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ASYMPTOTIC RESULTS FOR THE ABSORPTION TIMES OF RANDOM WALKS WITH A BARRIER

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Abstract. A sequence $R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\left\{R_{k-1}^{(n)}+\xi_k<n\right\}}$, $k \in \mathbf{N}$, $R_0^{(n)} := 0$, is called a random walk with a barrier $n \in \mathbf{N}$, where the ξ_k are independent copies of a random variable ξ assuming positive integer values. The asymptotic behavior of the absorption times is studied in the paper for a random walk with a barrier. This behavior depends on the properties of the tail of the distribution of the random variable ξ.

1. Introduction and main results

Let ξ be a random variable with a proper distribution

$$p_k := \mathsf{P}\{\xi = k\}, \qquad k \in \mathbf{N}.$$

For simplicity, we assume that $p_1 > 0$ in what follows.

For every $n \in \mathbf{N}$, we define the random walk with the barrier n to be

$$R_0^{(n)} := 0, \qquad R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\left\{R_{k-1}^{(n)}+\xi_k<n\right\}}, \quad k \in \mathbf{N},$$

where $\{\xi_k \colon k \in \mathbf{N}\}$ are independent copies of the random variables ξ. Let

$$T_n := \inf\left\{k \in \mathbf{N} \colon R_k^{(n)} = n-1\right\} = \sum_{l=1}^{\infty} 1_{\left\{R_l^{(n)}<n-1\right\}} + 1$$

be the absorption time. Note that $T_1 = 1$ almost surely, while $T_n < \infty$ almost surely for $n \ge 2$ (see Lemma 2.1).

Below we provide some examples of applications of random walks $\left\{R_k^{(n)} \colon k \in \mathbf{N}\right\}$ with a barrier.

Example 1. Let $\{Z_k \colon k \in \mathbf{N}\}$ be a death chain with the space of states $\mathbf{N}$. Let the transition probabilities be such that $\pi_{ij} > 0$ for $j < i$ and $\pi_{ij} = 0$ otherwise. Define the random variables

$$X_n := \inf\{k \ge 1 \colon Z_k = 1 \text{ given } Z_0 = n\},$$

which are called the absorption times of the death chain. It is shown in the paper [8] that if

$$\pi_{n,n-k} = \frac{\mathsf{P}\{\xi = k\}}{\mathsf{P}\{\xi < n-1\}},$$

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then the distribution of X_n for natural numbers $n \geq 1 + \sup\{k \in \mathbf{N}\colon \mathsf{P}\{\xi \geq k\} = 1\}$ coincides with the distribution of the random variable

$$M_n := \#\left\{i \in \mathbf{N}\colon R_{i-1}^{(n)} \neq R_i^{(n)}\right\} = \sum_{l=0}^{\infty} 1_{\left\{R_l^{(n)} + \xi_{l+1} < n\right\}}.$$

By Y_n, we denote the number of deleted edges in the procedure of separating the root of a random recursive tree with n vertices (this procedure is proposed in [10]). It is shown in the paper [9] that if

$$\mathsf{P}\{\xi = k\} := \frac{1}{k(k+1)}, \qquad k \in \mathbf{N},$$

then

$$Y_n \overset{\mathrm{d}}{\to} X_n \overset{\mathrm{d}}{\to} M_n, \qquad n \in \mathbf{N}.$$

Example 2. Assume that the wage fund of an employer is $50,000$ units of a currency per month. We also assume that the salary demands of job applicants are independent copies of a random variable ξ assuming values in multiples of 500 units. Then the random variable $M_{50\,001}$ means the number of employees, while the random variable $T_{50\,001}$ is the total number of seekers trying to get a job at this business.

Some further applications of random walks with a barrier are also considered in the paper [6].

In what follows, a law μ_α, $\alpha \in [1, 2)$, is called stable if its characteristic function is of the following form:

$$\begin{cases} \exp\bigl\{-|t|^\alpha \Gamma(1-\alpha)\bigl(\cos(\pi\alpha/2) + i\sin(\pi\alpha/2)\,\mathrm{sgn}(t)\bigr)\bigr\}, & 1 < \alpha < 2, \\ \exp\{-|t|(\pi/2 - i\ln|t|\,\mathrm{sgn}(t))\}, & \alpha = 1. \end{cases}$$

Now we provide the main results of the paper concerning the asymptotic behavior of the sequence $\{T_n\colon n \in \mathbf{N}\}$ as $n \to \infty$.

Theorem 1.1. *If $m := \mathsf{E}\,\xi < \infty$, then the following statements are equivalent.*

(i) *There are sequences $\{a(n), b(n)\colon n \in \mathbf{N}\}$, $a(n) > 0$ and $b(n) \in \mathbf{R}$ such that $(T_n - b(n))/a(n)$ weakly converges as $n \to \infty$ to a nondegenerate and proper probability distribution.*

(ii) *For some $\alpha \in [1, 2]$ and for some slowly varying at ∞ function L,*

$$\sum_{k=1}^{n} k^2 p_k \sim n^{2-\alpha} L(n), \qquad n \to \infty. \tag{1}$$

If $\sigma^2 := \mathrm{Var}\,\xi < \infty$, then one can choose $b(n) := n/m$ and $a(n) := \bigl(m^{-3}\sigma^2 n\bigr)^{1/2}$; the limit distribution in this case is normal with zero mean and variance 1.

If $\sigma^2 = \infty$ and (1) holds for $\alpha = 2$, then one can choose $b(n) := n/m$ and $a(n) := c(n)m^{-3/2}$, where the nonnegative sequence $c(n)$ is such that $\lim_{n\to\infty} nL(c(n))/c^2(n) = 1$. The limit distribution in this case is normal with zero mean and variance 1.

If (1) holds for $\alpha \in [1, 2)$, then one can choose $b(n) := n/m$ and $a(n) := c(n)/m^{(\alpha+1)/\alpha}$, where $c(n)$ is a nonnegative sequence such that

$$\lim_{n\to\infty} nL(c(n))/c^\alpha(n) = \frac{2-\alpha}{\alpha}.$$

The limit distribution in this case is μ_α.

Theorem 1.2. *Let, for* $\alpha \in (0,1)$,

$$\sum_{k=n}^{\infty} p_k \sim n^{-\alpha} L(n), \qquad n \to \infty, \tag{2}$$

where L *is a slowly varying at* ∞ *function. Then*

$$\frac{T_n}{a(n)} \xrightarrow{d} \int_0^{\infty} e^{-U(t)}\, dt, \qquad n \to \infty,$$

where $a(n) := n^{\alpha} L^{-1}(n)$ *and where* $\{U(t)\colon t \ge 0\}$ *is the subordinator with zero shift and Lévy measure*

$$\nu(dt) = \frac{e^{-t/\alpha}}{(1-e^{-t/\alpha})^{\alpha+1}}\, dt, \qquad t > 0.$$

Theorem 1.3. *Let* $\mathsf{E}\,\xi = \infty$ *and*

$$\sum_{k=n}^{\infty} p_k \sim L(n)/n, \qquad n \to \infty, \tag{3}$$

for some slowly varying at ∞ *function* L. *Let* $c(t)$ *be an arbitrary positive function such that* $\lim_{t\to\infty} tL(c(t))/c(t) = 1$. *Put* $\psi(t) := t\int_0^{c(t)} \mathsf{P}\{\xi > y\}\, dy$. *Let* $b(t)$ *be another arbitrary positive function such that*

$$b(\psi(t)) \sim \psi(b(t)) \sim t, \qquad t \to \infty.$$

Put $a(t) := t^{-1} b(t) c(b(t))$. *Then* $(T_n - b(n))/a(n)$ *weakly converges to the stable law* μ_1.

It is shown in the paper [8] that Theorems 1.1, 1.2, and 1.3 hold if T_n is changed for M_n. The results of our paper do not follow from results of [8]; however, our proofs are similar to those in [8].

2. Auxiliary results

For all $m \in \mathbf{N}$ and all $i \in \mathbf{N}$, let $\widehat{R}_0^{(m)}(i) := 0$,

$$\widehat{R}_k^{(m)}(i) := \widehat{R}_{k-1}^{(m)}(i) + \xi_{i+k} 1_{\left\{\widehat{R}_{k-1}^{(m)}(i) + \xi_{i+k} < m\right\}}, \qquad k \in \mathbf{N},$$

and

$$\widehat{T}_m(i) := \sum_{l=1}^{\infty} 1_{\left\{\widehat{R}_l^{(m)}(i) < m-1\right\}} + 1.$$

Note that the distributions of the random variables $\widehat{T}_m(i)$ and T_m are identical. Put $\widehat{T}_{\cdot} := \widehat{T}_{\cdot}(1)$.

Lemma 2.1. *The random variable* T_n *is finite almost surely for all finite* $n \in \mathbf{N}$.

Proof. We use the induction in n. It is clear that $T_1 = 1$ almost surely. Let $T_k < \infty$ almost surely for $k = 1, 2, \ldots, n-1$. We show that $T_n < \infty$ almost surely, too.

The sequence $\left\{\widehat{R}_k^{(n)}(1) \in \mathbf{N}\right\}$ as well as the sequence $\widehat{T}_n$ does not depend on ξ_1. Moreover,

$$T_n = \left(1 + \widehat{T}_{n-\xi_1}(1)\right) 1_{\{\xi_1 \le n-2\}} + 1_{\{\xi_1 = n-1\}} + \left(1 + \widehat{T}_n(1)\right) 1_{\{\xi_1 \ge n\}}.$$

By the induction assumption, $\widehat{T}_k < \infty$, $k = 1, 2, \ldots, n-1$. Thus

$$\widehat{T}_{n-\xi_1} 1_{\{\xi_1 \le n-2\}} = \sum_{k=1}^{n-2} \widehat{T}_{n-k} 1_{\{\xi_1 = k\}} < \infty.$$

Therefore

$$\mathsf{P}\{T_n = \infty\} = \mathsf{P}\left\{(1+\widehat{T}_n)1_{\{\xi_1 \ge n\}} = \infty\right\} = \mathsf{P}\{T_n = \infty\}\,\mathsf{P}\{\xi_1 \ge n\},$$

whence

$$\mathsf{P}\{T_n = \infty\} = 0, \tag{4}$$

since $\mathsf{P}\{\xi_1 \ge n\} < 1$. The lemma is proved. □

Put $S_0 := 0$ and $S_n := \xi_1 + \cdots + \xi_n$ for $n \in \mathbf{N}$. Let

$$N_n := \inf\{k \ge 1 \colon S_k \ge n\}.$$

We show that the sequences T_n, $n \in \mathbf{N}$, and $\mathsf{E}\,T_n$, $n \in \mathbf{N}$, satisfy certain recurrence relations.

Proposition 2.1. *For all $n \in \mathbf{N}$,*

$$T_n = N_n + \widehat{T}_{Y_n}(N_n) - 2 \cdot 1_{\{Y_n = 1\}} \tag{5}$$

with probability one, where $Y_n := n - S_{N_n - 1}$.

Proof. We have

$$\begin{aligned} T_n &= T_n\left(1_{\{Y_n=1\}} + 1_{\{Y_n \ne 1\}}\right) = (N_n - 1)1_{\{Y_n=1\}} + T_n 1_{\{Y_n \ne 1\}} \\ &= (N_n - 1)1_{\{Y_n=1\}} + \sum_{k=1}^{N_n - 1} 1_{\left\{R_k^{(n)} < n-1\right\}} 1_{\{Y_n \ne 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\left\{R_k^{(n)} < n-1\right\}} 1_{\{Y_n \ne 1\}} \\ &\quad + 2 \cdot 1_{\{Y_n \ne 1\}} \\ &= N_n - 1 + 2 \cdot 1_{\{Y_n \ne 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\left\{R_k^{(n)} < n-1\right\}} 1_{\{Y_n \ne 1\}}. \end{aligned}$$

If the random event $\{Y_n = 1, k \ge N_n + 1\}$ occurs, then $R_k^{(n)} = n - 1$. Thus

$$\sum_{k=N_n+1}^{\infty} 1_{\left\{R_k^{(n)} < n-1\right\}} 1_{\{Y_n \ne 1\}} = \sum_{k=N_n+1}^{\infty} 1_{\left\{R_k^{(n)} < n-1\right\}}.$$

Therefore,

$$\begin{aligned} T_n &= N_n - 1 + 2 \cdot 1_{\{Y_n \ne 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\left\{R_k^{(n)} < n-1\right\}} \\ &= N_n - 1 + 2 \cdot 1_{\{Y_n \ne 1\}} + \sum_{k=1}^{\infty} 1_{\left\{\widehat{R}_k^{(n - S_{N_n - 1})}(N_n) < n - S_{N_n - 1} - 1\right\}} \\ &= N_n + \widehat{T}_{n - S_{N_n - 1}}(N_n) - 2 \cdot 1_{\{Y_n = 1\}}, \end{aligned}$$

and this completes the proof of Proposition 2.1. □

Proposition 2.2. *For all $n \in \mathbf{N}$ and $k \in \mathbf{N}$,*

$$\mathsf{E}\,T_n^k = r_n \sum_{l=1}^{n-1} \mathsf{E}\,T_{n-l}^k p_l + k \cdot r_n\,\mathsf{E}\,T_n^{k-1} + D_k\left(\mathsf{E}\,T_n, \mathsf{E}\,T_n^2, \ldots, \mathsf{E}\,T_n^{k-2}\right), \tag{6}$$

where

$$r_n := \frac{1}{p_1 + p_2 + \cdots + p_{n-1}}, \qquad D_\cdot(x_1, x_2, \ldots, x_n) := \upsilon_\cdot^{(0)} + \sum_{k=1}^{n} \upsilon_\cdot^{(k)} x_k,$$

and $\left\{\upsilon_\cdot^{(k)}, k = 0, 1, 2, \ldots, n\right\}$ are some real numbers.

In particular, if $k = 1$, then

$$\mathsf{E}\, T_n = r_n \,\mathsf{P}\{\xi_1 \neq n-1\} + r_n \sum_{k=1}^{n-1} \mathsf{E}\, T_{n-k} p_k. \tag{7}$$

Proof. We have

$$\begin{aligned}
\mathsf{P}\{T_n = i\} &= \sum_{l=1}^{\infty} \mathsf{P}\{T_n = i, \xi_1 = l\} \\
&= \sum_{l=1}^{n-2} \mathsf{P}\{T_n = i, \xi_1 = l\} + \mathsf{P}\{T_n = i, \xi_1 = n-1\} + \mathsf{P}\{T_n = i, \xi_1 \geq n\} \\
&= \sum_{l=1}^{n-1} \mathsf{P}\{T_{n-l} = i-1\}\,\mathsf{P}\{\xi_1 = l\} - 1_{\{i=2\}}\,\mathsf{P}\{\xi_1 = n-1\} \\
&\quad + 1_{\{i=1\}}\,\mathsf{P}\{\xi_1 = n-1\} + \mathsf{P}\{T_n = i-1\}\,\mathsf{P}\{\xi_1 \geq n\}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathsf{E}\, T_n^k &= \sum_{i=0}^{\infty} i^k\,\mathsf{P}\{T_n = i\} \\
&= \sum_{i=0}^{\infty} i^k \Bigg[\sum_{l=1}^{n-1} \mathsf{P}\{T_{n-l} = i-1\}\,\mathsf{P}\{\xi_1 = l\} - 1_{\{i=2\}}\,\mathsf{P}\{\xi_1 = n-1\} \\
&\qquad\qquad + 1_{\{i=1\}}\,\mathsf{P}\{\xi_1 = n-1\} + \mathsf{P}\{T_n = i-1\}\,\mathsf{P}\{\xi_1 \geq n\}\Bigg].
\end{aligned}$$

Further

$$\sum_{i=0}^{\infty}(i-1+1)^k \sum_{l=1}^{n-1} \mathsf{P}\{T_{n-l} = i-1\}\,\mathsf{P}\{\xi_1 = l\} = \sum_{l=1}^{n-1} \mathsf{E}(T_{n-l})^k p_l + \sum_{j=1}^{n-1} \binom{k}{j} \mathsf{E}\, T_{n-l}^j + 1,$$

$$\sum_{i=0}^{\infty} i^k\,\mathsf{P}\{T_n = k-1\}\,\mathsf{P}\{\xi_1 \geq n\} = \left(\mathsf{E}(T_n)^k + \sum_{j=1}^{n-1} \binom{k}{j} \mathsf{E}\, T_n^j + 1\right) \sum_{l=n}^{\infty} p_l,$$

whence

$$\mathsf{E}\, T_n^k = r_n \sum_{l=1}^{n-1} \mathsf{E}\, T_{n-l}^k p_l + k r_n \left[\sum_{l=1}^{n-1} \mathsf{E}\, T_{n-l}^{k-1} p_l + \mathsf{E}\, T_n^{k-1} \sum_{l=n}^{\infty} p_l\right] + D_{k_1}\left(\mathsf{E}\, T_n, \mathsf{E}\, T_n^2, \dots, \mathsf{E}\, T_n^{k-2}\right).$$

Using the latter relation for the moment $\mathsf{E}\, T_n^{k-1}$, we complete the proof of (6). □

Lemma 2.2. *Let condition* (2) *hold for* $\alpha \in (0,1)$. *Then*

$$\lim_{n\to\infty} \frac{L(n)}{n^\alpha}\,\mathsf{E}\, T_n = \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{\Gamma(1-\alpha)\Gamma(1+\alpha) - 1}.$$

Proof. It is known that

$$\lim_{n\to\infty} \frac{L(n)}{n^\alpha}\,\mathsf{E}\, N_n = \frac{1}{\Gamma^(1-\alpha)\Gamma(1+\alpha)}$$

(see, for example, [8]). Put

$$b := \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{\Gamma(1-\alpha)\Gamma(1+\alpha) - 1}.$$

We show that

$$\varlimsup_{n\to\infty} \frac{\mathsf{E}\,T_n}{\mathsf{E}\,N_n} \le b. \tag{8}$$

Assume the converse. Then, for an arbitrary $\varepsilon > 0$, the inequality $\mathsf{E}\,T_n > (b+\varepsilon)\,\mathsf{E}\,N_n$ holds for infinitely many numbers n. Thus one can choose a positive number ε such that $\mathsf{E}\,T_n > (b+\varepsilon)\,\mathsf{E}\,N_n + c$ infinitely often for any fixed $c > 0$. Let

$$n_c := \inf\{n \ge 1\colon\ \mathsf{E}\,T_n > (b+\varepsilon)\,\mathsf{E}\,N_n + c\}.$$

Note that $\lim_{c\to\infty} n_c = \infty$. Then

$$\mathsf{E}\,T_n \le (b+\varepsilon)\,\mathsf{E}\,N_n + c, \qquad n \in \{1,2,\dots,n_c-1\}.$$

Therefore

$$\begin{aligned}(b+\varepsilon)\,\mathsf{E}\,N_{n_c} + c < \mathsf{E}\,T_{n_c} &\overset{(7)}{=} r_{n_c}\,\mathsf{P}\{\xi_1 \ne n_c - 1\} + r_{n_c} \sum_{k=1}^{n_c-1} \mathsf{E}\,T_{n_c-k} p_k \\ &\le r_{n_c}\,\mathsf{P}\{\xi_1 \ne n_c - 1\} + c + r_{n_c}(b+\varepsilon) \sum_{k=1}^{n_c-1} \mathsf{E}\,N_{n_c-k} p_k.\end{aligned}$$

Lemma 3.4 of [8] implies that $\mathsf{E}\,N_n$ is such that

$$\mathsf{E}\,N_n = 1 + \sum_{k=1}^{n-1} \mathsf{E}\,N_{n-k} p_k.$$

Hence

$$(b+\varepsilon)\,\mathsf{E}\,N_{n_c} + c < r_{n_c}\,\mathsf{P}\{\xi_1 \ne n_c - 1\} + c + r_{n_c}(b+\varepsilon)(\mathsf{E}\,N_{n_c} - 1)$$

or, equivalently,

$$0 < r_{n_c}\,\mathsf{P}\{\xi \ne n_c - 1\} - r_{n_c}(b+\varepsilon) + \mathsf{E}\,N_{n_c}(r_{n_c} - 1)(b+\varepsilon).$$

Since $r_n - 1 \sim n^{-\alpha} L(n)$, $n \to \infty$, we pass to the limit in the latter inequality as $c \to \infty$ and obtain

$$\frac{\varepsilon}{b} < b - 1 + 1 - b = 0.$$

This contradiction proves (8). Similar reasoning shows that the converse inequality holds for the lower limit as well. This proves the lemma. ☐

Theorem 2.3. *If*

$$\sum_{l=1}^{n} \sum_{k=l}^{\infty} p_k \sim L(n), \qquad n \to \infty, \tag{9}$$

for some slowly varying at ∞ function L, then

$$\frac{T_n}{\mathsf{E}\,T_n} \xrightarrow{\mathsf{P}} 1. \tag{10}$$

Moreover $\mathsf{E}\,T_n \sim n/L(n)$ as $n \to \infty$.

Proof. To prove the theorem, it is sufficient to show that

$$\lim_{n\to\infty} \frac{\mathsf{E}\,T_n^k}{\mathsf{E}\,N_n^k} = 1 \tag{11}$$

for all $k \in \mathbf{N}$, since $\mathsf{E}\,N_n^k \sim n^k/L^k(n)$ according to Remark 3.5 in [8].

We prove relation (11) by induction in k. Assume that it holds for $k \in \{1,2,\dots,m-1\}$, but, at the same time,

$$\varlimsup_{n\to\infty} \frac{\mathsf{E}\,T_n^k}{\mathsf{E}\,N_n^k} > 1.$$

Then, for an arbitrary $\varepsilon > 0$, the inequality $\mathsf{E}\,T_n^k > (\varepsilon+1)\,\mathsf{E}\,N_n^k$ holds for infinitely many numbers n. Then one can choose ε such that $\mathsf{E}\,T_n^k > (\varepsilon+1)\,\mathsf{E}\,N_n^k + c$ infinitely often for any fixed $c > 0$. Put

$$n_c := \inf\left\{n \geq 1\colon\ \mathsf{E}\,T_n^k > (\varepsilon+1)\,\mathsf{E}\,N_n^k + c\right\}.$$

Note that $\lim_{c\to\infty} n_c = \infty$. Then

$$\mathsf{E}\,T_n^k \leq (1+\varepsilon)\,\mathsf{E}\,N_n^k + c, \qquad n \in \{1, 2, \ldots, n_c - 1\}. \tag{12}$$

According to Lemma 3.4 of [8], the absorption times of the random variable N_n satisfy the relation

$$\mathsf{E}\,N_n^k = D_k\left(\mathsf{E}\,N_n, \ldots, \mathsf{E}\,N_n^{k-2}\right) + k\,\mathsf{E}\,N_n^{k-1} + \sum_{i=1}^{n-1} \mathsf{E}\,N_{n-i}^k p_i.$$

Thus

$$\begin{aligned}
(1+\varepsilon)\,\mathsf{E}\,N_{n_c}^k + c < \mathsf{E}\,T_{n_c}^k &\overset{(6)}{=} r_n \sum_{l=1}^{n_c-1} \mathsf{E}\,T_{n_c-l}^k p_l + k\cdot r_n\,\mathsf{E}\,T_{n_c}^{k-1} + D_k\left(\mathsf{E}\,T_{n_c}, \ldots, \mathsf{E}\,T_{n_c}^{k-2}\right)\\
&\overset{(12)}{\leq} r_n(1+\varepsilon) \sum_{l=1}^{n_c-1} \mathsf{E}\,N_{n_c-l}^k p_l + k\cdot r_n\,\mathsf{E}\,T_{n_c}^{k-1} + D_k\left(\mathsf{E}\,T_{n_c}, \ldots, \mathsf{E}\,T_{n_c}^{k-2}\right) + c\\
&= (1+\varepsilon)(r_{n_c}-1)\left(\mathsf{E}\,N_{n_c}^k - D_k(\mathsf{E}\,N_{n_c}) - k\,\mathsf{E}\,N_{n_c}^{k-1}\right) + (1+\varepsilon)\,\mathsf{E}\,N_{n_c}^k\\
&\quad - (1+\varepsilon)(D_k(\mathsf{E}\,N_{n_c}) + k\,\mathsf{E}\,N_{n_c}^{k-1}) + k\cdot r_n\,\mathsf{E}\,T_{n_c}^{k-1} + D_k(\mathsf{E}\,T_{n_c}) + c,
\end{aligned}$$

where $D_k(\mathsf{E}\,X) := D_k(\mathsf{E}\,X, \mathsf{E}\,X^2, \ldots, \mathsf{E}\,X^{k-2})$. Therefore

$$\begin{aligned}
0 &< (1+\varepsilon)(r_{n_c}-1)\left(\mathsf{E}\,N_{n_c}^k - D_k(\mathsf{E}\,N_{n_c}) - k\,\mathsf{E}\,N_{n_c}^{k-1}\right) - (1+\varepsilon)\left(D_k(\mathsf{E}\,N_{n_c}) + k\,\mathsf{E}\,N_{n_c}^{k-1}\right)\\
&\quad + kr_n\,\mathsf{E}\,T_{n_c}^{k-1} + D_k(\mathsf{E}\,T_{n_c}).
\end{aligned}$$

We divide the latter inequality by $\mathsf{E}\,N_{n_c}^{k-1}$ and pass to the limit as $c \to \infty$. In view of Remark 3.5 of [8], we have $\mathsf{E}\,N_n^{k-1} \sim n^{k-1}/L^{k-1}(n)$. Condition (9) and Theorem 1.7.2 of [2] imply that $\lim_{n\to\infty}(r_n-1)n/L(n) = 0$. By the assumption of the induction,

$$\lim_{c\to\infty} \frac{D_k\left(\mathsf{E}\,T_{n_c}, \ldots, \mathsf{E}\,T_{n_c}^{k-2}\right)}{\mathsf{E}\,N_n^{k-1}} = 0 \quad \text{and} \quad \lim_{c\to\infty} \frac{\mathsf{E}\,T_{n_c}^{k-1}}{\mathsf{E}\,N_{n_c}^{k-1}} = 1,$$

whence

$$0 < k - (1+\varepsilon)k = -\varepsilon k.$$

This contradiction proves that

$$\varlimsup_{n\to\infty} \frac{\mathsf{E}\,T_n^k}{\mathsf{E}\,N_n^k} \leq 1.$$

The inequality for the lower limit is proved similarly. Thus (11) holds for all $k \in \mathbf{N}$. The theorem is proved. □

Remark 2.1. One can see from the proof of Theorem 2.3 that

$$\frac{T_n}{N_n} \xrightarrow{\mathsf{P}} 1, \qquad n \to \infty, \tag{13}$$

provided condition (9) holds.

3. Proof of Theorem 1.1

It is known that

$$Y_n \xrightarrow{d} Y \quad \text{as } n \to \infty$$

if $m := \mathsf{E}\,\xi < \infty$ (see, for example, [8]), where the distribution of the random variable Y is $\mathsf{P}\{Y = k\} = m^{-1}\,\mathsf{P}\{\xi \geq k\}$. Thus we obtain from (5) and (15) that

$$T_n - N_n \xrightarrow{\mathrm{d}} T'_Y - 2 \cdot 1_{\{Y=1\}}, \qquad n \to \infty.$$

Considering an arbitrary sequence $d(n)$ such that $\lim_{n\to\infty} d(n) = \infty$, we get

$$\frac{T_n - N_n}{d(n)} \xrightarrow{\mathsf{P}} 0, \qquad n \to \infty.$$

Assume that the distribution of ξ does not belong to the domain of attraction of any stable law with the index $\alpha \in [1, 2]$. Then there are no subsequences $x(n) \in \mathbf{R}$ and $y(n) > 0$ such that $(S_n - x(n))/y(n)$ converges weakly to a nondegenerate proper probability law. Since

$$\mathsf{P}\{N_n > m\} = \mathsf{P}\{S_m \leq n - 1\},$$

the same property holds for N_n, too.

On the other hand, assume that (1) holds and let $\alpha \in [1, 2)$. Then

$$\sum_{k=n}^{\infty} p_k \sim \frac{\alpha}{2-\alpha} n^{-\alpha} L(n).$$

According to a theorem in the paper [4],

$$\frac{N_n - b(n)}{a(n)} \Rightarrow \mu_\alpha, \qquad n \to \infty,$$

where $a(n)$ and $b(n)$ are as defined in [4].

If $\alpha = 2$, then there exists a sequence of nonrandom numbers a_n such that

$$n a_n^{-2} L(a_n) \to 1 \quad \text{as } n \to \infty$$

and

$$\lim_{n\to\infty} \mathsf{P}\left\{\frac{S_n - nm}{a_n} \leq x\right\} = \mu_2(-\infty, x],$$

where μ_2 is the standard normal distribution (see, for example, [12]). Then we derive from Theorem 2 of [5] that

$$\frac{N_n - b(n)}{a(n)} \Rightarrow \mu_2.$$

4. Proof of Theorem 1.2

Repeating the reasoning of the proof of Theorem 1.4 in [8], one can show that the sequence $T_n/a(n)$ converges in distribution to some random variable T. Put

$$Y_n := n - S_{N_n - 1}.$$

Then

$$\frac{T_n}{a(n)} = \frac{\widehat{T}_{Y_n}(N_n)}{a(Y_n)} \frac{a(Y_n)}{a(n)} + \frac{N_n - 2 \cdot 1_{\{Y_n=1\}}}{a(n)}. \tag{14}$$

Let $T'_{\cdot}$ be the absorption time for the sequence $\{(R_l^{(\cdot)})' : l \in \mathbf{N}_0\}$, which is an independent copy of $\{R_l^{(\cdot)}, l \in \mathbf{N}_0\}$ and which is independent of (N_n, Y_n). For every fixed number $m \in \mathbf{N}_0$,

$$\begin{aligned}
\mathsf{P}\{\widehat{T}_{Y_n}(N_n) = m\} &= \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathsf{P}\{\widehat{T}_{n-j}(i) = m, N_n = i, S_{N_n-1} = j\} \\
&= \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathsf{P}\left\{\sum_{l=0}^{\infty} 1_{\left\{\widehat{R}_l^{(n-j)}(i) < n-j-1\right\}} = m-1, N_n = i, S_{N_n-1} = j\right\} \\
&= \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathsf{P}\left\{\sum_{l=0}^{\infty} 1_{\left\{\widehat{R}_l^{(n-j)}(i) < n-j-1\right\}} = m-1\right\} \mathsf{P}\{N_n = i, S_{N_n-1} = j\} \\
&= \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathsf{P}\left\{\sum_{l=0}^{\infty} 1_{\left\{(R_l^{(n-j)})' < n-j-1\right\}} = m-1\right\} \mathsf{P}\{N_n = i, S_{N_n-1} = j\} \\
&= \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathsf{P}\{T'_{n-j} = m\} \, \mathsf{P}\{N_n = i, S_{N_n-1} = j\} = \mathsf{P}\{T'_{Y_n} = m\}.
\end{aligned}$$

This proves the identity of the distributions, namely

$$\widehat{T}_{Y_n}(N_n) \stackrel{\mathrm{d}}{=} T'_{Y_n}, \tag{15}$$

where the sequence $\{T'_n : n \in \mathbf{N}\}$ does not depend on $(N_n, n - S_{N_n-1})$ and has the same distribution as

$$\{T_n : n \in \mathbf{N}\}.$$

Since $Y_n \xrightarrow{\mathsf{P}} \infty$, we have

$$\frac{T'_{Y_n}}{a(Y_n)} \xrightarrow{\mathsf{P}} T', \tag{16}$$

where T' has the same distribution as T. Proposition 3.9 of [8] implies that

$$\left(\frac{a(Y_n)}{a(n)}, \frac{N_n}{a(n)}\right) \xrightarrow{\mathrm{d}} \left(e^{-U(V)}, \int_0^V e^{-U(t)}\, dt\right), \qquad n \to \infty, \tag{17}$$

where V is a random variable with the density $e^{-x}1_{\{x>0\}}$ that does not depend on the subordinator $U(t)$. Put $X := \exp\{-U(V)\}$ and $Y := \int_0^V e^{-U(t)}\, dt$. Since the left hand side of (14) weakly converges, the random variables

$$\xi_n := \left(\frac{T'_{Y_n}}{a(Y_n)}, \frac{a(Y_n)}{a(n)}, \frac{N_n - 2 \cdot 1_{\{Y_n = 1\}}}{a(n)}\right)$$

weakly converge, too. Taking into account (14) and (17) we prove that

$$\xi_n \xrightarrow{\mathrm{d}} (T', X, Y)\,.$$

Passing to the limit as $n \to \infty$ we prove that

$$T \xrightarrow{\mathrm{d}} XT' + Y, \tag{18}$$

where T' does not depend on the pair (X, Y). To prove this relation, it suffices to show that

$$\lim_{n\to\infty} \left| \mathsf{E} \exp\left\{ i \left(t\frac{T'_{Y_n}}{a(Y_n)} + v\frac{a_{Y_n}}{a(n)} + u\frac{N_n - 2\cdot 1_{\{Y_n=1\}}}{a(n)} \right) \right\} - \mathsf{E}\, e^{itT'}\, \mathsf{E}\, e^{i(vX+uY)} \right| = 0$$

for all real t, u, and v.

In view of (16) and (17) and since $Y_n \to \infty$ as $n \to \infty$, we obtain that, for all $\varepsilon > 0$, there are $N_i = N_i(\varepsilon)$, $i = 1, 2, 3$, such that

$$\left|\mathsf{E}\exp\left\{it\frac{T'_{Y_n}}{a(Y_n)}\right\} - \mathsf{E}\, e^{itT'}\right| < \varepsilon, \qquad n > N_1, \tag{19}$$

$$\left|\mathsf{E}\exp\left\{i\left(v\frac{a_{Y_n}}{a(n)} + u\frac{N_n - 2\cdot 1_{\{Y_n=1\}}}{a(n)}\right)\right\} - \mathsf{E}\, e^{i(vX+uY)}\right| < \varepsilon, \qquad n > N_2, \tag{20}$$

$$\mathsf{P}\{Y_n \le \max\{N_1, N_2\}\} < \varepsilon, \qquad n > N_3.$$

Put $N := \max\{N_1, N_2\}$. Then

$$\begin{aligned}
&\left|\mathsf{E}\exp\left\{i\left(t\frac{T'_{Y_n}}{a(Y_n)} + v\frac{a_{Y_n}}{a(n)} + u\frac{N_n - 2\cdot 1_{\{Y_n=1\}}}{a(n)}\right)\right\} - \mathsf{E}\, e^{itT'}\,\mathsf{E}\, e^{i(vX+uY)}\right| \\
&\quad = \Bigg|\sum_{s=0}^{n-1}\sum_{l=1}^{s+1}\bigg(\mathsf{E}\exp\left\{i\bigg(t\frac{T'_{n-s}}{a(n-s)} + v\frac{a_{n-s}}{a(n)} + u\frac{l - 2\cdot 1_{\{n-s=1\}}}{a(n)}\bigg)\right\} \\
&\qquad\qquad - \mathsf{E}\, e^{itT'}\,\mathsf{E}\, e^{i(vX+uY)}\bigg)\mathsf{P}\{S_{l-1} = s, N_n = l\}\Bigg| \\
&\quad \le \left|\sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\dots\right| + \left|\sum_{s=n-N+1}^{n-1}\sum_{l=1}^{s+1}\dots\right|.
\end{aligned}$$

For all $n > \max(N, N_3)$, the second term is estimated as follows:

$$\begin{aligned}
&\Bigg|\sum_{s=n-N+1}^{n-1}\sum_{l=1}^{s+1}\bigg(\mathsf{E}\exp\left\{i\left(t\frac{T'_{n-s}}{a(n-s)} + v\frac{a_{n-k}}{a(n)} + u\frac{l - 2\cdot 1_{\{n-s=1\}}}{a(n)}\right)\right\} \\
&\qquad\qquad - \mathsf{E}\, e^{itT'}\,\mathsf{E}\, e^{i(vX+uY)}\bigg)\mathsf{P}\{S_{l-1} = s, N_n = l\}\Bigg| \\
&\quad \le 2\left|\mathsf{P}\{Y_n \le N\}\right| < 2\varepsilon.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&\Bigg|\sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\bigg(\mathsf{E}\exp\left\{i\left(t\frac{T'_{n-s}}{a(n-s)} + v\frac{a_{n-s}}{a(n)} + u\frac{l - 2\cdot 1_{\{n-s=1\}}}{a(n)}\right)\right\} \\
&\qquad\qquad - \mathsf{E}\, e^{itT'}\,\mathsf{E}\, e^{i(vX+uY)}\bigg)\mathsf{P}\{S_{l-1} = s, N_n = l\}\Bigg| \\
&\quad \le \sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\left|\left(\mathsf{E}\exp\left\{i\left(t\frac{T'_{n-s}}{a(n-s)}\right)\right\} - \mathsf{E}\, e^{itT'}\right)\mathsf{E}\, e^{i(vX+uY)}\right|\mathsf{P}\{S_{l-1} = s, N_n = l\} \\
&\qquad + \sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\Bigg|\mathsf{E}\exp\left\{i\frac{T'_{n-s}}{a(n-s)}\right\}\left(\mathsf{E}\exp\left\{i\left(v\frac{a_{n-s}}{a(n)} + u\frac{N_n - 2\cdot 1_{\{n-s=1\}}}{a(n)}\right)\right\}\right. \\
&\qquad\qquad \left. - \mathsf{E}\, e^{i(vX+uY)}\right)\Bigg| \\
&\qquad\qquad \times \mathsf{P}\{S_{l-1} = s, N_n = l\} \\
&\quad \overset{(19),(20)}{\le} 2\varepsilon.
\end{aligned}$$

Since T' does not depend on the pair (X,Y), equation (18) for the distributions has a unique solution

$$T \stackrel{\mathrm{d}}{\to} \int_0^\infty e^{-U(t)}\,dt$$

(see, for example, [11]). Thus

$$\frac{T_n}{a(n)} \stackrel{\mathrm{d}}{\to} \int_0^\infty e^{-U(t)}\,dt, \qquad n\to\infty.$$

The theorem is proved.

5. Proof of Theorem 1.3

Condition (3) implies that $m(x) := \int_0^x \mathsf{P}\{\xi > y\}\,dy$, $x>0$, belongs to the de Haan class Π, that is,

$$\lim_{t\to\infty} \frac{m(\lambda t)-m(t)}{L(t)} = \ln\lambda,$$

whence we obtain that $m(x)$ is slowly varying. Since $\sum_{l=1}^n \sum_{k=l}^\infty p_k \sim m(n)$, we get from Remark 2.1 that

$$\frac{T_n}{N_n-1} \to 1, \qquad n\to\infty.$$

In view of Theorem 3(c) and the formula in [1, p. 42], we deduce that

$$\frac{N_n-b(n)-1}{a(n)} \Rightarrow \mu_1.$$

Thus

$$\frac{T_n-b(n)}{a(n)} - \frac{T_n-N_n+1}{N_n-1}\frac{b(n)}{a(n)} \Rightarrow \mu_1.$$

Similarly to the proof of Theorem 1.5 in [8] one can show that the second term approaches zero in probability. Indeed, let

$$\frac{T_n-N_n+1}{N_n-1}\frac{b(n)}{a(n)} = \frac{\widehat{T}_{Y_n}+1-2\cdot 1_{\{Y_n=1\}}}{Y_n/m(Y_n)}\frac{m(n)}{m(Y_n)}\frac{b(n)Y_n}{na(n)}\frac{n}{m(n)(N_n-1)} =: \prod_{i=1}^4 K_i(n).$$

By Theorem 2.3, $m(n)T_n/n \stackrel{\mathsf{P}}{\to} 1$. Using the equality of distributions (5) and that $Y_n \stackrel{\mathsf{P}}{\to} \infty$ as $n\to\infty$, we have $K_1(n) \stackrel{\mathsf{P}}{\to} 1$. Theorem 6 of [7] implies that $K_2(n) \stackrel{\mathrm{d}}{\to} 1/R$, where R is a random variable with the uniform distribution on $[0,1]$. Proposition 3.7 and Corollary 3.6 imply that $K_3(n) \stackrel{\mathsf{P}}{\to} 0$ and $K_4(n) \stackrel{\mathsf{P}}{\to} 1$. The theorem is proved.

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Bibliography

1. N. H. Bingham, *Limit theorems for regenerative phenomena, recurrent events and renewal theory*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **21** (1972), 20–44. MR0353459 (50:5942)
2. N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1989. MR1015093 (90i:26003)
3. M. Drmota, A. Iksanov, M. Möhle, and U. Rösler, *A limiting distribution for the number of cuts needed to isolate the root of a random recursive tree*, http://www.unicyb.kiev.ua/∼iksan (2006).
4. W. Feller, *Fluctuation theory of recurrent events*, Trans. Amer. Math. Soc. **67** (1949), 98–119. MR0032114 (11:255c)

5. C. C. Heyde, *A limit theorem for random walks with drift*, J. Appl. Probab. **1** (1967), 144–150. MR0207061 (34:6877)
6. K. Hinderer and H. Walk, *Anwendung von Erneuerungstheoremen und Taubersätzen für eine Verallgemeinerung der Erneuerungsprozesse*, Math. Z. **126** (1972), 95–115. MR0300354 (45:9400)
7. K. B. Erickson, *Strong renewal theorems with infinite mean*, Trans. Amer. Math. Soc. **151** (1970), 263–291. MR0268976 (42:3873)
8. A. Iksanov and M. Möhle, *On a random recursion related to absorption times of death Markov chains*, http://www.unicyb.kiev.ua/∼iksan (2007).
9. A. Iksanov and M. Möhle, *A probabilistic proof of a weak limit law for the number of cuts needed to isolate the root of a random recursive tree*, Elect. Comm. in Probab. **12** (2007), 36–43. MR2407414 (2009e:05076)
10. A. Meir and J. W. Moon, *Cutting down recursive trees*, Math. Biosci. **21** (1974), 173–181.
11. W. Vervaat, *On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables*, Adv. Appl. Probab. **11** (1979), 750–783. MR544194 (81b:60064)
12. W. Feller, *An Introduction to Probability Theory and its Applications*, Second edition, vol. II, John Wiley & Sons, New York–London–Sydney, 1971. MR0270403 (42:5292)

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