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ON EXCESS-OF-LOSS REINSURANCE

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HANSJÖRG ALBRECHER AND JOZEF L. TEUGELS

Abstract. We discuss a unified framework to analyze the distribution of the number of claims and the aggregate claim sizes in an excess-of-loss reinsurance contract based upon the use of point processes and work out several examples explicitly. We first deal with a single excess-of-loss situation with an extra upper bound on the coverage of individual claims. Subsequently the results are extended to a reinsurance chain with k partners.

1. Introduction

Traditionally, an *excess-of-loss reinsurance* form covers the overshoot over a certain retention level M for all claims whether or not they are considered to be large. Among the insurance branches where it is used, we mention in particular general liability, and to a lesser extent motor liability ([25]) and windstorm reinsurance ([18]). Because of its transparency, an excess-of-loss treaty has been one of the main research objects in the reinsurance literature right from the beginning (see for instance [3, 4]). It is clear that excess-of-loss reinsurance limits the liability of the first line insurer but that he himself will cover all claims below the retention M. In this form, the excess-of-loss reinsurance treaty cover has a number of desirable theoretical properties as explained by Bühlmann in [5] and by Asmussen et al. in [2]. For a combination with quota-share reinsurance, see [6].

We will look at a more general form where each claim will be considered between two boundaries: we will call the lower retention u while the upper will be taken to be $u + v$, indicating by v the range for which the treaty is used. As illustrated below, the notation allows us to deal with any partner in an excess-loss reinsurance chain. If u and v depend on the claim orderings, then this reinsurance form is commonly called *drop-down-excess-of-loss reinsurance*. For a study of such a reinsurance form, we refer to Ladoucette et al. [12].

In the following we need some basic notation, first for the original portfolio.

- The *epochs of the claims* are denoted by $T_0 = 0, T_1, T_2, \dots$. Apart from the fact that the epochs form a nondecreasing sequence, we in general do not assume anything specific about their interdependence. The random variables defined by $W_0 = 0$ and $\{W_{i+1} := T_{i+1} - T_i; i = 0, 1, \dots\}$ are called the *waiting times* in between successive claims. In some particular cases it might be useful to assume that the sequence $\{W_i; i \geq 1\}$ consists of independent random variables all with a common distribution V as a random variable T, i.e. $\mathsf{P}(T \leq x) = V(x)$. In that

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case the claim epochs are the *renewal time points* of a renewal process generated by T or V.

- The *number of claims* up to time t is denoted and defined by

$$N(t) = \sup\{n\colon T_n \leq t, t \geq 0\}.$$

We denote its probabilities by $p_n(t) = \mathsf{P}(N(t) = n)$. For the generating function of the sequence $\{p_n(t)\}$ we use the notation ($|z| \leq 1$)

$$Q_t(z) := \sum_{n=0}^{\infty} p_n(t) z^n = \mathsf{E}\left\{z^{N(t)}\right\}.$$

- The claim occurring at time T_n has size X_n. The sequence $\{X_i; i = 1, 2, \ldots\}$ of consecutive *claim sizes* is assumed to be a renewal process; i.e., the claims are taken to be independent with common *claim size distribution* $F(x) = \mathsf{P}(X \leq x)$, where X is a generic claim size. We take for granted that the claim times process $\{T_{i+1} - T_i; i = 0, 1, \ldots\}$ and the claim sizes process $\{X_i; i = 1, 2, \ldots\}$ are independent.
- The *total claim amount* or the *aggregate claim amount* at time t is defined and denoted by $X(t) = \sum_{i=1}^{N(t)} X_i$ if $N(t) \geq 1$, while $X(t) = 0$ if $N(t) = 0$. For its distribution we write

$$\mathsf{P}(X(t) \leq x) =: G_t(x).$$

Under the assumption of independent claim number and claim size processes, $G_t(x) = \sum_{n=0}^{\infty} p_n(t) F^{*n}(x)$, where $*n$ refers to the n-th convolution of F with itself. For the Laplace transform we then have ($s \geq 0$)

$$\hat{G}_t(s) = \mathsf{E}\left\{e^{-sX(t)}\right\} = \sum_{n=0}^{\infty} p_n(t) \hat{F}^n(s) = Q_t\big(\hat{F}(s)\big), \tag{1}$$

where $\hat{F}$ is the Laplace transform of F.

- In the sequel we will use the same notation for the reinsured part as for the original one but supplied with $\tilde{}$. For example, the number of reinsured claims up to time t will be denoted by $\tilde{N}(t)$. Whenever the time parameter t is unimportant, it will be dropped in the sequel.

Several papers and textbooks are available in the literature that deal with risk analysis in an excess-of-loss setup under certain model assumptions (see for instance Klugman et al. [10] for a survey). In this paper, we intend to provide a general yet transparent approach to the study of some random quantities in connection with excess-of-loss contracts based on the use of point processes. In Section 2 we first deal with the number of reinsured claims for a single reinsurer together with a wide set of examples from the actuarial literature. In Section 3 we then turn to reinsurance chains with more than three partners. Section 4 finally deals with the aggregate claim amounts carried by the individual partners in the reinsurance chain.

2. Number of reinsured claims

2.1. **General properties.** Consider the bivariate point process with points

$$\{(T_k, X_k),\ 1 \leq k \leq N(t)\}.$$

To deal with the number of reinsured claims, let A be any Borel set in $(0, t] \times \mathbb{R}^+$ and denote by $\tilde{N} := \tilde{N}(A)$ the number of points from the bivariate point process that fall into the set A. To make the calculations more transparent we define

$$Y_n := \begin{cases} 0 & \text{if } (T_n, X_n) \notin A, \\ 1 & \text{if } (T_n, X_n) \in A. \end{cases}$$

Hence

$$\tilde{N}(t) = \#\{k\colon 1 \le k \le N(t) \mid (T_k, X_k) \in A\} = \sum_{n=1}^{\infty} Y_n.$$

In this section, we will choose A to be of the form $A = (0,t] \times (u, u+v]$. The reason for allowing arbitrariness in the second component of A is that situations vary between first and second line insurance. For a first line insurance, $u = 0$ while $v = M$, the retention. For the first reinsurer, however, $u = M$ while v may take any positive value. If the first line reinsurer does not shift part of the risk to a second reinsurer, then $v = \infty$. If the first line insurer also buys an excess-of-loss reinsurance at a second company, then v equals the extra retention on top of the one for the first line reinsurance company and so on. By not specifying u and v the treatment below applies to any company in an excess-of-loss reinsurance chain. Of course, any reinsurer can apply the type of reinsurance of his choice, irrespective of what the former insurer has been doing.

By the underlying assumptions about the claim number and the claim times processes we find the following expression ($t \ge 0$, $n \ge 0$):

$$\tilde{p}_n(t) := \mathsf{P}\big(\tilde{N}(t) = n\big) = \sum_{k=n}^{\infty} p_k(t) \binom{k}{n} r^n (1-r)^{k-n},$$

where we simplified the notation by introducing the abbreviation

$$r := r(u,v) = \mathsf{P}\{u < X \le u+v\} = F(u+v) - F(u),$$

the probability of a claim size larger than the value u but not overshooting the value $u+v$. Indeed, among the k claims that arrived by time t, exactly n could be called *successes* when we interpret an event to be a success if the allowed time slot is $(0,t]$ while the value slot is $(u, u+v]$.

An alternative expression can be obtained if we look at the generating function $\tilde{Q}_t(z)$ of the distribution $\{\tilde{p}_n(t)\}$. It is straightforward to derive that this is related to the generating function $Q_t(z)$ of the distribution $\{p_n(t)\}$ by the equation

$$\tilde{Q}_t(z) = Q_t((1-r) + rz) \tag{2}$$

(see also [10, p. 266]). Written in this fashion, the variable $\tilde{N}(t)$ can be considered as a *thinned* version of the original $N(t)$. This also means that, up to the determination of the quantity r, both contain similar statistical information. For example, if the insurer has a statistical estimate of $Q_t(z)$, then the same is true for the reinsurer since

$$\tilde{Q}_t(w) = Q_t((1-r) + rw).$$

But conversely, if $\tilde{Q}_t(w)$ is known, then

$$Q_t(z) = \tilde{Q}_t\left(\frac{z-(1-r)}{r}\right).$$

Hence, only the extra parameter r needs to be known or estimated.

From the above, one can quickly derive information on the moments of $\tilde{N}(t)$. For any nonnegative integer k,

$$\mathsf{E}\binom{\tilde{N}(t)}{k} = \frac{1}{k!}\tilde{Q}_t^{(k)}(1) = r^k\, \mathsf{E}\binom{N(t)}{k}. \tag{3}$$

In particular

$$\mathsf{E}\,\tilde{N}(t) = r\,\mathsf{E}\,N(t),$$
$$\operatorname{Var}\tilde{N}(t) = r^2 \operatorname{Var} N(t) + r(1-r)\,\mathsf{E}\,N(t).$$

For the measures of dispersion one has

$$\tilde{I}(t) := \frac{\operatorname{Var}\tilde{N}(t)}{\mathsf{E}\,\tilde{N}(t)} = rI(t) + (1-r).$$

Mack [13] notices that $\tilde{I}(t)-1 = r(I(t)-1)$, showing that the sign of the dispersion for the original claim number process remains the same for the reinsurance process. Moreover, the smaller r (e.g. by lowering the height v of the layer), the closer the dispersion gets to that of the Poisson case. This phenomenon is well known within the context of renewal theory. See for example [16].

2.2. **Examples.** Let us illustrate the above procedure for a wide set of examples traditional in the actuarial literature. For that purpose consider the general case of holonomic generating functions $Q_t(z)$; i.e., $Q_t(z)$ is a smooth function satisfying a linear homogenous differential equation with polynomial coefficients

$$\sum_{i=0}^{k} Q_t^{(i)}(z) P_i(z) = 0. \tag{4}$$

Here $Q_t^{(i)}(z)$ denotes the i-th derivative (with $Q_t^{(0)}(z) := Q_t(z)$) and $P_i(z) = \sum_{j=0}^{d_i} a_{ij} z^j$ is a polynomial of degree d_i with real coefficients a_{ij}. This class contains most of the claim number distributions that allow for effective recursive calculations of aggregate claim distributions; see for instance Wang and Sobrero [26] and Albrecher and Pirsic [1]. From (2) we immediately derive

$$\sum_{i=0}^{k} Q_t^{(i)}(1-r+rz) P_i(1-r+rz) = \sum_{i=0}^{k} \tilde{Q}_t^{(i)}(z) \tilde{P}_i(z) = 0$$

with

$$\tilde{P}_i(z) = \sum_{j=0}^{d_i} \tilde{a}_{ij}\, z^j, \quad \text{where } \tilde{a}_{ij} = r^{j-i} \sum_{m=j}^{d_i} a_{im} \binom{m}{j} (1-r)^{m-j}. \tag{5}$$

Hence the degree of each polynomial P_i is preserved when switching from the insured to the reinsured claim numbers and as we shall see in worked out examples below, for many special cases the type of claim number distribution is also preserved, with just the parameters being modified (for the first example, namely the Sundt–Jewell class, this has already been worked out in detail in Klugman et al. [10, Sec. 3.10], but we still include it here again for completeness and in particular for emphasizing the common pattern among all these examples).

(1) The *Sundt–Jewell class.*

This very popular set of claim number distributions was introduced in 1981 by Jewell and Sundt [21]. It is based on the simple recursion

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \qquad n \in \{2, 3, \ldots\}, \tag{6}$$

where the quantities a and b may depend on the (usually fixed) time variable t (since the time parameter is not important in this example, we omit it throughout for each of the quantities a, b, and $Q(z)$). The above class has been introduced in an attempt to gather a variety of classical claim number distributions under the same umbrella. Later Willmot [28] reconsidered the equation and added a number of overlooked solutions. Note that the recursion does not specify the quantities p_1 and p_0. Nevertheless the requirement $\sum_{n=0}^{\infty} p_n = 1$ eliminates one of the latter two parameters. As a result the Sundt–Jewell class may be used as a three-parameter class (a, b, p_0) for data-fitting. As will be shown in a later section, the recursion is also instrumental in the numerical calculation of the distribution of the total claim amount $G_t(x)$.

The solution of the above relation (6) can be obtained in a variety of ways. We use generating functions. Then (6) turns into the first order differential equation

$$(1 - az)Q'(z) = (a + b)Q(z) + p_1 - (a + b)\rho, \tag{7}$$

where

$$\rho := p_0 = \mathsf{P}(N = 0).$$

Solving this equation with the side condition $Q(1) = 1$ is standard. Note that apart from the quantities a and b, the other two parameters can be retrieved from $\rho = Q(0)$ and $p_1 = Q'(0)$.

We turn to the reinsured quantities. Clearly, (7) is a special case of (4) (just differentiate (7) w.r.t. z to see that $Q(z)$ is holonomic). Hence it is easily calculated that (after normalization) $\tilde{P}_1(z) = 1 - \tilde{a}z$ and $\tilde{P}_0(z) = \tilde{a} + \tilde{b}$, where

$$\tilde{a} := \frac{ar}{1 - a(1 - r)}, \qquad \tilde{b} := \frac{br}{1 - a(1 - r)}. \tag{8}$$

For the inhomogeneous term, we have $\tilde{\rho} := \tilde{p}_0 = \tilde{Q}(0) = Q(1 - r)$ while $\tilde{p}_1 := \tilde{Q}'(0) = rQ'(1 - r)$. Inserting $z = 1 - r$ in (7) it follows that the differential equation for $\tilde{Q}(\cdot)$ is also given by

$$(1 - \tilde{a}z)\tilde{Q}'(z) = \big(\tilde{a} + \tilde{b}\big)\tilde{Q}(z) + \tilde{p}_1 - (\tilde{a} + \tilde{b})\tilde{\rho}, \tag{9}$$

which imitates (7) perfectly. Comparing (7) with (9) we conclude that the probabilities for the reinsured quantity $\{\tilde{p}_n\}$ satisfy a relation of the form (6), i.e.

$$\tilde{p}_n = \left(\tilde{a} + \frac{\tilde{b}}{n}\right)\tilde{p}_{n-1}, \qquad n \in \{2, 3, \ldots\},$$

with new parameters $\tilde{a}$ and $\tilde{b}$ defined above.

Let us look at a number of special cases of the relations (6) and (7).

- Case 1: $a = \tilde{a} = 0$.
  This case is rather easy and leads quickly to the well-known observation that a thinned (shifted) *Poisson variable* is again (shifted) Poisson:
  $$p_n = \frac{1 - \rho}{1 - e^{-b}} e^{-b} \frac{b^n}{n!},$$
  where the quantity b is replaced by $\tilde{b} = br$.
- Case 2: $a \neq 0$, $\tilde{a} \neq 0$.
  It is advantageous to use the auxiliary quantity $\Delta := 1 + b/a = 1 + \tilde{b}/\tilde{a}$ that remains invariant under thinning.
  - Subcase (a): $\Delta \neq 0$. Here we find the *shifted Pascal (negative binomial) distributions*
    $$p_n = \frac{1 - \rho}{(1 - a)^{-\Delta} - 1} \binom{\Delta + n - 1}{n} a^n, \qquad n \geq 1, \tag{10}$$
    and the same for $\{\tilde{p}_n\}$ with a replaced by $\tilde{a}$ and ρ by $\tilde{\rho}$.

– Subcase (b): $\Delta = 0$. Now the solution is the *shifted logarithmic distribution*

$$p_n = \frac{1-\rho}{-\log(1-a)}\frac{a^n}{n}, \qquad n \geq 1,\ 0 < a < 1, \tag{11}$$

again with the same expression for $\{\tilde{p}_n\}$ with a replaced by $\tilde{a}$ and ρ by $\tilde{\rho}$.

A few observations are in order.

(i) The cases $a = \tilde{a} = 0$ and $\Delta = 0$ are the natural limits of the general case of the shifted Pascal distributions.

(ii) Apart from the above three distributions, also other cases are *invariant* under reinsurance thinning.

– If $\Delta = -m$, a negative integer, then we run into the *shifted binomial distribution* both for the original and for the thinned process.

– If $\Delta = -\theta \in (-1, 0)$, a similar calculation yields a *shifted Engen distribution* (see [28])

$$p_n = \frac{1-\rho}{1-(1-a)^\theta}\frac{a^n\Gamma(n-\theta)}{n!\,\Gamma(1-\theta)}, \qquad n \geq 1,$$

with the same expression for $\{\tilde{p}_n\}$ with a replaced by $\tilde{a}$ and ρ by $\tilde{\rho}$.

(iii) It turns out that in all of these cases the generating function $Q(z)$ can be written in the form $Q(z) = \rho + (1-\rho)R(z)$, where $R(z)$ is again a generating function of a discrete probability distribution $\{q_n; n \in \mathbb{N}\}$. The effect of the parameter ρ is to introduce an eventual extra weight at the point 0. This is often done by *truncation* in the sense that $\rho = \mathsf{P}(N = 0)$ while $R(z)$ is the generating function of the probabilities $\mathsf{P}(N = n \mid N > 0)$. If we shift the distributions and take $\rho = 0$, then we of course get the classical unshifted distributions. By the above, this remark also applies to the thinned process of the reinsured claims.

Other, more involved holonomic functions $Q(z)$ will be discussed in the framework of mixed Poisson processes below. Note that the $Q(z)$ satisfying (4) generalize (6) by allowing recursions for $\{p_n\}$ of higher order and by replacing the factor $a+b/n$ by general rational functions (see also [19, 23, 30] and more recently [8]).

(2) The *mixed Poisson process.*

A far-reaching generalization of the ordinary Poisson process is obtained when the parameter λ is replaced by a random variable Λ with *mixing* or *structure distribution* H. In particular we can think of situations where the counting process consists of various subprocesses that individually behave as a Poisson process with a specific parameter value. For a textbook treatment, see [7]. We get the following characteristics:

$$p_n(t) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,dH(\lambda),$$

which yields

$$Q_t(z) = \int_0^\infty e^{-\lambda t(1-z)}\,dH(\lambda).$$

If we look at the reinsurance process, then by (2), clearly,

$$
\begin{aligned}
\tilde{Q}_t(z) &= \int_0^\infty \exp\{-\lambda t(1-((1-r)+rz))\}\, dH(\lambda) \\
&= \int_0^\infty \exp\{-\lambda tr(1-z)\}\, dH(\lambda) = Q_{rt}(z).
\end{aligned} \tag{12}
$$

This simple relation shows that we only need to replace the time variable t by rt. Other quantities related to the original mixed Poisson process transform in an analogous fashion. In particular,

$$
\mathsf{E}\binom{\tilde{N}(t)}{k} = \frac{(rt)^k}{k!}\,\mathsf{E}\{\Lambda^k\}.
$$

As an example, consider H to be a gamma distribution with parameters α and b. Then we end up with the *Pascal process*, originally introduced by Thyrion in [24]. Here

$$
\tilde{p}_n(t) = \binom{\alpha+n-1}{n}\left(\frac{b}{rt+b}\right)^{\alpha}\left(\frac{rt}{rt+b}\right)^{n}, \tag{13}
$$

which is another way of writing (10).

Whereas from a conceptual point of view, the identity (12) is simple and transparent, it still leaves open the question in what way the resulting distribution changes for the reinsured quantities, once the mixed Poisson distribution is calibrated to a given data set. One can either answer this question by investigating the structure of the mixing density directly or use the approach via holonomic functions: Each of the following examples of mixed Poisson distributions (taken from Willmot [29]) has a holonomic generating function. Therefore (5) can be applied to show that, due to the uniqueness of the probability generating function, for many concrete cases the time shift (and hence reinsurance thinning) leaves the claim number distribution invariant with just the involved parameters adjusted accordingly. Here is a set of concrete examples.

(i) When the mixing distribution is generalized inverse Gaussian with density

$$
f(x) = \frac{\mu^{-\lambda}x^{\lambda-1}\exp\left(-\left(x^2+\mu^2\right)/(2\beta x)\right)}{2\,K_\lambda(\mu\beta^{-1})}, \qquad x>0,
$$
$$
\mu,\beta>0, \qquad \lambda\in\mathbb{R},
$$

we obtain the *Sichel distribution* characterized by

$$
2\beta z\, Q_t''(z) + \mu^2\, Q_t(z) + 2\beta(\lambda+1)\, Q_t'(z) - (1+2\beta)Q_t''(z) = 0
$$

and from (5) it is readily verified that $\tilde{Q}_t(z)$ satisfies the same equation with $\tilde{\beta}=\beta r$ and $\tilde{\mu}=\mu r$.

(ii) If the mixing distribution is a Beta distribution with density

$$
f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{x^{\alpha-1}(\mu-x)^{\beta-1}}{\mu^{\alpha+\beta-1}}, \qquad 0<x<\mu,
$$
$$
\alpha,\beta,\mu>0,
$$

then $Q_t(z)$ satisfies

$$
(1-z)Q_t''(z) + Q_t'(z)(z\mu-(\alpha+\beta+\mu)) + \mu\alpha Q_t(z) = 0,
$$

and correspondingly $\tilde{Q}_t(z)$ satisfies the same equation with $\tilde{\mu}=\mu\, r$, $\tilde{\alpha}=\alpha$ and $\tilde{\beta}=\beta$.

(iii) If the mixing distribution is a transformed Beta distribution with density

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{c\,\mu^\alpha\, x^{c\beta-1}}{(\mu+x^c)^{\alpha+\beta}}, \qquad x>0,$$
$$\alpha,\beta,\mu>0, \qquad c\in\mathbb{N},$$

then $Q_t(z)$ satisfies

$$(z-1)Q_t^{(c+1)}(z) + c(1-\alpha)Q_t^{(c)}(z) + \mu(z-1)Q_t'(z) + c\beta\mu Q_t(z) = 0.$$

From (5) we immediately deduce that $\tilde{Q}_t(z)$ is of the same form with parameters $\tilde{\mu} = \mu r^c$, $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \beta$. Note that this example is fairly general, since for $c = 1$ we retrieve the case of a generalized Pareto mixing distribution (and further the Pareto case if in addition $\beta = 1$), and the Burr mixing distribution for $\beta = 1$ (and further the log-logistic distribution if in addition $\alpha = 1$).

(iv) If the mixing distribution is a transformed Gamma distribution with density

$$f(x) = \frac{\mu^\alpha\, c\, x^{c\alpha-1}\, e^{-\mu x^c}}{\Gamma(\alpha)}, \qquad x>0,$$
$$\mu,\alpha>0,\ c\in\mathbb{N},$$

then $Q_t(z)$ satisfies

$$(z-1)Q_t'(z) + c\alpha Q_t(z) - \mu c Q_t^{(c)}(z) = 0.$$

Correspondingly, $\tilde{Q}_t(z)$ is of the same form with $\tilde{\mu} = \mu/r^c$ and $\tilde{\alpha} = \alpha$. Again, several well-known cases are contained in this example as the Weibull case ($\alpha = 1$) and the Gamma mixing distribution ($c = 1$) leading to the Pascal distribution for p_n (cf. (13)).

(v) As another example in this context, we mention the inverse Gamma mixing distribution with density function

$$f(x) = \frac{\mu^\alpha x^{-(\alpha+1)} e^{-\mu/x}}{\Gamma(\alpha)}, \quad x>0, \qquad \mu,\alpha>0,$$

leading to

$$(z-1)Q_t''(z) + (1-\alpha)Q_t'(z) + \mu Q_t(z) = 0,$$

which also holds for $\tilde{Q}_t(z)$ with $\tilde{\mu} = \mu r$ and $\tilde{\alpha} = \alpha$.

(vi) Finally, for an exponential-inverse Gaussian mixing distribution with density

$$\mu(1+2\beta x)^{-1/2} e^{(\mu/\beta)(1-(1+2\beta x)^{1/2})}, \quad x>0, \qquad \mu,\beta>0,$$

we have

$$2\beta\left(1-2z+z^2\right)Q_t'(z) + \left(z^2-(2-\beta)z-\left(\mu^2+\beta-1\right)\right)Q_t(z) = \mu-\mu^2-\mu z.$$

The solution of the latter equation is also holonomic (since the inhomogeneous term can be removed by differentiating twice) and hence we can again apply formula (5) to deduce that $(\tilde{p}_n)$ has the same distribution as p_n with $\tilde{\mu} = \mu/r$ and $\tilde{\beta} = \beta/r$.

(3) *Infinitely divisible processes.*

If the claims are arriving with stationary and independent increments, then the resulting claim number process is an infinitely divisible process. The representation of the corresponding probability generating function looks as follows:

$$Q_t(z) = e^{-\lambda t(1-g(z))},$$

where $g(z)$ is the probability generating function of a discrete random variable — say G — with masses on the strictly positive integers $g_n = \mathsf{P}(G = n)$. As such the counting process could also be called a *discrete compound Poisson process.* The probabilities are given in the form

$$p_n(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} g_n^{*k},$$

where $\{g_n^{*k}; n = 1, 2, \ldots\}$ is the k-fold convolution of the distribution $\{g_n\}$ with probability generating function $g(z) = \mathsf{E}(z^G)$. For an interesting interpretation in an actuarial context, see [11]. Needless to say that the explicit evaluation of the above probabilities is mostly impossible because of the complicated nature of the convolutions.

We again look at what happens with the reinsurance process. From (2) we get

$$\tilde{Q}_t(z) = e^{-\lambda t(1-g((1-r)+rz))},$$

which can be rewritten in the form

$$\tilde{Q}_t(z) = e^{-\tilde{\lambda} t(1-\tilde{g}(z))},$$

where $\tilde{\lambda} = \lambda(1 - g(1-r))$ and

$$\tilde{g}(z) = \frac{g((1-r)+rz) - g(1-r)}{1 - g(1-r)}.$$

It does not take much to see that $\tilde{g}(z) = \mathsf{E}(z^{\tilde{G}})$, where $(m > 0)$

$$\tilde{g}_m := \mathsf{P}(\tilde{G} = m) = \frac{1}{1 - g(1-r)} \sum_{n=m}^{\infty} \binom{n}{m} (1-r)^{n-m} r^m \, \mathsf{P}(G = n).$$

The latter formula can in itself be interpreted as a thinning of the original variable G. We therefore see that the thinned process remains infinitely divisible.

Here are a few more explicit cases that have appeared in the actuarial literature.

(i) If $g(z) = z$, then we arrive at the Poisson process which appeared already in the Sundt–Jewell example.

(ii) As a further particular case one finds a *generalized Poisson–Pascal process* introduced by [9], where $g(z) = (1 - \beta(z-1))^{-\alpha}$ is the generating function of a Pascal variable. Again $\tilde{g}(z)$ is of (shifted) Pascal type with $\tilde{\beta} = r\beta$.

(iii) If $g(z) = (e^{\theta z} - 1)/(e^{\theta} - 1)$, then G is truncated Poisson and the corresponding claim process is of *Neyman type A*. The filtered process is of the same type but needs $\tilde{\theta} = r\theta$ as a new parameter.

(iv) If $g(z) = ((1-\theta)z)/(1-\theta z)$ with $0 < \theta < 1$, then G is truncated geometric. The generating function is

$$Q_t(z) = \exp\left\{-\lambda t \left(1 - \frac{(1-\theta)z}{1-\theta z}\right)\right\}.$$

The probabilities can be evaluated in terms of Laguerre polynomials

$$p_n(t) = \mathsf{P}\{N(t)=n\} = \begin{cases} e^{-\lambda t} & \text{if } n=0, \\ e^{-\lambda t}\theta^n L_n^{-1}\left(-\frac{\lambda t(1-\theta)}{\theta}\right) & \text{if } n \geq 1. \end{cases}$$

The latter counting process is called the *Pólya–Aeppli process*. The reinsurance process is again of the same type with $\tilde{\theta} = \theta r/(1-\theta(1-r))$.

(4) The *Sparre Andersen model*.

In renewal theory, filtering occurs when one deletes each time point of the renewal process with a fixed probability, independently of the renewal process. The thinned process is a counting process that jumps at the time points

$$T_1 = \inf\{i \geq 1 \colon Y_i = 1\}$$

and for $n \geq 1$, $T_{n+1} = \inf\{i > T_n \colon Y_i = 1\}$. It is worthwhile to remark that the thinned process $\{\tilde{W}_j := T_j - T_{j-1}; j \in \mathbb{N}\}$ with $T_0 = 0$ is also a renewal process but now generated by the mixture

$$\mathsf{P}\{\tilde{W}_1 \leq x\} = \sum_{j=1}^{\infty} r(1-r)^{j-1} F^{*j}(x).$$

3. An excess-of-loss chain

We extend the above procedure to a reinsurance chain where, depending on the size of the individual claims, a participating company takes up part of the responsibility. Let

$$u_0 := 0 < u_1 < u_2 < \cdots < u_{k-1} < \infty := u_k$$

be a sequence of values that break up the positive half-line in k disjoint pieces. Put $a_+ := \max(a, 0)$. Assume that a claim of size X occurs. The first line insurer takes up $\min(X, u_1)$, the second line insurer $\min((X-u_1)_+, u_2-u_1)$, etc.; the last is responsible for $(X-u_{k-1})_+$. Although for each participant in the chain, the number of claims to face can be calculated by the method of the previous section by choosing the appropriate value of r, we now extend this analysis to see how the $N(t)$ claims distribute over the k partners within the reinsurance chain. Let $N_i(t)$, $i=1,\ldots,k$, denote the number of claims for which the claim size ends up in the interval $(u_{i-1}, u_i]$. For any claim, the probability that this happens is equal to

$$r_i := F(u_i) - F(u_{i-1}), \qquad i = 1, 2, \ldots, k.$$

The results of the previous section correspond to $u_1 = u$, $u_2 = u+v$ and $k=3$. Note that $\sum_{i=1}^k r_i = 1$.

In the argument of Section 2 we replace the binomial distribution by a multinomial distribution to see that the probability generating function of the vector

$$\vec{N}(t) := (N_1(t), N_2(t), \ldots, N_k(t))$$

is given by

$$\tilde{Q}_t(z_1, z_2, \ldots, z_k) := \mathsf{E}\left[\prod_{i=1}^k z_i^{N_i(t)}\right] = Q_t(z_1 r_1 + z_2 r_2 + \cdots + z_k r_k) = Q_t(\vec{z}\cdot\vec{r}), \tag{14}$$

where $\vec{z}\cdot\vec{r}$ denotes the inner product of the two vectors $\vec{z} = (z_1, z_2, \ldots, z_k)$ and

$$\vec{r} = (r_1, r_2, \ldots, r_k).$$

As before it is easy to derive the first few moments of the quantities $\{N_i(t)\}$. We have

$$\mathsf{E}\, N_i(t) = r_i \,\mathsf{E}\, N(t),$$
$$\operatorname{Var}(N_i(t)) = r_i^2 \operatorname{Var}(N(t)) + r_i(1-r_i)\,\mathsf{E}\, N(t),$$

and for $i \neq j$,

$$\operatorname{Cov}(N_i(t), N_j(t)) = r_i r_j [\operatorname{Var}(N(t)) - \mathsf{E}\, N(t)]. \tag{15}$$

Recall that for the multinomial distribution covariances are negative. Here however the additional variability caused by the counting process $N(t)$ has an influence on the sign of the covariances. If the counting process is underdispersed, then the variance $\operatorname{Var} N(t)$ is lower than the expectation $\mathsf{E}\, N(t)$ and so one is *more certain* about the overall number of claims. The subdivision onto the various layers is such that a high number in one layer forces a low number in the other one to make up for the overall *almost fixed* sum, and hence the covariances remain negative. On the other hand, in the over-dispersed case where $\operatorname{Var} N(t) > \mathsf{E}\, N(t)$, the variability of the overall number of claims is so large that the realization of the sum $N(t)$ determines the *scale* of the subdivision and large values in one layer come together with large values in the other layer. In some sense, the change-point for the dominance of one effect over the other is just the Poisson case. Since we are only capturing dependence using the first two moments, the phenomenon is, while not fully surprising, simple and explicit.

Turning to examples, the corresponding adaptations are readily made. If we restrict attention to N_i of the single layer i, then $\tilde{Q}_i(z_i) := \tilde{Q}(1, 1, \ldots, z_i, \ldots, 1)$ and, due to

$$\vec{z} \cdot \vec{r} = r_1 + r_2 + \cdots + r_{i-1} + z_i r_i + r_{i+1} + \cdots + r_k = 1 - r_i + z_i r_i$$

in this case, we are back in the binomial case of Section 2 with $r = r_i$.

Since the i-th partner in the chain in fact faces a claim as soon as the individual claim exceeds u_{i-1}, the total number of claims for partner i is given by

$$\tilde{N}_i(t) = \sum_{l=i}^{k} N_l(t).$$

Using the abbreviation $r'_j := \sum_{l=j}^{k} r_l$, we immediately see that $\tilde{N}_i(t)$ can be identified with $\tilde{N}(t)$ from Section 2 with $r = r'_i = 1 - F(u_i)$ (just put $z_1 = z_2 = \cdots = z_i = 1$ and $z_{i+1} = \cdots = z_k = z$ in the generating function (14)) and all the corresponding distributional results carry over. In particular,

$$\mathsf{E}\big(\tilde{N}_i(t)\big) = r'_i\, \mathsf{E}(N(t))$$

and

$$\operatorname{Var}\big(\tilde{N}(t)\big) = r_i'^2 \operatorname{Var}(N(t)) + r'_i(1 - r'_i)\, \mathsf{E}(N(t)).$$

Moreover, the covariance between two partners i and j ($j > i$) in the chain can be determined:

$$\begin{aligned}
\operatorname{Cov}\big(\tilde{N}_i(t), \tilde{N}_j(t)\big) &= \sum_{l=j}^{k} \operatorname{Var}(N_l) + \sum_{l=i}^{k} \sum_{\substack{m=j \\ m \neq l}}^{k} \operatorname{Cov}(N_l, N_m) \\
&= r'_j (r'_i - r_{j-1}) \operatorname{Var}(N(t)) + r'_j (1 - r'_i + r_{j-1})\, \mathsf{E}(N(t)) \\
&= r'_j (r'_i - r_{j-1}) [\operatorname{Var}(N(t)) - \mathsf{E}(N(t))] + r'_j\, \mathsf{E}(N(t)).
\end{aligned}$$

The last line shows the influence of the quantities r'_j in the aggregate case.

4. The incurred claims

We now turn to the claim amounts that the individual partners in the reinsurance chain are carrying. An alternative to the approach of the previous section, where a separate claim number process for each partner in the chain was considered, is to partition each claim of the original process according to its contribution to the respective layers. For a claim X_j, let $Y_{j,i}$ denote the part that is shifted to the i-th partner in the chain. This gives rise to a vector $\vec{Y_j} := (Y_{j,1}, Y_{j,2}, \ldots, Y_{j,k})$, where

$$Y_{j,i} = \min((X_j - u_{i-1})_+, u_i - u_{i-1}), \qquad 1 \le i \le k,$$

with our interpretation $u_0 = 0$ and $u_k = \infty$. It is easy to check that $X_j = \sum_{i=1}^k Y_{j,i}$. We derive an expression for the vector Laplace transform of $\vec{Y_j}$. Let $\vec{s} := (s_1, s_2, \ldots, s_k)$. Then

$$\begin{aligned}
\mathsf{E}\left\{e^{-\vec{s}\cdot\vec{Y_j}}\right\} &= \int_0^{u_k} \mathsf{E}\left\{e^{-s_1 Y_{j,1} - s_2 Y_{j,2} - \cdots - s_k Y_{j,k}} \mid X = v\right\} dF(v) \\
&= \int_0^{u_1} \mathsf{E}\left\{e^{-s_1 X_j} \mid X_j = v\right\} dF(v) \\
&\quad + \int_{u_1}^{u_2} \mathsf{E}\left\{e^{-s_1 u_1 - s_2 (X_j - u_1)} \mid X_j = v\right\} dF(v) + \cdots \\
&\quad + \int_{u_{k-1}}^{\infty} \mathsf{E}\left\{e^{-s_1 u_1 - s_2(u_2 - u_1) - \cdots - s_{k-1}(u_{k-1} - u_{k-2}) - s_k (X_j - u_{k-1})} \mid X_j = v\right\} dF(v) \\
&= \sum_{i=1}^k \exp\left\{-\sum_{r=1}^{i-1} s_r (u_r - u_{r-1})\right\} \int_{u_{i-1}}^{u_i} e^{-s_i(v - u_{i-1})}\, dF(v).
\end{aligned}$$

4.1. **The aggregate claim amount in a reinsurance layer.** Now let

$$X_i(t) = \sum_{j=1}^{N(t)} Y_{j,i} \tag{16}$$

denote the total claim amount for the i-th partner. Then

$$\vec{s} \cdot \vec{X}(t) = \sum_{i=1}^k s_i X_i(t) = \sum_{j=1}^{N(t)} \sum_{i=1}^k s_i Y_{j,i}$$

for $\vec{X}(t) := (X_1(t), X_2(t), \ldots, X_k(t))$ and hence by a conditioning argument

$$\mathsf{E}\left\{e^{-\vec{s}\cdot\vec{X}(t)}\right\} = Q_t\left(\mathsf{E}\left\{e^{-\vec{s}\cdot\vec{Y}}\right\}\right).$$

It is easy to check that by taking $s_1 = s_2 = \cdots = s_k = s$ one retains (1). If we restrict attention to the i-th partner, then

$$\mathsf{E}\left\{e^{-s_i X_i(t)}\right\} = Q_t\big(\hat{F}_i(s_i)\big), \tag{17}$$

where $\hat{F}_i(\cdot)$ is the Laplace transform

$$\hat{F}_i(s) = F(u_{i-1}) + \int_{u_{i-1}}^{u_i} e^{-s(v-u_{i-1})}\, dF(v) + (1 - F(u_i)) e^{-s(u_i - u_{i-1})}. \tag{18}$$

With our conventions $u_0 = 0$ and $u_k = \infty$ the above expression is also valid for the first and for the last partner in the chain. We perform an integration by parts in (18) and obtain

$$\hat{F}_i(s) = 1 - s \int_0^{v_i} (1 - F(u_{i-1} + y))\, e^{-sy}\, dy, \tag{19}$$

where $v_i := u_i - u_{i-1}$ denotes the span of the range for which the i-th partner carries the responsibility. The last expression suggests to introduce an equilibrium distribution

$$\mathbf{F}_i(x) := \frac{1}{\mathsf{E}\,Y_i} \int_0^x (1 - F(u_{i-1} + y))\,dy$$

with Laplace transform

$$\hat{\mathbf{F}}_i(s) = \frac{1 - \hat{F}_i(s)}{s\,\mathsf{E}\,Y_i},$$

for then

$$\hat{F}_i(s) = 1 - \mathsf{E}\,Y_i s \int_0^{v_i} e^{-sy} d\mathbf{F}_i(s).$$

As an alternative to (16) with its distribution determined by (17) and (19), we can also write the total claim amount for the i-th partner ($i < k$) as

$$X_i(t) = \sum_{j=1}^{N_i(t)} \check{X}_j + v_i \tilde{N}_{i+1}(t), \tag{20}$$

where $\check{X}_j = X_j - u_{i-1} \mid u_{i-1} < X_j \leq u_i$ and it depends on the distribution F of the claim size X_j how tractable this alternative expression is.

4.2. **The Sundt–Jewell class.** For simplicity we return to the binomial case of Section 2. In [15] Panjer used the Sundt–Jewell recursion (6) to derive recursions for the total claim distribution. If the individual claim size distribution F has a derivative f, then the density $g_t(x) = g(x)$ is determined by the sum

$$g(x) = \sum_{n=0}^{\infty} p_n f^{*n}(x).$$

He showed that the following equation holds:

$$g(x) = \int_0^x g(x-y) \left\{a + b\frac{y}{x}\right\} f(y)\,dy + \delta f(x), \tag{21}$$

where $\delta = p_1 - (a+b)\rho$.

In the general case for F, there is always the possibility of a jump at the origin since

$$G_t(0) = \lim_{x \downarrow 0} \sum_{n=0}^{\infty} p_n(t) F^{*n}(x) = \sum_{n=0}^{\infty} p_n(t) F^n(0) = Q_t(F(0)).$$

Apart from that and as proved in [17], the distribution G satisfies the integral equation

$$G(x) = \int_0^x dF(y) \int_0^{x-y} \left\{a + \frac{by}{y+z}\right\} dG(z) + \delta F(x).$$

If F happens to have a density f, then we fall back on the Panjer equation. If F has a discrete distribution, then we find a discrete analogue of the Panjer equation.

Let us turn to the reinsured total claim amount. There is a possibility that the reinsurer does not incur any claims, in which case $\tilde{G}_t(0) = \tilde{Q}_t(\tilde{F}(0)) = Q_t(1-r)$. Referring to other results above we can infer that the total reinsured amount can be calculated recursively as

$$\tilde{G}(x) = \int_0^x d\tilde{F}(y) \int_0^{x-y} \left\{\tilde{a} + \frac{\tilde{b}y}{y+z}\right\} d\tilde{G}(z) + \tilde{\delta}\tilde{F}(x), \tag{22}$$

where $\tilde{\delta} = \tilde{p}_1 - (\tilde{a} + \tilde{b})\tilde{p}_0$. Here $\tilde{F}$ is the conditional distribution of X given that the claim exceeds u. More precisely

$$\tilde{F}(x) = \mathsf{P}(\min(X - u, v) \leq x \mid u < X).$$

Formula (22) is helpful in dealing with the *burning cost problem*. It should be clear that the key problem lies in the determination of the distribution $\tilde{F}$. Four cases seem to emerge naturally.

(1) For the reinsurer the worst case scenario is that he gets no information whatsoever neither on the counting process $\{N(t)\}$ nor the claim sizes $\{X_i\}$. In this situation the reinsurer faces exactly the same problems as the first insurer. This means that the reinsurer has to estimate $\tilde{a}$, $\tilde{b}$ and $\tilde{F}$ from the claims that are passed on to him by the first insurer.

(2) If the insurer provides information on $\{N(t)\}$, then the reinsurer knows the quantities a and b and he can therefore use the expressions (8) if he knows the quantity r. But by (2) both insurer and reinsurer have the same kind of information on the counting process once the value of r is known. If the reinsurer has sufficient experience with the practical aspects of the portfolio, then he might be able to get an estimate for r from relations of the type (3). Of course, the main assignment remains to get a reliable estimate of $\tilde{F}$. Again, experience and comparisons with similar portfolios should be helpful here. Of course, the reinsurer should realize that he can rely only on fewer data than the first insurer as the only available data are those that fall above the retention u.

(3) If the first insurer passes on information on the claim sizes $\{X_i\}$, then the estimation of $\tilde{F}$ should be simplified. However he will not gain too much insight into the process $\{\tilde{N}(t)\}$ since the only information on $\{N(t\}$ that is available will be its value at the time of the transaction. Also here, familiarity with the portfolio under consideration or similarity with other portfolios might provide crucial help in the estimation of $\tilde{a}$ and $\tilde{b}$. In the latter quantities the crucial quantity r appears too and so r needs to be estimated from the observed claim sizes.

(4) In the unlikely situation that the insurer passes on all information on both $\{N(t)\}$ and $\{X_i\}$, then also the estimation of the $\tilde{F}$ boils down to a standard actuarial activity.

4.3. **Some remarks on asymptotics.** In practice, it may not always be easy to evaluate the distribution of $X_i(t)$ through (16) (together with (17) and (19)) or, alternatively, through (20). However, asymptotic results for the tail of $X_i(t)$ can often be determined in a simple way using general analytic methods. For the last (i.e. k-th) partner in the chain, the ultimate tail of the distribution remains in play and we refer to the standard literature for the treatment of light-tailed and heavy-tailed claim sizes. For $i < k$, the Laplace transform $\hat{F}_i(s)$ is an entire function of s and we always end up in the light-tailed case. As surveyed in [17], such a condition is useful in a vast number of claim-counting situations. As a concrete example, consider the Pascal process (13) that appeared both as a mixed Poisson process and as a special case of the Sundt–Jewell class. Following [17], one can show that, for the i-th partner ($i < k$), asymptotically as $x \to \infty$,

$$\mathsf{P}(X_i(t) > x) \sim \left(\frac{b}{t\left|\hat{F}_i'(\theta_i)\right|}\right)^{\alpha} \frac{1}{|\theta_i|\Gamma(\alpha)} e^{-|\theta_i|x} x^{\alpha-1},$$

where $\theta_i < 0$ depends on t and is the unique value that satisfies $\hat{F}_i(\theta_i) = 1 + b/t$. Similarly, under the logarithmic distribution (11) from the Sundt–Jewell class one derives that, as $x \to \infty$,

$$\mathsf{P}(X_i(t) > x) \sim \big(-\log(1-a(t))|\theta_i|\big)^{-1} e^{-|\theta_i|x} x^{-1},$$

where $\theta_i < 0$ depends on t and is the unique solution of $\hat{F}_i(\theta_i) = 1/a(t)$.

For the above examples one can derive some more explicit information on the quantities $|\theta_i|$. Assume that we have to solve the equation $\hat{F}_i(\theta_i) = c$ for $c > 1$. Then from (19) one derives for $s < 0$ that

$$1 - (1 - e^{-sv_i})(1 - F(u_i)) \le \hat{F}_i(s) \le 1 - (1 - e^{-sv_i})(1 - F(u_{i-1})).$$

This then quickly leads to the inequalities

$$\log \frac{c - F(u_{i-1})}{1 - F(u_{i-1})} \le |\theta_i| v_i \le \log \frac{c - F(u_i)}{1 - F(u_i)}.$$

Of course, one cannot expect these inequalities to be sharp. Nevertheless they are quite instructive as they show how the unknown quantities $|\theta_i|v_i$ are ordered by the quantities

$$\tau_i := \log \frac{c - F(u_i)}{1 - F(u_i)}$$

that are either given or estimated from given data. Hence, $\tau_{i-1} \le |\theta_i| v_i \le \tau_i$ and the maximal length σ_i of the interval in which $|\theta_i|$ is located is given by

$$\sigma_i \le \frac{1}{v_i}(\tau_i - \tau_{i-1}) \le \frac{c-1}{v_i} \frac{r_i}{(1 - F(u_i))^2}.$$

In practice, there is often an upper limit of coverage in terms of m *reinstatements* of the i-th partner in the reinsurance chain; i.e., the actual coverage is given by

$$\overline{X}_i(t) = \min\{X_i(t), mv_i\}$$

with $m \in \mathbb{N}$. Most of the results in the literature on fair pricing of reinstatement contracts are based on the assumption that the claim number process $N(t)$ is in the Sundt–Jewell class (see for instance Sundt [22] or Mata [14]). The results in this paper in principle allow us to extend this type of analysis to more general situations. Moreover, the asymptotic results given above are particularly well-suited to determine the remaining risk for the i-th layer ($i < k$) staying at the first-line insurer in a contract with reinstatements, since the relevant value of x will then typically be large.

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Radon Institute, Austrian Academy of Sciences, Linz, Austria, and University of Linz, Altenbergerstrasse 69, A-4040 Linz, Austria

E-mail address: hansjoerg.albrecher@ricam.oeaw.ac.at

EURANDOM, Technische Universiteit Eindhoven, The Netherlands, and Katholieke Universiteit Leuven, Leuven Center for Statistics, Celestijnenlaan 200B, B-3001 Heverlee, Belgium

E-mail address: jef.teugels@wis.kuleuven.be

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