

**PARAMETRIC ESTIMATION FOR LINEAR SYSTEM
OF STOCHASTIC DIFFERENTIAL EQUATIONS
DRIVEN BY FRACTIONAL BROWNIAN MOTIONS
WITH DIFFERENT HURST INDICES**

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ABSTRACT. We consider the problem of maximum likelihood estimation of the common trend parameter for a linear system of stochastic differential equations driven by two independent fractional Brownian motions possibly with different Hurst indices. Asymptotic properties of the maximum likelihood estimator are discussed.

1. INTRODUCTION

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by a Wiener process is now well known [6]. There has been some recent interest to study similar problems for stochastic processes driven by fractional Brownian motion (fBm) in view of their applications to modelling of internet traffic and study of fluctuation in market share with long range dependence. Properties of maximum likelihood estimators for linear stochastic differential equations driven by fractional Brownian motion (fBm) are investigated in [1] and [8, 9, 10]. Geometric Brownian motion has been widely used for modelling fluctuations of share prices in the stock market, and geometric fractional Brownian motion, that is, a process governed by a stochastic differential equation of the type

$$(1.1) \quad dX(t) = \theta X(t) dt + \sigma_1 X(t) dW^h(t), \quad X(0) = x_0 \in \mathbb{R}, \quad 0 \leq t \leq T,$$

has also been studied as a model for modelling fluctuations in share prices in the stock market in the presence of long range dependence. In the present scenario where the fluctuations in share prices in one country are influenced by the same in another country or within the same country from different regions, it is reasonable to model the share prices by a system of stochastic differential equations driven by noise components coming from different environments which could be dependent or independent.

We discuss estimation of the trend for a linear system of stochastic differential equations in Section 2 and specialize the results to a linear system of geometric fractional Brownian motion in Section 3.

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2. GENERAL CASE

Let (Ω, \mathcal{F}, P) be a complete probability space with a filtration $\{\mathcal{F}_t, t \geq 0\}$. Consider the linear stochastic differential system

$$(2.1) \quad \begin{aligned} dX(t) &= \theta a_1(t, X(t)) dt + b_1(t, X(t)) dW^H(t), & X(0) &= x_0 \in \mathbb{R}, \quad 0 \leq t \leq T_1, \\ dY(t) &= \theta a_2(t, Y(t)) dt + b_2(t, Y(t)) dW^h(t), & Y(0) &= y_0 \in \mathbb{R}, \quad 0 \leq t \leq T_2, \end{aligned}$$

where $\theta \in \Theta \subset \mathbb{R} \setminus \{0\}$. The functions b_1 and b_2 are assumed to be known and non-vanishing. We further assume that the functions a_1 and a_2 are also known and that the fractional Brownian motions $\{W^h(t), 0 \leq t \leq T\}$ and $\{W^H(t), 0 \leq t \leq T\}$ with known Hurst indices $h \in [\frac{1}{2}, 1)$ and $H \in [\frac{1}{2}, 1)$ are independent and adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$. The latter can be achieved if necessary by choosing \mathcal{F}_t to be the σ -algebra generated by the family

$$\{W^h(s), 0 \leq s \leq t; W^H(s), 0 \leq s \leq t\}.$$

We use pathwise construction of the stochastic integral with respect to the fBm discussed in [12]. Suppose that system (2.1) has a unique pathwise solution

$$\{X(s), 0 \leq s \leq T_1; Y(s), 0 \leq s \leq T_2\}.$$

Sufficient conditions for the existence and uniqueness of the solution are given in [5]. In addition to these conditions, we assume that

$$\frac{a_i(t, X(t))}{b_i(t, X(t))}$$

is Lebesgue integrable on $[0, T_i]$, $i = 1, 2$. Let P_t^Z be the measure generated by a process Z on $[0, t]$. We will now calculate the Radon–Nikodym derivative for probability measures Q on (Ω, \mathcal{F}) such that P_t^X is equivalent to Q_t^X , $0 \leq t \leq T_1$, and the process X has zero drift under the measure Q . Let

$$\phi_t \equiv \psi(t, X(t)) = -\theta \frac{a_1(t, X(t))}{b_1(t, X(t))}.$$

Define, for $0 < s < t \leq T_1$,

$$k_H(t, s) = \begin{cases} \kappa_H^{-1} s^{1/2-H} (t-s)^{1/2-H}, & 0 \leq s \leq t, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$(2.2) \quad \kappa_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right).$$

Suppose $\psi(t, x) \in C^{1,2}([0, T_1] \times \mathbb{R})$. Then, by [3, Lemma 1], there exists another \mathcal{F}_t -predictable process $\{\delta_s, 0 \leq s \leq T_1\}$ such that

$$(2.3) \quad \int_0^t \delta_s ds < \infty \quad \text{P-a.s.}, \quad 0 \leq t \leq T_1,$$

and

$$(2.4) \quad J_t := \int_0^t k_H(t, s) \phi_s ds = \int_0^t \delta_s ds, \quad 0 \leq t \leq T_1.$$

It is proved in [4] that

$$(2.5) \quad \int_0^t k_H(t, s) dW_s^H = \int_0^t s^{1/2-H} d\widetilde{W}_s, \quad 0 \leq t \leq T_1,$$

where the stochastic integral on the left side exists as a pathwise integral with respect to the fBm W^H ,

$$\widetilde{W}_t = \int_0^t s^{H-1/2} d\widetilde{M}_s, \quad 0 \leq t \leq T_1,$$

and

$$\widetilde{M}_t = \int_0^t k_H(t, s) dW_s^H, \quad 0 \leq t \leq T_1.$$

Furthermore the process $\{\widetilde{W}_s, 0 \leq t \leq T_1\}$ is a standard Wiener process. Suppose that

$$(2.6) \quad \mathbb{E} \left(\int_0^{T_1} s^{2H-1} \delta_s^2 ds \right) < \infty.$$

Define

$$\widetilde{L}_t = \int_0^t s^{H-1/2} \delta_s d\widetilde{W}_s, \quad 0 \leq t \leq T_1.$$

Under assumption (2.6), the process $\{\widetilde{L}_t, \mathcal{F}_t, 0 \leq t \leq T_1\}$ is a square integrable martingale. Suppose the martingale $\{\widetilde{L}_t, \mathcal{F}_t, 0 \leq t \leq T_1\}$ satisfies the condition

$$\mathbb{E} \left[\exp \left\{ \widetilde{L}_t - \frac{1}{2} \langle \widetilde{L} \rangle_t \right\} \right] = 1, \quad 0 \leq t \leq T_1.$$

Then it is known that the process

$$B_t^H = W_t^H - \int_0^t \phi_s ds, \quad 0 \leq t \leq T_1,$$

is an fBm with respect to the probability measure Q_H defined on (Ω, \mathcal{F}) by

$$\frac{dQ_H}{dP} \Big|_{\mathcal{F}_t} = \exp \left\{ \widetilde{L}_t - \frac{1}{2} \langle \widetilde{L} \rangle_t \right\}, \quad 0 \leq t \leq T_1.$$

Note that

$$\frac{dQ_H}{dP} \Big|_{\mathcal{F}_t} = \exp \left\{ \int_0^t s^{H-1/2} \delta_s d\widetilde{W}_s - \frac{1}{2} \int_0^t s^{2H-1} \delta_s^2 ds \right\}, \quad 0 \leq t \leq T_1.$$

Analogously to the above discussion, we construct another probability measure Q_h defined on (Ω, \mathcal{F}) such that

$$\frac{dQ_h}{dP} \Big|_{\mathcal{F}_t} = \exp \left\{ \int_0^t s^{h-1/2} \psi_s d\widehat{W}_s - \frac{1}{2} \int_0^t s^{2h-1} \psi_s^2 ds \right\}, \quad 0 \leq t \leq T_2.$$

Here \widehat{W} is the Wiener process corresponding to the fBm W^h , and $\{\psi_t, 0 \leq t \leq T_2\}$ and $\{\eta_t, 0 \leq t \leq T_2\}$ are processes such that

$$(2.7) \quad I_t := \int_0^t k_h(t, s) \eta_s ds = \int_0^t \psi_s ds, \quad 0 \leq t \leq T_2.$$

Observe that

$$B_t^h = W_t^h - \int_0^t \eta_s ds, \quad 0 \leq t \leq T_2,$$

is an fBm with respect to the probability measure Q_h defined by

$$\frac{dQ_h}{dP} \Big|_{\mathcal{F}_t} = \exp \left\{ \int_0^t s^{h-1/2} \psi_s d\widehat{W}_s - \frac{1}{2} \int_0^t s^{2h-1} \psi_s^2 ds \right\},$$

$$0 \leq t \leq T_2.$$

With respect to the probability measures Q_H and Q_h , the trend term should be zero for the first equation in the system over the interval $[0, T_1]$, and it should be zero for the second equation in the system over the interval $[0, T_2]$. Hence

$$b_1(t, X(t))\phi_t = -\theta a_1(t, X(t))$$

and

$$b_2(t, Y(t))\eta_t = -\theta a_2(t, Y(t)).$$

Observe that (here g'_t denotes the derivative of g with respect to t evaluated at t)

$$\begin{aligned} \frac{dQ}{dP} &= \frac{dQ_h}{dP} \Big|_{\mathcal{F}_{T_2}} \frac{dQ_H}{dP} \Big|_{\mathcal{F}_{T_1}} \\ &= \exp \left\{ \int_0^{T_2} s^{h-1/2} J'_s d\widehat{W}_s - \frac{1}{2} \int_0^{T_2} s^{2h-1} J_s'^2 ds \right\} \\ &\quad \times \exp \left\{ \int_0^{T_1} s^{H-1/2} I'_s d\widetilde{W}_s - \frac{1}{2} \int_0^{T_1} s^{2H-1} I_s'^2 ds \right\} \\ &= \exp \left\{ \int_0^{T_2} s^{h-1/2} J'_s d\widehat{W}_s + \int_0^{T_1} s^{H-1/2} I'_s d\widetilde{W}_s \right. \\ &\quad \left. - \frac{1}{2} \left[\int_0^{T_2} s^{2h-1} J_s'^2 ds + \int_0^{T_1} s^{2H-1} I_s'^2 ds \right] \right\}. \end{aligned}$$

Note that

$$\begin{aligned} J'_t &= \left(\int_0^t k_h(t, s) \eta_s ds \right)'_t \\ &= \left(\int_0^t k_h(t, s) \frac{-\theta a_2(s, Y(s))}{b_2(s, Y(s))} ds \right)'_t \\ &= -\theta \left(\int_0^t k_h(t, s) \frac{a_2(s, Y(s))}{b_2(s, Y(s))} ds \right)'_t \\ &=: -\theta \Delta_t. \end{aligned}$$

Similarly

$$I'_t = -\theta \left(\int_0^t k_H(t, s) \frac{a_1(s, X(s))}{b_1(s, X(s))} ds \right)'_t =: -\theta \Gamma_t.$$

From the above relations, we get that

$$\begin{aligned} \log \frac{dQ}{dP} &= \int_0^{T_2} s^{h-1/2} J'_s d\widehat{W}_s + \int_0^{T_1} s^{H-1/2} I'_s d\widetilde{W}_s \\ &\quad - \frac{1}{2} \left[\int_0^{T_1} s^{2h-1} J_s'^2 ds + \int_0^{T_2} s^{2H-1} I_s'^2 ds \right] \\ &= \int_0^{T_2} s^{h-1/2} (-\theta \Delta_s) d\widehat{W}_s + \int_0^{T_1} s^{H-1/2} (-\theta \Gamma_s) d\widetilde{W}_s \\ &\quad - \frac{1}{2} \left[\int_0^{T_2} s^{2h-1} (-\theta \Delta_s)^2 ds + \int_0^{T_1} s^{2H-1} (-\theta \Gamma_s)^2 ds \right]. \end{aligned}$$

In order to estimate the parameter θ based on the observations of the process

$$\{X(s), 0 \leq s \leq T_1\}$$

and of the process $\{Y(s), 0 \leq s \leq T_2\}$, we maximize the function dQ/dP or equivalently $\log(dQ/dP)$. Differentiating the function $\log(dQ/dP)$ with respect to θ and equating the derivative to zero, we obtain the likelihood equation

$$\begin{aligned} & \theta \left[\int_0^{T_1} s^{2H-1} \Gamma_s^2 ds + \int_0^{T_2} s^{2h-1} \Delta_s^2 ds \right] \\ &= - \left[\int_0^{T_1} s^{H-1/2} \Gamma_s d\widetilde{W}_s + \int_0^{T_2} s^{h-1/2} \Delta_s d\widehat{W}_s \right]. \end{aligned}$$

The solution $\widehat{\theta}_{T_1, T_2}$ of this equation is given by

$$(2.8) \quad \widehat{\theta}_{T_1, T_2} = - \frac{\left[\int_0^{T_1} s^{H-1/2} \Gamma_s d\widetilde{W}_s + \int_0^{T_2} s^{h-1/2} \Delta_s d\widehat{W}_s \right]}{\left[\int_0^{T_1} s^{2H-1} \Gamma_s^2 ds + \int_0^{T_2} s^{2h-1} \Delta_s^2 ds \right]},$$

which is the maximum likelihood estimator in this general case. It can be checked that (see [3, Eq. (14)])

$$d\widetilde{W}_s = dW_s^{(1)} - \theta s^{H-1/2} \Gamma(s) ds$$

and

$$d\widehat{W}_s = dW_s^{(2)} - \theta s^{h-1/2} \Delta(s) ds,$$

where $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes under the measure Q . Using these relations, it follows that

$$(2.9) \quad \widehat{\theta}_{T_1, T_2} - \theta = - \frac{\left[\int_0^{T_1} s^{H-1/2} \Gamma_s dW_s^{(1)} + \int_0^{T_2} s^{h-1/2} \Delta_s dW_s^{(2)} \right]}{\left[\int_0^{T_1} s^{2H-1} \Gamma_s^2 ds + \int_0^{T_2} s^{2h-1} \Delta_s^2 ds \right]}.$$

Let

$$J(T_1, T_2) \equiv \int_0^{T_1} s^{2H-1} \Gamma_s^2 ds + \int_0^{T_2} s^{2h-1} \Delta_s^2 ds.$$

Then we get that

$$J(T_1, T_2)(\widehat{\theta}_{T_1, T_2} - \theta) = - \left[\int_0^{T_1} s^{H-1/2} \Gamma_s dW_s^{(1)} + \int_0^{T_2} s^{h-1/2} \Delta_s dW_s^{(2)} \right].$$

Define

$$(2.10) \quad M^{(1)}(t) = \int_0^t s^{H-1/2} \Gamma_s dW_s^{(1)}, \quad 0 \leq t \leq T_1,$$

$$(2.11) \quad M^{(2)}(t) = \int_0^t s^{h-1/2} \Delta_s dW_s^{(2)}, \quad 0 \leq t \leq T_2,$$

and let $\mathbf{M}(T_1, T_2)$ be the diagonal matrix with the diagonal elements

$$(M^{(1)}(T_1), M^{(2)}(T_2)).$$

Note that $M^{(i)}$, $i = 1, 2$, are independent continuous local martingales with quadratic variations

$$\langle M^{(1)} \rangle_t = \int_0^t s^{2H-1} \Gamma_s^2 ds, \quad 0 \leq t \leq T_1,$$

and

$$\langle M^{(2)} \rangle_t = \int_0^t s^{2h-1} \Delta_s^2 ds, \quad 0 \leq t \leq T_2.$$

Let $\langle \mathbf{M} \rangle_{t,u}$ be a diagonal matrix with the diagonal elements $(\langle M^{(1)} \rangle_t, \langle M^{(2)} \rangle_u)$. Suppose there exists a vector-valued function (k_{1,t_1}, k_{2,t_2}) such that $k_{i,t_i} > 0$, $i = 1, 2$, increasing to

infinity as $t_i \rightarrow \infty$. Let $K_{t,u}$ be a diagonal matrix with the diagonal elements $(k_{1,t}, k_{2,u})$. Suppose that

$$(2.12) \quad K_{T_1, T_2}^{-1} \mathbf{M}(T_1, T_2) K_{T_1, T_2}^{-1} \xrightarrow{P} \eta^2 \quad \text{as } T_1 \text{ and } T_2 \rightarrow \infty,$$

where η^2 is a random positive diagonal matrix. Following the results in [7, Theorem 1.50] and [11, Theorem A.1], it follows that

$$K_{T_1, T_2}^{-1} \langle \mathbf{M} \rangle_{T_1, T_2} \xrightarrow{L} \mathbf{Z} \eta \quad \text{as } T_1 \text{ and } T_2 \rightarrow \infty,$$

where \mathbf{Z} is a diagonal matrix with the diagonal elements independent standard normal random variables and the random matrices \mathbf{Z} and η are independent. As a consequence of this result, we can give a set of sufficient conditions for asymptotic normality of the estimator $\hat{\theta}_{T_1, T_2}$ as $T_1 \rightarrow \infty$ and $T_2 \rightarrow \infty$. We will now discuss a special case.

Special case. The asymptotic properties of the estimator $\hat{\theta}_{T_1, T_2}$ depend on the processes Δ_s and Γ_s which in turn depend on the functions a_1, b_1, a_2, b_2 , the process $\{X(s), 0 \leq s \leq T_1\}$ and the process $\{Y(s), 0 \leq s \leq T_2\}$. However, if these functions and the processes X and Y are such that $\Delta_s = \alpha s^{1-2h}$ and $\Gamma_s = \beta s^{1-2H}$ for some constants α and β , then it follows that

$$(2.13) \quad \begin{aligned} \hat{\theta}_{T_1, T_2} - \theta &= - \frac{\left[\int_0^{T_1} s^{H-1/2} \beta s^{1-2H} dW_s^{(1)} + \int_0^{T_2} s^{h-1/2} \alpha s^{1-2h} dW_s^{(2)} \right]}{\left[\int_0^{T_1} s^{2H-1} \beta^2 s^{2-4H} ds + \int_0^{T_2} s^{2h-1} \alpha^2 s^{2-4h} ds \right]} \\ &= - \frac{\left[\beta \int_0^{T_1} s^{1/2-H} dW_s^{(1)} + \alpha \int_0^{T_2} s^{1/2-h} dW_s^{(2)} \right]}{\left[\beta^2 \int_0^{T_1} s^{1-2H} ds + \alpha^2 \int_0^{T_2} s^{1-2h} ds \right]} \\ &= - \frac{\left[\beta \int_0^{T_1} s^{1/2-H} dW_s^{(1)} + \alpha \int_0^{T_2} s^{1/2-h} dW_s^{(2)} \right]}{\left[\beta^2 T_1^{2-2H} (2-2H)^{-1} + \alpha^2 T_2^{2-2h} (2-2h)^{-1} \right]}. \end{aligned}$$

Since the processes $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes, it is easy to see that the estimator $\hat{\theta}_{T_1, T_2}$ has a normal distribution with mean θ and variance

$$\left[\beta^2 T_1^{2-2H} (2-2H)^{-1} + \alpha^2 T_2^{2-2h} (2-2h)^{-1} \right]^{-1}.$$

It is clear that the processes $\Delta_s = \alpha s^{1-2h}$ and $\Gamma_s = \beta s^{1-2H}$ for some constants α and β if $a_1 = \beta b_1$ and $a_2 = \alpha b_2$ hold.

3. GEOMETRIC FRACTIONAL BROWNIAN MOTION

We now specialize the results in Section 2 to a linear system generated by geometric fractional Brownian motions.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. Consider the linear system of stochastic differential equations

$$(3.1) \quad \begin{aligned} dX(t) &= \theta X(t) dt + \sigma_1 X(t) dW^H(t), & X(0) &= x_0 \in \mathbb{R}, \quad 0 \leq t \leq T_1, \\ dY(t) &= \theta Y(t) dt + \sigma_2 Y(t) dW^h(t), & Y(0) &= y_0 \in \mathbb{R}, \quad 0 \leq t \leq T_2, \end{aligned}$$

defined on $(\Omega, \mathcal{F}, \mathbf{P})$, where $\{\theta, \sigma_1, \sigma_2\} \subset \mathbb{R} \setminus \{0\}$; the fractional Brownian motions

$$\{W^h(t), 0 \leq t \leq T_1\} \quad \text{and} \quad \{W^H(t), 0 \leq t \leq T_2\}$$

with known Hurst indices $h \in [\frac{1}{2}, 1)$ and $H \in [\frac{1}{2}, 1)$ respectively are independent. We further assume that the parameters σ_1 and σ_2 are known positive constants.

Following the notation introduced in Section 2, let Q be the product measure of the probability measures Q^h induced on $C[0, T_1]$ and Q^H induced on $C[0, T_2]$. With respect

to this probability measure Q , the processes $\{\phi_s\}$ and $\{\eta_s\}$ chosen above should be such that the trend term should be zero in order that the process

$$\{X(s), 0 \leq s \leq T_1\}$$

be a solution of the first stochastic differential equation in the system given by (3.1) on the interval $[0, T_1]$ and the trend term should be zero in order that the process

$$\{Y(s), 0 \leq s \leq T_2\}$$

be a solution of the second stochastic differential equation in the system given by (3.1) on the interval $[0, T_2]$. Hence

$$\sigma_1 \int_0^t \phi_s ds = -\theta t, \quad 0 \leq t \leq T_1,$$

and

$$\sigma_2 \int_0^t \eta_s ds = -\theta t, \quad 0 \leq t \leq T_2.$$

As before,

$$\begin{aligned} \frac{dQ}{dP} &= \frac{dQ_H}{dP} \Big|_{\mathcal{F}_{T_1}} \frac{dQ_h}{dP} \Big|_{\mathcal{F}_{T_2}} \\ &= \exp \left\{ \int_0^{T_1} s^{H-1/2} \delta_s d\widetilde{W}_s - \frac{1}{2} \int_0^{T_1} s^{2H-1} \delta_s^2 ds \right\} \\ (3.2) \quad &\times \exp \left\{ \int_0^{T_2} s^{h-1/2} \psi_s d\widehat{W}_s - \frac{1}{2} \int_0^{T_2} s^{2h-1} \psi_s^2 ds \right\} \\ &= \exp \left\{ \int_0^{T_2} s^{h-1/2} \psi_s d\widehat{W}_s + \int_0^{T_1} s^{H-1/2} \delta_s d\widetilde{W}_s \right. \\ &\quad \left. - \frac{1}{2} \left[\int_0^{T_2} s^{2h-1} \psi_s^2 ds + \int_0^{T_1} s^{2H-1} \delta_s^2 ds \right] \right\}. \end{aligned}$$

Note that

$$(3.3) \quad \delta_t = \left(\int_0^t k_H(t, s) \phi_s ds \right)'_t = -\frac{\theta}{\sigma_1} \left(\int_0^t k_H(t, s) ds \right)'_t$$

and

$$(3.4) \quad \psi_t = \left(\int_0^t k_h(t, s) \eta_s ds \right)'_t = -\frac{\theta}{\sigma_2} \left(\int_0^t k_h(t, s) ds \right)'_t.$$

It is easy to see, from the computations given in [4], that

$$\int_0^t k_H(t, s) ds = D_H^2 t^{2-2H},$$

where

$$D_H = \frac{C_H}{2H(2-2H)^{1/2}}$$

and

$$C_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{1/2}.$$

Hence

$$\left(\int_0^t k_H(t, s) ds \right)'_t = D_H^2 (2-2H)t^{1-2H}.$$

Let

$$\gamma_H(t) = D_H^2(2 - 2H)t^{1-2H}.$$

It can be checked that

$$\gamma_H(t) = \frac{\Gamma(\frac{3}{2} - H)}{2H\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}t^{1-2H} =: J_H t^{1-2H}.$$

From the above relations, we get that

$$\begin{aligned} \log \frac{dQ}{dP} &= -\frac{\theta}{\sigma_1} \int_0^{T_1} s^{H-1/2} \gamma_H(s) d\widetilde{W}_s - \frac{\theta}{\sigma_2} \int_0^{T_2} s^{h-1/2} \gamma_h(s) d\widehat{W}_s \\ (3.5) \quad &\quad - \frac{\theta^2}{2\sigma_1^2} \int_0^{T_1} s^{2H-1} \gamma_H^2(s) ds - \frac{\theta^2}{2\sigma_2^2} \int_0^{T_2} s^{2h-1} \gamma_h^2(s) ds. \end{aligned}$$

Estimation. Note that the Radon–Nikodym derivative dQ/dP obtained above is the Radon–Nikodym derivative of the product measure of the probability measure generated by the process $\{X(s), 0 \leq s \leq T_1\}$ on the space $C[0, T_1]$ and the probability measure generated by the independent process $\{Y(s), 0 \leq s \leq T_2\}$. In order to estimate the parameter θ based on the observation of the process $\{X(s), 0 \leq s \leq T_1\}$ and of the process $\{Y(s), 0 \leq s \leq T_2\}$, we maximize the likelihood function dQ/dP or equivalently $\log(dQ/dP)$. Differentiating the function $\log(dQ/dP)$ with respect to θ and equating the derivative to zero, we obtain the likelihood equation

$$\theta \left[\frac{J_H^2 T_1^{2-2H}}{\sigma_1^2 2 - 2H} + \frac{J_h^2 T_2^{2-2h}}{\sigma_2^2 2 - 2h} \right] = - \left[\frac{J_H}{\sigma_1} \int_0^{T_1} s^{1/2-H} d\widetilde{W}_s + \frac{J_h}{\sigma_2} \int_0^{T_2} s^{1/2-h} d\widehat{W}_s \right],$$

which leads to the estimator, hereafter called the maximum likelihood estimator (MLE), given by

$$\widehat{\theta}_{T_1, T_2} = - \frac{\left[\frac{J_H}{\sigma_1} \int_0^{T_1} s^{1/2-H} d\widetilde{W}_s + \frac{J_h}{\sigma_2} \int_0^{T_2} s^{1/2-h} d\widehat{W}_s \right]}{\left[\frac{J_H^2 T_1^{2-2H}}{\sigma_1^2 2 - 2H} + \frac{J_h^2 T_2^{2-2h}}{\sigma_2^2 2 - 2h} \right]}.$$

It can be checked that (see [3, Eq. (14)])

$$d\widetilde{W}_s = dW_s^{(1)} - \frac{\theta}{\sigma_1} s^{H-1/2} \gamma_H(s) ds$$

and

$$d\widehat{W}_s = dW_s^{(2)} - \frac{\theta}{\sigma_2} s^{h-1/2} \gamma_h(s) ds,$$

where $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes under the measure Q . Using these relations, it follows that

$$\widehat{\theta}_{T_1, T_2} - \theta = - \frac{\left[\frac{J_H}{\sigma_1} \int_0^{T_1} s^{1/2-H} dW_s^{(1)} + \frac{J_h}{\sigma_2} \int_0^{T_2} s^{1/2-h} dW_s^{(2)} \right]}{\left[\frac{J_H^2 T_1^{2-2H}}{\sigma_1^2 2 - 2H} + \frac{J_h^2 T_2^{2-2h}}{\sigma_2^2 2 - 2h} \right]}.$$

In particular, it follows that the estimator $\widehat{\theta}_{T_1, T_2} - \theta$ has the normal distribution with mean zero and variance

$$\left[\frac{J_H^2 T_1^{2-2H}}{\sigma_1^2 2 - 2H} + \frac{J_h^2 T_2^{2-2h}}{\sigma_2^2 2 - 2h} \right]^{-1}.$$

Suppose $h \geq H$. Further suppose that we observe the process X governed by the first equation in the system up to time $T_1 = T$ and observe the process Y governed by the second equation in the system up to time

$$T_2 = T^{\frac{1-H}{1-h}}.$$

Then the variance of the MLE is given by

$$\left[\frac{J_H^2 T_1^{2-2H}}{\sigma_1^2 2-2H} + \frac{J_h^2 T_2^{2-2h}}{\sigma_2^2 2-2h} \right]^{-1},$$

which is of the order $O(T^{2H-2})$. A better estimator with smaller variance can be obtained by suitably choosing $T_1 = T$ and

$$T_2 = cT^{\frac{1-H}{1-h}},$$

where c is defined by the relation

$$\frac{J_H^2 T^{2-2H}}{\sigma_1^2 2-2H} = c^{2-2h} \frac{J_h^2 T_2^{2-2h}}{\sigma_2^2 2-2h}.$$

Remarks. The methods of this paper can be extended to study the problem of estimation of the parameter θ for more general linear systems of the type

$$\begin{aligned} dX_i(t) &= [\theta a_i(t, X_i(t)) + c_i(t, X_i(t))] dt + b_i(t, X_i(t)) dW^{H_i}(t), \\ 0 \leq t \leq T_i, \quad 1 \leq i \leq n. \end{aligned}$$

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