frac{T'_{n-s}}{a(n-s)} \right\} \left( \mathbb{E} \exp \left\{ i \left( v \frac{a_{n-s}}{a(n)} + u \frac{N_n - 2 \cdot 1_{\{n-s=1\}}}{a(n)} \right) \right\} \right. \right. \\ & \quad \left. \left. - \mathbb{E} e^{i(vX+uY)} \right) \right| \\ & \quad \times \mathbb{P}\{S_{l-1} = s, N_n = l\} \\ &\stackrel{(19),(20)}{\leq} 2\varepsilon. \end{aligned}$$

Since  $T'$  does not depend on the pair  $(X, Y)$ , equation (18) for the distributions has a unique solution

$$T \stackrel{d}{=} \int_0^\infty e^{-U(t)} dt$$

(see, for example, [11]). Thus

$$\frac{T_n}{a(n)} \stackrel{d}{=} \int_0^\infty e^{-U(t)} dt, \quad n \rightarrow \infty.$$

The theorem is proved.

## 5. PROOF OF THEOREM 1.3

Condition (3) implies that  $m(x) := \int_0^x \mathbb{P}\{\xi > y\} dy$ ,  $x > 0$ , belongs to the de Haan class II, that is,

$$\lim_{t \rightarrow \infty} \frac{m(\lambda t) - m(t)}{L(t)} = \ln \lambda,$$

whence we obtain that  $m(x)$  is slowly varying. Since  $\sum_{l=1}^n \sum_{k=l}^\infty p_k \sim m(n)$ , we get from Remark 2.1 that

$$\frac{T_n}{N_n - 1} \rightarrow 1, \quad n \rightarrow \infty.$$

In view of Theorem 3(c) and the formula in [1, p. 42], we deduce that

$$\frac{N_n - b(n) - 1}{a(n)} \Rightarrow \mu_1.$$

Thus

$$\frac{T_n - b(n)}{a(n)} - \frac{T_n - N_n + 1}{N_n - 1} \frac{b(n)}{a(n)} \Rightarrow \mu_1.$$

Similarly to the proof of Theorem 1.5 in [8] one can show that the second term approaches zero in probability. Indeed, let

$$\frac{T_n - N_n + 1}{N_n - 1} \frac{b(n)}{a(n)} = \frac{\widehat{T}_{Y_n} + 1 - 2 \cdot 1_{\{Y_n=1\}}}{Y_n/m(Y_n)} \frac{m(n)}{m(Y_n)} \frac{b(n)Y_n}{na(n)} \frac{n}{m(n)(N_n - 1)} =: \prod_{i=1}^4 K_i(n).$$

By Theorem 2.3,  $m(n)T_n/n \xrightarrow{P} 1$ . Using the equality of distributions (5) and that  $Y_n \xrightarrow{P} \infty$  as  $n \rightarrow \infty$ , we have  $K_1(n) \xrightarrow{P} 1$ . Theorem 6 of [7] implies that  $K_2(n) \xrightarrow{d} 1/R$ , where  $R$  is a random variable with the uniform distribution on  $[0, 1]$ . Proposition 3.7 and Corollary 3.6 imply that  $K_3(n) \xrightarrow{P} 0$  and  $K_4(n) \xrightarrow{P} 1$ . The theorem is proved.

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