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PARAMETRIC ESTIMATION FOR LINEAR SYSTEM OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS WITH DIFFERENT HURST INDICES

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Abstract. We consider the problem of maximum likelihood estimation of the common trend parameter for a linear system of stochastic differential equations driven by two independent fractional Brownian motions possibly with different Hurst indices. Asymptotic properties of the maximum likelihood estimator are discussed.

1. Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by a Wiener process is now well known [6]. There has been some recent interest to study similar problems for stochastic processes driven by fractional Brownian motion (fBm) in view of their applications to modelling of internet traffic and study of fluctuation in market share with long range dependence. Properties of maximum likelihood estimators for linear stochastic differential equations driven by fractional Brownian motion (fBm) are investigated in [1] and [8, 9, 10]. Geometric Brownian motion has been widely used for modelling fluctuations of share prices in the stock market, and geometric fractional Brownian motion, that is, a process governed by a stochastic differential equation of the type

$$dX(t) = \theta X(t)\,dt + \sigma_1 X(t)\,dW^h(t), \qquad X(0) = x_0 \in \mathbb{R},\ 0 \le t \le T, \tag{1.1}$$

has also been studied as a model for modelling fluctuations in share prices in the stock market in the presence of long range dependence. In the present scenario where the fluctuations in share prices in one country are influenced by the same in another country or within the same country from different regions, it is reasonable to model the share prices by a system of stochastic differential equations driven by noise components coming from different environments which could be dependent or independent.

We discuss estimation of the trend for a linear system of stochastic differential equations in Section 2 and specialize the results to a linear system of geometric fractional Brownian motion in Section 3.

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2. General case

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t, t \geq 0\}$. Consider the linear stochastic differential system

$$\begin{aligned} dX(t) &= \theta a_1(t, X(t))\, dt + b_1(t, X(t))\, dW^H(t), \qquad X(0) = x_0 \in \mathbb{R},\ 0 \leq t \leq T_1, \\ dY(t) &= \theta a_2(t, Y(t))\, dt + b_2(t, Y(t))\, dW^h(t), \qquad Y(0) = y_0 \in \mathbb{R},\ 0 \leq t \leq T_2, \end{aligned} \tag{2.1}$$

where $\theta \in \Theta \subset \mathbb{R} \setminus \{0\}$. The functions b_1 and b_2 are assumed to be known and non-vanishing. We further assume that the functions a_1 and a_2 are also known and that the fractional Brownian motions $\{W^h(t), 0 \leq t \leq T\}$ and $\{W^H(t), 0 \leq t \leq T\}$ with known Hurst indices $h \in [\frac{1}{2}, 1)$ and $H \in [\frac{1}{2}, 1)$ are independent and adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$. The latter can be achieved if necessary by choosing $\mathcal{F}_t$ to be the σ-algebra generated by the family

$$\left\{W^h(s),\ 0 \leq s \leq t;\ W^H(s),\ 0 \leq s \leq t\right\}.$$

We use pathwise construction of the stochastic integral with respect to the fBm discussed in [12]. Suppose that system (2.1) has a unique pathwise solution

$$\{X(s),\ 0 \leq s \leq T_1;\ Y(s),\ 0 \leq s \leq T_2\}.$$

Sufficient conditions for the existence and uniqueness of the solution are given in [5]. In addition to these conditions, we assume that

$$\frac{a_i(t, X(t))}{b_i(t, X(t))}$$

is Lebesgue integrable on $[0, T_i]$, $i = 1, 2$. Let P_t^Z be the measure generated by a process Z on $[0, t]$. We will now calculate the Radon–Nikodym derivative for probability measures Q on $(\Omega, \mathcal{F})$ such that P_t^X is equivalent to Q_t^X, $0 \leq t \leq T_1$, and the process X has zero drift under the measure Q. Let

$$\phi_t \equiv \psi(t, X(t)) = -\theta \frac{a_1(t, X(t))}{b_1(t, X(t))}.$$

Define, for $0 < s < t \leq T_1$,

$$k_H(t, s) = \begin{cases} \kappa_H^{-1} s^{1/2-H}(t-s)^{1/2-H}, & 0 \leq s \leq t, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\kappa_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right). \tag{2.2}$$

Suppose $\psi(t, x) \in C^{1,2}([0, T_1] \times \mathbb{R})$. Then, by [3, Lemma 1], there exists another $\mathcal{F}_t$-predictable process $\{\delta_s, 0 \leq s \leq T_1\}$ such that

$$\int_0^t \delta_s\, ds < \infty \quad \mathsf{P}\text{-a.s.}, \qquad 0 \leq t \leq T_1, \tag{2.3}$$

and

$$J_t := \int_0^t k_H(t, s)\phi_s\, ds = \int_0^t \delta_s\, ds, \qquad 0 \leq t \leq T_1. \tag{2.4}$$

It is proved in [4] that

$$\int_0^t k_H(t, s)\, dW_s^H = \int_0^t s^{1/2-H}\, d\widetilde{W}_s, \qquad 0 \leq t \leq T_1, \tag{2.5}$$

where the stochastic integral on the left side exists as a pathwise integral with respect to the fBm W^H,

$$\widetilde{W}_t = \int_0^t s^{H-1/2}\, d\widetilde{M}_s, \qquad 0 \le t \le T_1,$$

and

$$\widetilde{M}_t = \int_0^t k_H(t,s)\, dW_s^H, \qquad 0 \le t \le T_1.$$

Furthermore the process $\{\widetilde{W}_s, 0 \le t \le T_1\}$ is a standard Wiener process. Suppose that

$$\mathsf{E}\left(\int_0^{T_1} s^{2H-1}\delta_s^2\, ds\right) < \infty. \tag{2.6}$$

Define

$$\widetilde{L}_t = \int_0^t s^{H-1/2}\delta_s\, d\widetilde{W}_s, \qquad 0 \le t \le T_1.$$

Under assumption (2.6), the process $\{\widetilde{L}_t, \mathcal{F}_t, 0 \le t \le T_1\}$ is a square integrable martingale. Suppose the martingale $\{\widetilde{L}_t, \mathcal{F}_t, 0 \le t \le T_1\}$ satisfies the condition

$$\mathsf{E}\left[\exp\left\{\widetilde{L}_t - \frac{1}{2}\langle \widetilde{L}\rangle_t\right\}\right] = 1, \qquad 0 \le t \le T_1.$$

Then it is known that the process

$$B_t^H = W_t^H - \int_0^t \phi_s\, ds, \qquad 0 \le t \le T_1,$$

is an fBm with respect to the probability measure Q_H defined on $(\Omega, \mathcal{F})$ by

$$\left.\frac{dQ_H}{d\mathsf{P}}\right|_{\mathcal{F}_t} = \exp\left\{\widetilde{L}_t - \frac{1}{2}\langle \widetilde{L}\rangle_t\right\}, \qquad 0 \le t \le T_1.$$

Note that

$$\left.\frac{dQ_H}{d\mathsf{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t s^{H-1/2}\delta_s\, d\widetilde{W}_s - \frac{1}{2}\int_0^t s^{2H-1}\delta_s^2\, ds\right\}, \qquad 0 \le t \le T_1.$$

Analogously to the above discussion, we construct another probability measure Q_h defined on $(\Omega, \mathcal{F})$ such that

$$\left.\frac{dQ_h}{d\mathsf{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t s^{h-1/2}\psi_s\, d\widehat{W}_s - \frac{1}{2}\int_0^t s^{2h-1}\psi_s^2\, ds\right\}, \qquad 0 \le t \le T_2.$$

Here $\widehat{W}$ is the Wiener process corresponding to the fBm W^h, and $\{\psi_t, 0 \le t \le T_2\}$ and $\{\eta_t, 0 \le t \le T_2\}$ are processes such that

$$I_t := \int_0^t k_h(t,s)\eta_s\, ds = \int_0^t \psi_s\, ds, \qquad 0 \le t \le T_2. \tag{2.7}$$

Observe that

$$B_t^h = W_t^h - \int_0^t \eta_s\, ds, \qquad 0 \le t \le T_2,$$

is an fBm with respect to the probability measure Q_h defined by

$$\left.\frac{dQ_h}{d\mathsf{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t s^{h-1/2}\psi_s\, d\widehat{W}_s - \frac{1}{2}\int_0^t s^{2h-1}\psi_s^2\, ds\right\},$$
$$0 \le t \le T_2.$$

With respect to the probability measures Q_H and Q_h, the trend term should be zero for the first equation in the system over the interval $[0,T_1]$, and it should be zero for the second equation in the system over the interval $[0,T_2]$. Hence

$$b_1(t,X(t))\phi_t = -\theta a_1(t,X(t))$$

and

$$b_2(t,Y(t))\eta_t = -\theta a_2(t,Y(t)).$$

Observe that (here g'_t denotes the derivative of g with respect to t evaluated at t)

$$\begin{aligned}\frac{dQ}{d\,\mathsf{P}} &= \frac{dQ_h}{d\,\mathsf{P}}\bigg|_{\mathcal{F}_{T_2}} \frac{dQ_H}{d\,\mathsf{P}}\bigg|_{\mathcal{F}_{T_1}}\\ &= \exp\bigg\{\int_0^{T_2} s^{h-1/2} J'_s\, d\widehat{W}_s - \frac{1}{2}\int_0^{T_2} s^{2h-1} {J'_s}^2\, ds\bigg\}\\ &\quad\times \exp\bigg\{\int_0^{T_1} s^{H-1/2} I'_s\, d\widetilde{W}_s - \frac{1}{2}\int_0^{T_1} s^{2H-1} {I'_s}^2\, ds\bigg\}\\ &= \exp\bigg\{\int_0^{T_2} s^{h-1/2} J'_s\, d\widehat{W}_s + \int_0^{T_1} s^{H-1/2} I'_s\, d\widetilde{W}_s\\ &\qquad - \frac{1}{2}\bigg[\int_0^{T_2} s^{2h-1} {J'_s}^2 ds + \int_0^{T_1} s^{2H-1} {I'_s}^2\, ds\bigg]\bigg\}.\end{aligned}$$

Note that

$$\begin{aligned}J'_t &= \bigg(\int_0^t k_h(t,s)\eta_s\, ds\bigg)'_t\\ &= \bigg(\int_0^t k_h(t,s)\frac{-\theta a_2(s,Y(s))}{b_2(s,Y(s))}\, ds\bigg)'_t\\ &= -\theta\bigg(\int_0^t k_h(t,s)\frac{a_2(s,Y(s))}{b_2(s,Y(s))}\, ds\bigg)'_t\\ &=: -\theta\Delta_t.\end{aligned}$$

Similarly

$$I'_t = -\theta\bigg(\int_0^t k_H(t,s)\frac{a_1(s,X(s))}{b_1(s,X(s))}\, ds\bigg)'_t =: -\theta\Gamma_t.$$

From the above relations, we get that

$$\begin{aligned}\log\frac{dQ}{d\,\mathsf{P}} &= \int_0^{T_2} s^{h-1/2} J'_s\, d\widehat{W}_s + \int_0^{T_1} s^{H-1/2} I'_s\, d\widetilde{W}_s\\ &\quad - \frac{1}{2}\bigg[\int_0^{T_1} s^{2h-1} {J'_s}^2\, ds + \int_0^{T_2} s^{2H-1} {I'_s}^2\, ds\bigg]\\ &= \int_0^{T_2} s^{h-1/2}(-\theta\Delta_s)\, d\widehat{W}_s + \int_0^{T_1} s^{H-1/2}(-\theta\Gamma_s)\, d\widetilde{W}_s\\ &\quad - \frac{1}{2}\bigg[\int_0^{T_2} s^{2h-1}(-\theta\Delta_s)^2\, ds + \int_0^{T_1} s^{2H-1}(-\theta\Gamma_s)^2\, ds\bigg].\end{aligned}$$

In order to estimate the parameter θ based on the observations of the process

$$\{X(s), 0 \le s \le T_1\}$$

and of the process $\{Y(s), 0 \le s \le T_2\}$, we maximize the function $dQ/d\,\mathsf{P}$ or equivalently $\log(dQ/d\,\mathsf{P})$. Differentiating the function $\log(dQ/d\,\mathsf{P})$ with respect to θ and equating the derivative to zero, we obtain the likelihood equation

$$\theta\Big[\int_0^{T_1} s^{2H-1}\Gamma_s^2\,ds + \int_0^{T_2} s^{2h-1}\Delta_s^2\,ds\Big] = -\Big[\int_0^{T_1} s^{H-1/2}\Gamma_s\,d\widetilde{W}_s + \int_0^{T_2} s^{h-1/2}\Delta_s\,d\widehat{W}_s\Big].$$

The solution $\widehat{\theta}_{T_1,T_2}$ of this equation is given by

$$\widehat{\theta}_{T_1,T_2} = -\frac{\Big[\int_0^{T_1} s^{H-1/2}\Gamma_s\,d\widetilde{W}_s + \int_0^{T_2} s^{h-1/2}\Delta_s\,d\widehat{W}_s\Big]}{\Big[\int_0^{T_1} s^{2H-1}\Gamma_s^2\,ds + \int_0^{T_2} s^{2h-1}\Delta_s^2\,ds\Big]}, \tag{2.8}$$

which is the maximum likelihood estimator in this general case. It can be checked that (see [3, Eq. (14)])

$$d\widetilde{W}_s = dW_s^{(1)} - \theta s^{H-1/2}\Gamma(s)\,ds$$

and

$$d\widehat{W}_s = dW_s^{(2)} - \theta s^{h-1/2}\Delta(s)\,ds,$$

where $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes under the measure Q. Using these relations, it follows that

$$\widehat{\theta}_{T_1,T_2} - \theta = -\frac{\Big[\int_0^{T_1} s^{H-1/2}\Gamma_s\,dW_s^{(1)} + \int_0^{T_2} s^{h-1/2}\Delta_s\,dW_s^{(2)}\Big]}{\Big[\int_0^{T_1} s^{2H-1}\Gamma_s^2\,ds + \int_0^{T_2} s^{2h-1}\Delta_s^2\,ds\Big]}. \tag{2.9}$$

Let

$$J(T_1,T_2) \equiv \int_0^{T_1} s^{2H-1}\Gamma_s^2\,ds + \int_0^{T_2} s^{2h-1}\Delta_s^2\,ds.$$

Then we get that

$$J(T_1,T_2)(\widehat{\theta}_{T_1,T_2} - \theta) = -\Big[\int_0^{T_1} s^{H-1/2}\Gamma_s\,dW_s^{(1)} + \int_0^{T_2} s^{h-1/2}\Delta_s\,dW_s^{(2)}\Big].$$

Define

$$M^{(1)}(t) = \int_0^t s^{H-1/2}\Gamma_s\,dW_s^{(1)}, \qquad 0 \le t \le T_1, \tag{2.10}$$

$$M^{(2)}(t) = \int_0^t s^{h-1/2}\Delta_s\,dW_s^{(2)}, \qquad 0 \le t \le T_2, \tag{2.11}$$

and let $\mathbf{M}(T_1,T_2)$ be the diagonal matrix with the diagonal elements

$$\big(M^{(1)}(T_1), M^{(2)}(T_2)\big).$$

Note that $M^{(i)}$, $i = 1, 2$, are independent continuous local martingales with quadratic variations

$$\big\langle M^{(1)}\big\rangle_t = \int_0^t s^{2H-1}\Gamma_s^2\,ds, \qquad 0 \le t \le T_1,$$

and

$$\big\langle M^{(2)}\big\rangle_t = \int_0^t s^{2h-1}\Delta_s^2\,ds, \qquad 0 \le t \le T_2.$$

Let $\langle\mathbf{M}\rangle_{t,u}$ be a diagonal matrix with the diagonal elements $\big(\langle M^{(1)}\rangle_t, \langle M^{(2)}\rangle_u\big)$. Suppose there exists a vector-valued function (k_{1,t_1}, k_{2,t_2}) such that $k_{i,t_i} > 0$, $i = 1, 2$, increasing to infinity as $t_i \to \infty$. Let $K_{t,u}$ be a diagonal matrix with the diagonal elements $(k_{1,t}, k_{2,u})$. Suppose that

$$K_{T_1,T_2}^{-1}\mathbf{M}(T_1,T_2)K_{T_1,T_2}^{-1} \xrightarrow{\mathsf{P}} \eta^2 \quad \text{as } T_1 \text{ and } T_2 \to \infty, \tag{2.12}$$

where η^2 is a random positive diagonal matrix. Following the results in [7, Theorem 1.50] and [11, Theorem A.1], it follows that

$$K_{T_1,T_2}^{-1}\langle \mathbf{M}\rangle_{T_1,T_2} \xrightarrow{\mathcal{L}} \mathbf{Z}\eta \quad \text{as } T_1 \text{ and } T_2 \to \infty,$$

where $\mathbf{Z}$ is a diagonal matrix with the diagonal elements independent standard normal random variables and the random matrices $\mathbf{Z}$ and η are independent. As a consequence of this result, we can give a set of sufficient conditions for asymptotic normality of the estimator $\widehat{\theta}_{T_1,T_2}$ as $T_1 \to \infty$ and $T_2 \to \infty$. We will now discuss a special case.

Special case. The asymptotic properties of the estimator $\widehat{\theta}_{T_1,T_2}$ depend on the processes Δ_s and Γ_s which in turn depend on the functions a_1, b_1, a_2, b_2, the process $\{X(s), 0 \le s \le T_1\}$ and the process $\{Y(s), 0 \le s \le T_2\}$. However, if these functions and the processes X and Y are such that $\Delta_s = \alpha s^{1-2h}$ and $\Gamma_s = \beta s^{1-2H}$ for some constants α and β, then it follows that

$$\begin{aligned}
\widehat{\theta}_{T_1,T_2} - \theta &= -\frac{\left[\int_0^{T_1} s^{H-1/2}\beta s^{1-2H}\, dW_s^{(1)} + \int_0^{T_2} s^{h-1/2}\alpha s^{1-2h}\, dW_s^{(2)}\right]}{\left[\int_0^{T_1} s^{2H-1}\beta^2 s^{2-4H}\, ds + \int_0^{T_2} s^{2h-1}\alpha^2 s^{2-4h}\, ds\right]} \\
&= -\frac{\left[\beta\int_0^{T_1} s^{1/2-H}\, dW_s^{(1)} + \alpha\int_0^{T_2} s^{1/2-h}\, dW_s^{(2)}\right]}{\left[\beta^2\int_0^{T_1} s^{1-2H}\, ds + \alpha^2\int_0^{T_2} s^{1-2h}\, ds\right]} \\
&= -\frac{\left[\beta\int_0^{T_1} s^{1/2-H}\, dW_s^{(1)} + \alpha\int_0^{T_2} s^{1/2-h}\, dW_s^{(2)}\right]}{\left[\beta^2 T_1^{2-2H}(2-2H)^{-1} + \alpha^2 T_2^{2-2h}(2-2h)^{-1}\right]}.
\end{aligned} \tag{2.13}$$

Since the processes $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes, it is easy to see that the estimator $\widehat{\theta}_{T_1,T_2}$ has a normal distribution with mean θ and variance

$$\left[\beta^2 T_1^{2-2H}(2-2H)^{-1} + \alpha^2 T_2^{2-2h}(2-2h)^{-1}\right]^{-1}.$$

It is clear that the processees $\Delta_s = \alpha s^{1-2h}$ and $\Gamma_s = \beta s^{1-2H}$ for some constants α and β if $a_1 = \beta b_1$ and $a_2 = \alpha b_2$ hold.

3. Geometric fractional Brownian motion

We now specialize the results in Section 2 to a linear system generated by geometric fractional Brownian motions.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space. Consider the linear system of stochastic differential equations

$$\begin{aligned}
dX(t) &= \theta X(t)\, dt + \sigma_1 X(t)\, dW^H(t), \qquad X(0) = x_0 \in \mathbb{R},\ 0 \le t \le T_1, \\
dY(t) &= \theta Y(t)\, dt + \sigma_2 Y(t)\, dW^h(t), \qquad Y(0) = y_0 \in \mathbb{R},\ 0 \le t \le T_2,
\end{aligned} \tag{3.1}$$

defined on $(\Omega, \mathcal{F}, \mathsf{P})$, where $\{\theta, \sigma_1, \sigma_2\} \subset \mathbb{R} \setminus \{0\}$; the fractional Brownian motions

$$\{W^h(t), 0 \le t \le T_1\} \quad \text{and} \quad \{W^H(t), 0 \le t \le T_2\}$$

with known Hurst indices $h \in [\frac{1}{2}, 1)$ and $H \in [\frac{1}{2}, 1)$ respectively are independent. We further assume that the parameters σ_1 and σ_2 are known positive constants.

Following the notation introduced in Section 2, let Q be the product measure of the probability measures Q^h induced on $C[0, T_1]$ and Q^H induced on $C[0, T_2]$. With respect to this probability measure Q, the processes $\{\phi_s\}$ and $\{\eta_s\}$ chosen above should be such that the trend term should be zero in order that the process

$$\{X(s), 0 \le s \le T_1\}$$

be a solution of the first stochastic differential equation in the system given by (3.1) on the interval $[0, T_1]$ and the trend term should be zero in order that the process

$$\{Y(s), 0 \le s \le T_2\}$$

be a solution of the second stochastic differential equation in the system given by (3.1) on the interval $[0, T_2]$. Hence

$$\sigma_1 \int_0^t \phi_s \, ds = -\theta t, \qquad 0 \le t \le T_1,$$

and

$$\sigma_2 \int_0^t \eta_s \, ds = -\theta t, \qquad 0 \le t \le T_2.$$

As before,

$$\begin{aligned} \frac{dQ}{d\mathsf{P}} &= \frac{dQ_H}{d\mathsf{P}}\Big|_{\mathcal{F}_{T_1}} \frac{dQ_h}{d\mathsf{P}}\Big|_{\mathcal{F}_{T_2}} \\ &= \exp\left\{ \int_0^{T_1} s^{H-1/2}\delta_s \, d\widetilde{W}_s - \frac{1}{2}\int_0^{T_1} s^{2H-1}\delta_s^2 \, ds \right\} \\ &\quad \times \exp\left\{ \int_0^{T_2} s^{h-1/2}\psi_s \, d\widehat{W}_s - \frac{1}{2}\int_0^{T_2} s^{2h-1}\psi_s^2 \, ds \right\} \\ &= \exp\left\{ \int_0^{T_2} s^{h-1/2}\psi_s \, d\widehat{W}_s + \int_0^{T_1} s^{H-1/2}\delta_s \, d\widetilde{W}_s \right. \\ &\qquad \left. - \frac{1}{2}\left[\int_0^{T_2} s^{2h-1}\psi_s^2 \, ds + \int_0^{T_1} s^{2H-1}\delta_s^2 \, ds \right] \right\}. \end{aligned} \tag{3.2}$$

Note that

$$\delta_t = \left(\int_0^t k_H(t,s)\phi_s \, ds \right)_t' = -\frac{\theta}{\sigma_1}\left(\int_0^t k_H(t,s) \, ds \right)_t' \tag{3.3}$$

and

$$\psi_t = \left(\int_0^t k_h(t,s)\eta_s \, ds \right)_t' = -\frac{\theta}{\sigma_2}\left(\int_0^t k_h(t,s) \, ds \right)_t'. \tag{3.4}$$

It is easy to see, from the computations given in [4], that

$$\int_0^t k_H(t,s) \, ds = D_H^2 t^{2-2H},$$

where

$$D_H = \frac{C_H}{2H(2-2H)^{1/2}}$$

and

$$C_H = \left(\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma(2-2H)} \right)^{1/2}.$$

Hence

$$\left(\int_0^t k_H(t,s) \, ds \right)_t' = D_H^2 (2-2H) t^{1-2H}.$$

Let

$$\gamma_H(t) = D_H^2(2-2H)t^{1-2H}.$$

It can be checked that

$$\gamma_H(t) = \frac{\Gamma\left(\frac{3}{2}-H\right)}{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}t^{1-2H} =: J_H t^{1-2H}.$$

From the above relations, we get that

$$\begin{aligned}\log\frac{dQ}{d\mathsf{P}} = &-\frac{\theta}{\sigma_1}\int_0^{T_1} s^{H-1/2}\gamma_H(s)\,d\widetilde{W}_s - \frac{\theta}{\sigma_2}\int_0^{T_2} s^{h-1/2}\gamma_h(s)\,d\widehat{W}_s \\ &-\frac{\theta^2}{2\sigma_1^2}\int_0^{T_1} s^{2H-1}\gamma_H^2(s)\,ds - \frac{\theta^2}{2\sigma_2^2}\int_0^{T_2} s^{2h-1}\gamma_h^2(s)\,ds.\end{aligned} \tag{3.5}$$

Estimation. Note that the Radon–Nikodym derivative $dQ/d\mathsf{P}$ obtained above is the Radon–Nikodym derivative of the product measure of the probability measure generated by the process $\{X(s), 0 \le s \le T_1\}$ on the space $C[0,T_1]$ and the probability measure generated by the independent process $\{Y(s), 0 \le s \le T_2\}$. In order to estimate the parameter θ based on the observation of the process $\{X(s), 0 \le s \le T_1\}$ and of the process $\{Y(s), 0 \le s \le T_2\}$, we maximize the likelihood function $dQ/d\mathsf{P}$ or equivalently $\log(dQ/d\mathsf{P})$. Differentiating the function $\log(dQ/d\mathsf{P})$ with respect to θ and equating the derivative to zero, we obtain the likelihood equation

$$\theta\left[\frac{J_H^2}{\sigma_1^2}\frac{T_1^{2-2H}}{2-2H} + \frac{J_h^2}{\sigma_2^2}\frac{T_2^{2-2h}}{2-2h}\right] = -\left[\frac{J_H}{\sigma_1}\int_0^{T_1} s^{1/2-H}\,d\widetilde{W}_s + \frac{J_h}{\sigma_2}\int_0^{T_2} s^{1/2-h}\,d\widehat{W}_s\right],$$

which leads to the estimator, hereafter called the maximum likelihood estimator (MLE), given by

$$\widehat{\theta}_{T_1,T_2} = -\frac{\left[\frac{J_H}{\sigma_1}\int_0^{T_1} s^{1/2-H}\,d\widetilde{W}_s + \frac{J_h}{\sigma_2}\int_0^{T_2} s^{1/2-h}\,d\widehat{W}_s\right]}{\left[\frac{J_H^2}{\sigma_1^2}\frac{T_1^{2-2H}}{2-2H} + \frac{J_h^2}{\sigma_2^2}\frac{T_2^{2-2h}}{2-2h}\right]}.$$

It can be checked that (see [3, Eq. (14)])

$$d\widetilde{W}_s = dW_s^{(1)} - \frac{\theta}{\sigma_1}s^{H-1/2}\gamma_H(s)\,ds$$

and

$$d\widehat{W}_s = dW_s^{(2)} - \frac{\theta}{\sigma_2}s^{h-1/2}\gamma_h(s)\,ds,$$

where $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes under the measure Q. Using these relations, it follows that

$$\widehat{\theta}_{T_1,T_2} - \theta = -\frac{\left[\frac{J_H}{\sigma_1}\int_0^{T_1} s^{1/2-H}dW_s^{(1)} + \frac{J_h}{\sigma_2}\int_0^{T_2} s^{1/2-h}\,dW_s^{(2)}\right]}{\left[\frac{J_H^2}{\sigma_1^2}\frac{T_1^{2-2H}}{2-2H} + \frac{J_h^2}{\sigma_2^2}\frac{T_2^{2-2h}}{2-2h}\right]}.$$

In particular, it follows that the estimator $\widehat{\theta}_{T_1,T_2} - \theta$ has the normal distribution with mean zero and variance

$$\left[\frac{J_H^2}{\sigma_1^2}\frac{T_1^{2-2H}}{2-2H} + \frac{J_h^2}{\sigma_2^2}\frac{T_2^{2-2h}}{2-2h}\right]^{-1}.$$

Suppose $h \ge H$. Further suppose that we observe the process X governed by the first equation in the system up to time $T_1 = T$ and observe the process Y governed by the second equation in the system up to time

$$T_2 = T^{\frac{1-H}{1-h}}.$$

Then the variance of the MLE is given by

$$\left[\frac{J_H^2}{\sigma_1^2}\frac{T_1^{2-2H}}{2-2H}+\frac{J_h^2}{\sigma_2^2}\frac{T_2^{2-2h}}{2-2h}\right]^{-1},$$

which is of the order $O\big(T^{2H-2}\big)$. A better estimator with smaller variance can be obtained by suitably choosing $T_1 = T$ and

$$T_2 = cT^{\frac{1-H}{1-h}},$$

where c is defined by the relation

$$\frac{J_H^2}{\sigma_1^2}\frac{T^{2-2H}}{2-2H} = c^{2-2h}\frac{J_h^2}{\sigma_2^2}\frac{T_2^{2-2h}}{2-2h}.$$

Remarks. The methods of this paper can be extended to study the problem of estimation of the parameter θ for more general linear systems of the type

$$dX_i(t) = \big[\theta a_i(t, X_i(t)) + c_i(t, X_i(t))\big]\,dt + b_i(t, X_i(t))\,dW^{H_i}(t),$$
$$0 \le t \le T_i, \qquad 1 \le i \le n.$$

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