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THE DISTRIBUTION OF THE SUPREMUM OF Θ-PRE-GAUSSIAN SHOT NOISE PROCESSES

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Abstract. Estimates for the distribution of the supremum of Θ-pre-Gaussian shot noise stochastic processes are obtained in the paper for both cases of finite and infinite intervals.

1. Introduction

Stochastic processes represented in the form of stochastic integrals over processes with independent increments are studied in the paper. These processes are often called shot noise stochastic processes. Under the conditions found in [1] such processes are pre-Gaussian. Some estimates for the distribution of the supremum of Θ-pre-Gaussian stochastic processes are obtained in [2]. A generalization to the case of random fields is considered in [6]; the paper [4] deals with an even wider class of random structures. Continuing the studies originated in [1]–[6], we obtain conditions imposed on the response function $g(t,s)$ of a shot noise stochastic process $X(t)$ under which X belongs to the class of Θ-pre-Gaussian stochastic processes and find estimates for the distribution of the supremum of shot noise stochastic processes on finite intervals. Similar questions are studied for processes on infinite intervals.

The paper contains four sections. Some auxiliary results related to Θ-pre-Gaussian stochastic processes are collected in Section 2. Theorem 2.2 in Section 2 allows one to treat shot noise processes as Θ-pre-Gaussian.

Examples of measures satisfying some special assumptions are exhibited in Section 3.

Some bounds for the distribution of the supremum of shot noise stochastic processes are given for the case of a finite interval in Section 4 and for the case of an infinite interval in Section 5.

2. Auxiliary results

Let $(\Omega, \mathfrak{F}, \mathsf{P})$ be a standard probability space. A centered random variable $\xi = \xi(\omega)$, $\mathsf{E}\,\xi = 0$, is called pre-Gaussian if there are two numbers $H > 0$ and $b > 0$ such that the inequality

$$\mathsf{E}\exp\{\lambda\xi\} \le \exp\left\{\frac{b^2\lambda^2}{2}\right\}$$

holds for all $\lambda \in (-H, H)$. The class of pre-Gaussian random variables defined on the standard probability space is denoted by $\operatorname{Prg}(\Omega)$.

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Definition 2.1 (see [1])**.** A stochastic process $X = \{X(t), t \in T\}$ is called pre-Gaussian if all random variables $X(t)$, $t \in T$, are pre-Gaussian.

Note that the increments of pre-Gaussian stochastic processes are also pre-Gaussian, since $X(t) - X(s) \in \mathrm{Prg}(\Omega)$, $t, s \in T$.

Definition 2.2 (see [1])**.** A functional $\Theta \equiv \Theta(\xi) \geq 0$ defined for $\xi \in \mathrm{Prg}(\Omega)$ is called a prenorm if

1) $\Theta(0) = 0$,
2) $\Theta(-\xi) = \Theta(\xi)$, $\xi \in \mathrm{Prg}(\Omega)$.

Let $X = \{X(t), t \in T\}$ be a pre-Gaussian stochastic process and let $\Theta \equiv \Theta(\xi)$ be some prenorm on $\mathrm{Prg}(\Omega)$. Put

$$\theta_1(t) \equiv \Theta(X(t)), \qquad \theta(t,s) \equiv \Theta(X(t) - X(s)), \qquad t, s \in T.$$

Definition 2.3 (see [1])**.** We say that characteristics of a pre-Gaussian stochastic process X are subordinate to a prenorm Θ if there are two constants $\gamma > 0$ and $a \geq 1$ such that, for $|\lambda| < \gamma\theta_1^{-a}(t)$,

$$\mathsf{E}\exp\{\lambda X(t)\} \leq \exp\left\{\frac{\lambda^2\theta_1^2(t)}{2}\right\}, \tag{1}$$

while, for $|\lambda| < \gamma\theta^{-a}(t,s)$,

$$\mathsf{E}\exp\{\lambda(X(t) - X(s))\} \leq \exp\left\{\frac{\lambda^2\theta^2(t,s)}{2}\right\}, \qquad t, s \in T. \tag{2}$$

The processes whose characteristics are subordinate to a prenorm Θ are called $\Theta(a)$-pre-Gaussian.

Definition 2.4 ([1])**.** A function $\rho\colon T \times T \to [0, \infty)$ is called a pseudometric on a set T if

1) $\rho(t,t) = 0$, $t \in T$,
2) $\rho(t,s) = \rho(s,t)$, $s, t \in T$,
3) $\rho(t,s) \leq \rho(t,v) + \rho(v,s)$, $t, v, s \in T$.

Let (T, ρ) be a nonempty pseudometric space. Assume that there exists a minimal covering of the space T by closed balls whose radii do not exceed ε. If a finite ε-covering of the set T exists, then $N(\varepsilon)$ denotes the number of elements in the minimal ε-covering. The number $N(\varepsilon)$ is called the metric capacity, while $H(\varepsilon) \equiv \ln N(\varepsilon)$ is called the metric entropy.

Since the characteristic function $\varphi_\xi(t)$, $t \in \mathbb{R}$, of a random variable ξ is continuous on $\mathbb{R}$ and $\varphi_\xi(0) = 1$, the function $\log\varphi_\xi(t)$ is well defined for sufficiently small $|t|$. If this function possesses the derivative of order $k \geq 1$ at the point $t = 0$, then the real number

$$\varkappa_k \equiv \varkappa_k(\xi) = \frac{1}{i^k}\frac{d}{dt^k}(\log\varphi_\xi(t))\bigg|_{t=0}$$

is called the semi-invariant of order k of the random variable ξ.

Theorem 2.1. *Let ξ be a centered random variable that has the semi-invariants $\varkappa_k$ for all $k \geq 2$ and such that*

$$\beta(\xi) = \sup_{k\geq 2}\left[\frac{2|\varkappa_k(\xi)|}{k!}\right]^{\frac{1}{ak-2(a-1)}} < +\infty \tag{3}$$

for some $a \geq 1$*. Then* ξ *is a pre-Gaussian random variable and*

$$\mathsf{E}\exp\{\lambda\xi\} \leq \exp\left\{\frac{\lambda^2\beta^2}{2(1-c)}\right\} \tag{4}$$

for $|\lambda| < c/(\beta^a)$*, where* $\beta \equiv \beta(\xi)$ *and* $0 < c < 1$*.*

Proof. Inequality (3) implies that

$$\varkappa_k \leq \frac{1}{2}\beta^{ak-2(a-1)}k!.$$

Since $\varkappa_0 = \varkappa_1 = 0$, we have

$$\begin{aligned}\ln\mathsf{E}\exp\{\lambda\xi\} &= \sum_{k=2}^{\infty}\frac{\varkappa_k\lambda^k}{k!} \leq \frac{1}{2}\sum_{k=2}^{\infty}\frac{\beta^{ak-2(a-1)}k!\,|\lambda|^k}{k!} = \frac{1}{2}\sum_{k=2}^{\infty}\left(\beta^a|\lambda|\right)^{k-2}\beta^2\lambda^2\\ &= \frac{1}{2}\frac{\beta^2\lambda^2}{1-\beta^a|\lambda|} < \frac{\lambda^2\beta^2}{2(1-c)}\end{aligned}$$

for $\beta^a|\lambda| < c < 1$ (in other words, for $|\lambda| < c/(\beta^a)$). Thus inequality (4) holds. □

Note that Theorem 2.1 generalizes Theorem 1.7.1 of [1] (the latter is a particular case of Theorem 2.1 for $a = 1$).

Definition 2.5 (see [1])**.** A stochastic process

$$X(t) = \int_{-\infty}^{\infty} g(t,s)\,d\xi(s), \qquad t \in \mathbb{R}, \tag{5}$$

is called a shot noise stochastic process, where $\xi = (\xi(t), t \in \mathbb{R})$ is a real-valued centered homogeneous stochastic process with independent increments and $g = (g(t,s), t, s \in \mathbb{R})$ is a real-valued function such that

$$\int_{-\infty}^{\infty} g^2(t,s)\,ds < +\infty. \tag{6}$$

The function $g(t,s)$ is called the response function.

Note that the integral in (5) is defined in [1] as the integral with respect to a process with independent increments.

It is known (see, for example, [1, 9]) that any real-valued stochastically continuous and right continuous with probability one stochastic process $\xi(t)$ with independent increments defined on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ can be represented as follows for $t \geq 0$:

$$\xi(t) = a(t) + \xi_g(t) + \int_{|x|\leq 1} x\big[\nu([0,t],dx) - \Pi([0,t],dx)\big] + \int_{|x|>1} x\nu([0,t],dx). \tag{7}$$

If $t < 0$, then the interval $[0,t]$ should be replaced by $[t,0]$. In the above representation, $a(t)$, $t \in \mathbb{R}$, is a continuous nonrandom function, $\xi_g(t)$, $t \in \mathbb{R}$, is a centered and continuous with probability one Gaussian stochastic process with independent increments that does not depend on the random Poisson measure $\nu(dt,dx)$ constructed from the jumps of the process ξ, namely

$$\nu([0,t],dx) = \int_0^t \nu(dt,dx), \qquad \Pi(dt,dx) = \mathsf{E}\,\nu(dt,dx), \qquad \Pi([0,t],dx)] = \int_0^t \Pi(dt,dx).$$

In what follows we consider the case where $\xi(t)$ is a homogeneous stochastic process without Gaussian component, that is $a(t) = at$, $a \in \mathbb{R}$, $\mathsf{E}\,\xi_g^2 = 0$, and

$$\Pi(dt,dx) = dt \times \Pi(dx),$$

where $\Pi(dx)$ is a Borel measure on $\mathbb{R}$.

The values of a shot noise process $X(t)$, $t \in \mathbb{R}$, have semi-invariants of an arbitrary order under appropriate assumptions imposed on the measure Π and the response function g. The precise statement is given in Lemma 2.1 below. Note that $\varkappa_1(X(t)) = \mathsf{E}\, X(t) = 0$ for $t \in \mathbb{R}$.

The following result is a combination of Lemmas 5.2.2 and 5.2.3 of [1].

Lemma 2.1. *Assume that*

$$\bar{\pi}_k \equiv \int_{\mathbb{R}} |x|^k \, \Pi(dx) < +\infty, \qquad k \geq 1, \tag{8}$$

$$\bar{G}_k(t) \equiv \int_{\mathbb{R}} |g(t,s)|^k \, ds < +\infty, \qquad k \geq 2. \tag{9}$$

Then, for all $t, s \in \mathbb{R}$ and $k \geq 2$, the semi-invariants $\varkappa_k(X(t))$ and $\varkappa_k(X(t) - X(s))$ exist and moreover

$$\varkappa_k(X(t)) = \pi_k G_k(t), \tag{10}$$

$$\varkappa_k(X(t) - X(s)) = \pi_k G_k(t,s), \tag{11}$$

where

$$G_k(t,s) = \int_{-\infty}^{\infty} |g(t,\tau) - g(s,\tau)|^k \, d\tau. \tag{12}$$

The following result is similar to Theorem 5.2.1 of [1].

Theorem 2.2. *Let conditions* (8) *and* (9) *hold. A shot noise process $X(t)$ is $\Theta(a)$-pre-Gaussian with parameters*

$$\theta_1(t) = \frac{\beta_X(t)}{1-c}, \qquad \theta(t,s) = \frac{\beta_X(t,s)}{1-c}, \qquad 0 < c < 1,$$

if

$$\beta_X(t) = \sup_{k \geq 2} \left[\frac{2|\varkappa_k(X(t))|}{k!} \right]^{\frac{1}{ak-2(a-1)}} = \sup_{k \geq 2} \left[\frac{2|\pi_k G_k(t)|}{k!} \right]^{\frac{1}{ak-2(a-1)}} < +\infty \tag{13}$$

for all $t \in \mathbb{R}$. Moreover

$$\mathsf{E} \exp\{\lambda X(t)\} \leq \exp\left\{ \frac{\lambda^2 \beta_X^2(t)}{2(1-c)} \right\} \tag{14}$$

for $|\lambda| < c/(\beta_X^a(t))$ and

$$\mathsf{E} \exp\{\lambda (X(t) - X(s))\} \leq \exp\left\{ \frac{\lambda^2 \beta_X^2(t,s)}{2(1-c)} \right\} \tag{15}$$

for $|\lambda| < c/(\beta_X^a(t,s))$, where

$$\beta_X(t,s) = \sup_{k \geq 2} \left[\frac{2|\varkappa_k(X(t) - X(s))|}{k!} \right]^{\frac{1}{ak-2(a-1)}} = \sup_{k \geq 2} \left[\frac{2|\pi_k G_k(t,s)|}{k!} \right]^{\frac{1}{ak-2(a-1)}}. \tag{16}$$

Proof. Inequality (14) follows from equality (10) and Theorem 2.1. Inequality (15) also follows from Theorem 2.1 and Lemma 2.1. □

Putting $a = 2$ in Theorem 2.2 we obtain Theorem 5.2.1 of [1].

It is obvious that condition (13) holds if

$$\sup_{k\ge 2}\left[\left(\frac{2}{k!}\right)^{1-u}\int_{\mathbb{R}}|x|^k\,\Pi(dx)\right]^{\frac{1}{ak-2(a-1)}}<+\infty, \tag{17}$$

$$\sup_{k\ge 2}\left[\left(\frac{2}{k!}\right)^{u}\int_{\mathbb{R}}|g(t,s)|^k\,ds\right]^{\frac{1}{ak-2(a-1)}}<+\infty \tag{18}$$

for some $0\le u\le 1$.

3. Measures satisfying condition (17)

The next result is similar to that of Example 5.2.2 in [1].

Example 3.1. Let $\xi=(\xi(t),t\in\mathbb{R})$ be a centered Poisson process whose jumps are of height $h>0$ and whose intensity is $\nu>0$. Then $\bar{\pi}_k=h^k\nu$, $k\ge 2$.

It is obvious that

$$\sup_{k\ge 2}\left|\frac{\bar{\pi}_k}{k!}\right|=\nu\sup_{k\ge 2}\frac{h^k}{k!}=\frac{\nu h^{[h-1]+1}}{([h-1]+1)!}\le \nu e^h,$$

where $[\cdot]$ is the integer part of a number.

Thus the shot noise process X generated by the process ξ and response function g satisfying condition (9) is $\Theta(a)$-pre-Gaussian if

$$\sup_{k\ge 2}\left[\frac{2|G_k(t)|}{k!}\right]^{\frac{1}{ak-2(a-1)}}<+\infty.$$

Now we consider a special measure satisfying condition (17).

Let A be a Borel set and

$$\Pi^*(A)=\int_A p^*(x)\,dx,\qquad F^*(x)\equiv\Pi^*([-x,x]).$$

Then

$$F^*(+\infty)-F^*(x)=\Pi^*(|u|\ge x).$$

Example 3.2. Let a measure $\Pi^*(\cdot)$ possess the density p^*. Let

$$p^*(x)=\frac{1}{4D^{3a-2}}\exp\left\{-\frac{|x|}{D^a}\right\},\qquad x\in\mathbb{R},$$

and assume that condition (17) holds for $u=0$, where

$$D\equiv\sup_{k\ge 2}\left[\frac{2}{k!}\int_{\mathbb{R}}|x|^k\,\Pi(dx)\right]^{\frac{1}{ak-2(a-1)}}.$$

Proof of Example 3.2. Indeed, put

$$I_k\equiv\int_{\mathbb{R}}|x|^k\,\Pi(dx).$$

Then condition (17) reduces to

$$\sup_{k\ge 2}\left[\frac{2}{k!}I_k\right]^{\frac{1}{ak-2(a-1)}}=D<+\infty$$

holding if

$$I_k\le\frac{k!}{2}D^{ak-2(a-1)}.$$

Therefore

$$I_k = \frac{1}{4D^{3a-2}} \int_{\mathbb{R}} |x|^k \exp\left\{-\frac{|x|}{D^a}\right\} dx = \frac{1}{4D^{3a-2}} 2k! D^{ak+a} = \frac{1}{2} k! D^{ak-2(a-1)}. \qquad \square$$

This example implies that condition (17) holds for all $p(x) \le p^*(x)$, $x \in \mathbb{R}$, and $u = 0$. Moreover, the consideration in the latter example suggests a more general assumption under which the measure Π satisfies condition (17).

Theorem 3.1. *Let $\Pi(\cdot)$ be a finite Borel measure on $\mathbb{R}$ such that $F(x) \equiv \Pi([-x, x])$. The measure Π satisfies condition* (17) *if*

$$F(+\infty) - F(x) \le F^*(+\infty) - F^*(x) \tag{19}$$

for all $x > 0$.

Proof. Since the measure Π is finite,

$$\lim_{x \to +\infty} \Pi(|t| \ge x) = 0$$

and

$$\lim_{x \to +\infty} (F^*(+\infty) - F^*(x))|x|^k = 0, \tag{20}$$

whence

$$\lim_{x \to +\infty} (F(+\infty) - F(x))|x|^k = 0. \tag{21}$$

Then

$$\begin{aligned}\int_{\mathbb{R}} |x|^k \, \Pi(dx) &= \int_0^{+\infty} x^k \, dF(x) = -\int_0^{+\infty} x^k \, d(F(+\infty) - F(x)) \\ &= -x^k (F(+\infty) - F(x))\Big|_0^{+\infty} + k \int_0^{+\infty} x^{k-1} (F(+\infty) - F(x)) \, dx \\ &= k \int_0^{+\infty} x^{k-1} (F(+\infty) - F(x)) \, dx.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{R}} |x|^k \, \Pi(dx) &\le k \int_0^{+\infty} x^{k-1} (F^*(+\infty) - F^*(x)) \, dx = \int_{\mathbb{R}} |x|^k p^*(x) \, dx \\ &= \frac{k!}{2} D^{ak-2(a-1)}.\end{aligned}$$

This means that the measure Π satisfies condition (17) if condition (19) holds and this is what had to be proved. $\square$

4. Estimates for the distribution of the supremum over a finite interval

We use Theorem 2.9 of [2] which allows us to estimate the distribution of the supremum of a shot noise process by considering it as a $\Theta(a)$-pre-Gaussian stochastic process.

Theorem 4.1 (see [2])**.** *Let $X \equiv (X(t), t \in T)$ be a $\Theta(a)$-pre-Gaussian stochastic process and let $\sigma = \{\sigma(h), h > 0\}$, $\sigma(0) = 0$, be a continuous increasing function such that*

$$\sup_{\{t,s \in T \colon \rho(t,s) \le h\}} \Theta(X(t) - X(s)) = \sup_{\{t,s \in T \colon \rho(t,s) \le h\}} \theta(t,s) \le \sigma(h). \tag{22}$$

Assume that

$$\int_0^{\varepsilon} H^{1/\delta}\left(\sigma^{(-1)}(u)\right) du < +\infty \tag{23}$$

for $\delta = \min(a, 2)$ and for all $\varepsilon > 0$.

Then

$$\mathsf{P}\left\{\sup_{t\in T} X(t) > x\right\} \le G(x), \tag{24}$$

$$\mathsf{P}\left\{\inf_{t\in T} X(t) < -x\right\} \le G(x), \tag{25}$$

$$\mathsf{P}\left\{\sup_{t\in T} |X(t)| > x\right\} \le 2G(x) \tag{26}$$

for all $x > 0$, where

$$G(x) \equiv \begin{cases} 1, & 0 < x < \Delta_1, \\ \exp\left\{-\frac{(x-\Delta_1)^2}{2R_1^2}\right\}, & \Delta_1 \le x \le \Delta_1 + R_1^2\Delta_2, \\ \exp\left\{-\Delta_2\left(x - \frac{R_1^2\Delta_2}{2} - \Delta_1\right)\right\}, & x > \Delta_1 + R_1^2\Delta_2, \end{cases}$$

$$R_1 \equiv \sqrt{\frac{1}{1-p}\left(\theta_1^2 + \frac{w^2 p}{1-p}\right)}, \qquad \Delta_1 \equiv \frac{\psi_1(p)}{\sqrt{2}(1-p)},$$

$$\Delta_2 \equiv \begin{cases} \gamma(1-p)\min\left(\frac{1}{w^a}, \frac{1}{\beta_1^a}\right)\sqrt{1-A}, & 1 \le a \le 2, \\ \gamma(1-p)\min\left(\frac{1}{\beta_1^a}\sqrt{1-A\beta_1^{2(a-2)}}; \frac{1}{w^a}\sqrt{1-Aw^{2(a-2)}}\right), & a > 2, \end{cases}$$

the value of the constant A depends on a, namely $A \in \left(0, \min\left(\beta_1^{2(2-a)}, w^{2(2-a)}\right)\right)$ for $a > 2$, and $A \in (0,1)$ for $1 \le a \le 2$,

$$\begin{aligned} \psi_1(p) \equiv \frac{1}{p}\max\left(1; \frac{\sqrt{2}}{\sqrt{A}\gamma}\left(\frac{1}{p}\int_0^{wp} H^{\frac{1}{\delta}}\left(\sigma^{(-1)}(u)\right)du\right)^{\delta-1}\right)\int_0^{wp^2} H^{\frac{1}{\delta}}\left(\sigma^{(-1)}(u)\right)du \\ + \frac{1}{p}\int_0^{wp^2} H^{1-\frac{1}{\delta}}\left(\sigma^{(-1)}(u)\right)du \\ + \theta_1(1-p)\Bigg(\min\left(H^{1-\frac{1}{\delta}}\left(\sigma^{(-1)}(wp)\right); \sqrt{A}\frac{\gamma}{\sqrt{2}}w^{1-\delta}\right) \\ + \frac{H\left(\sigma^{(-1)}(wp)\right)}{\min\left(H^{1-\frac{1}{\delta}}\left(\sigma^{(-1)}(wp)\right); \sqrt{A}\frac{\gamma}{\sqrt{2}}w^{1-\delta}\right)}\Bigg), \end{aligned} \tag{27}$$

and where

$$w = \sigma(\beta), \qquad \beta \le \inf_{t\in T}\sup_{s\in T}\rho(t,s), \qquad \beta_1 \equiv (1-c)^{-1}\sup_{t\in T}\beta_X(t), \qquad \{c,p\} \subset (0,1),$$

$\sigma^{(-1)}$ being the converse function to σ.

Example 4.1. Let $T = [a,b]$ where $\frac{1}{2}(b-a) > 1$,

$$g(t,s) = g(|t-s|) = \exp\left\{-(t-s)^2\right\},$$

and let $\xi(t)$ be a centered Poisson process whose jumps are of the height $h > 0$ and whose intensity is $\nu > 0$.

The conditions of Theorem 2.2 are satisfied for $\bar{\pi}_k = h^k \nu$. Since

$$\int_{\mathbb{R}} |g(x)|^k \, dx = \int_{\mathbb{R}} \exp\left\{-kx^2\right\} \, dx = \sqrt{2\pi}\sqrt{\frac{1}{2k}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2k}}} \exp\left\{-\frac{x^2}{2} 2k\right\} \, dx$$

$$= \sqrt{2\pi}\sqrt{\frac{1}{2k}} = \frac{\sqrt{\pi}}{\sqrt{k}},$$

we have

$$\bar{G}_k(t) = \frac{\sqrt{\pi}}{\sqrt{k}}, \qquad k \geq 2,$$

and

$$\beta_X(t) = \sup_{k \geq 2} \left[\frac{2|\pi_k G_k(t)|}{k!}\right]^{\frac{1}{ak-2(a-1)}} = \sup_{k \geq 2} \left[2\sqrt{\pi}\nu \frac{h^k}{\sqrt{k}k!}\right]^{\frac{1}{ak-2(a-1)}} < +\infty.$$

This means that the shot noise process $X(t)$ generated by the response function $g(t)$ and stochastic process $\xi(t)$ is $\Theta(a)$-pre-Gaussian, and one can use Theorem 4.1 for this process.

Theorem 4.2 allows one to find estimates for the distribution of the supremum of a shot noise stochastic process.

Theorem 4.2. *Let $X \equiv (X(t), t \in \mathbb{R})$ be a shot noise process. Assume that conditions* (17) *and* (18) *hold for $0 \leq u \leq 1$. We further assume that the response function $g(t,s)$, $t, s \in \mathbb{R}$, is such that*

$$|g(t,\tau) - g(s,\tau)| \leq \varrho(|t-s|)\hat{c}(\tau). \tag{28}$$

Here $\hat{c}(\tau) \geq 0$ is such that

$$\int_{\mathbb{R}} |\hat{c}(\tau)|^k \, d\tau < +\infty$$

for all $k \geq 2$ and

$$\sup_{k \geq 2} \left[\left(\frac{2}{k!}\right)^u \int_{\mathbb{R}} |\hat{c}(\tau)|^k \, d\tau\right]^{\frac{1}{ak-2(a-1)}} < +\infty. \tag{29}$$

The function $\varrho(h)$, $h > 0$, on the right hand side of (28) *is continuous, increasing, and such that $\varrho(0) = 0$ and the integral*

$$\int_0^{\varepsilon} \left[\ln\left(\frac{T}{2\sigma^{(-1)}(u)} + 1\right)\right]^{1/\delta} du \tag{30}$$

converges for $\delta = \min(a, 2)$ and for all $\varepsilon > 0$, where

$$\begin{gathered} \sigma(h) = \begin{cases} C_1 \varrho^{1/(a)}(h), & \varrho(h) < 1, \\ C_1 \varrho(h), & \varrho(h) \geq 1, \end{cases} \\ C_1 \equiv \frac{1}{1-c} \sup_{k \geq 2} \left[\frac{2}{k!}|\pi_k| \int_{\mathbb{R}} |\hat{c}(\tau)|^k \, d\tau\right]^{\frac{1}{ak-2(a-1)}}. \end{gathered} \tag{31}$$

Then inequalities (24)–(26) *hold for all $x > 0$.*

Proof. Note that $C_1 < +\infty$ as is seen from conditions (17) and (29). Now we check that (17) and (28) imply (22):

$$\begin{aligned}
\sup_{\{t,s\in T\colon \rho(t,s)\le h\}} \theta(t,s) &= \frac{1}{1-c} \sup_{\{t,s\in T\colon \rho(t,s)\le h\}} \beta_X(t,s) \\
&= \frac{1}{1-c} \sup_{\{t,s\in T\colon \rho(t,s)\le h\}} \left(\sup_{k\ge 2} \left[\frac{2}{k!} |\pi_k G_k(t,s)| \right]^{\frac{1}{ak-2(a-1)}} \right) \\
&\le \frac{1}{1-c} \sup_{k\ge 2} \left[\left(\frac{2}{k!}\right)^{1-u} |\pi_k| \right]^{\frac{1}{ak-2(a-1)}} \\
&\quad \times \sup_{\{t,s\in T\colon \rho(t,s)\le h\}} \left(\sup_{k\ge 2} \left[\left(\frac{2}{k!}\right)^{u} |G_k(t,s)| \right]^{\frac{1}{ak-2(a-1)}} \right) \\
&\le \frac{1}{1-c} \sup_{k\ge 2} \left[\left(\frac{2}{k!}\right)^{1-u} |\pi_k| \right]^{\frac{1}{ak-2(a-1)}} \\
&\quad \times \sup_{\{t,s\in T\colon \rho(t,s)\le h\}} \left(\sup_{k\ge 2} \left[\left(\frac{2}{k!}\right)^{u} \varrho^k(t-s) \int_{\mathbb{R}} |\hat{c}(\tau)|^k d\tau \right]^{\frac{1}{ak-2(a-1)}} \right).
\end{aligned} \tag{32}$$

Since the function $k/(ak-2(a-1))$ decreases for $k \ge 2$,

$$(\varrho(h))^{\frac{k}{ak-2(a-1)}} < (\varrho(h))^{1/a}$$

for $\varrho(h) < 1$, whence relation (31) follows.

By the definition of the function $\sigma(h)$ in (31) and since

$$H(\varepsilon) = \ln N(\varepsilon), \qquad N(\varepsilon) < T/(2\varepsilon) + 1$$

(see [1]), condition (23) of Theorem 4.1 coincides with (30).

Further, bounds (17) and (18) imply condition (13). Similarly (18) implies (8) and (17) implies (9). Therefore all the assumptions of Theorem 2.2 are satisfied and X is a $\Theta(a)$-pre-Gaussian stochastic process. Thus Theorem 4.2 follows from Theorems 2.2 and 4.1. □

Example 4.2. Let $T = [a,b]$ be an interval such that $\frac{1}{2}(b-a) = 1$ and let

$$g(t,\tau) = \left(1 + t^2 + \tau^2\right)^{-1}.$$

Assume that a measure Π satisfies condition (17).

Let

$$I(\tau,k) \equiv \int_{\mathbb{R}} |g(t,\tau)|^k \, dt = \int_{\mathbb{R}} \frac{dt}{(1+\tau^2+t^2)^k}, \qquad k \ge 2.$$

Then

$$\sup_{k\ge 2} \left[\left(\frac{2}{k!}\right)^u I(\tau,k) \right]^{\frac{1}{ak-2(a-1)}} \le \sqrt{I(\tau,2)}. \tag{33}$$

Note that

$$\sup_{0\le t\le T} \frac{\partial g(t,\tau)}{\partial t} \le \hat{c}(\tau)$$

for $\varrho(h) = h$ and

$$|g(t,\tau) - g(s,\tau)| \le \sup_{0\le t\le T} \left| \frac{\partial g(t,\tau)}{\partial t} \right| |t-s| \le |t-s| \hat{c}(\tau).$$

Now we choose

$$\hat{c}(\tau) = \sup_t \left| \frac{\partial g(t,\tau)}{\partial t} \right| = \frac{3\sqrt{3}}{8(1+\tau^2)^{3/2}}.$$

Note that

$$\int_{\mathbb{R}} |\hat{c}(\tau)|^k \, d\tau \equiv I(k) < +\infty, \qquad k \geq 1,$$

and

$$F(a,u) \equiv \sup_{k\geq 2} \left[\left(\frac{2}{k!} \right)^u I(k) \right]^{\frac{1}{ak-2(a-1)}} < 1 < +\infty$$

for all $a \geq 1$ and $0 \leq u \leq 1$; that is, condition (29) holds.

Consider the integral in (30). Since it has a singularity at the point 0, one may assume that

$$\sigma(h) = C_1 \varrho^{1/a}(h)$$

for sufficiently small $\varepsilon > 0$ and that

$$\int_0^\varepsilon \left[\ln\left(1 + \frac{b-a}{2} \frac{1}{(u/C_1)^a} \right) \right]^{1/\delta} du \leq \int_0^\varepsilon \left[\frac{1}{\alpha} \left(C_2 \frac{1}{u^a} \right)^\alpha \right]^{1/\delta} du$$
$$= \alpha^{-1/\delta} C_2^{\alpha/\delta} \frac{\delta}{\delta - a\alpha} \varepsilon^{(\delta - a\alpha)/\delta}$$

for $0 < \alpha < 1$, where

$$C_2 \equiv \frac{1}{2} C_1^a (b-a).$$

Thus Theorem 4.2 implies inequality (24) for all $x > 0$.

Theorem 4.2 becomes simpler if the function $g(t,\tau)$ depends on the difference of arguments only.

Theorem 4.3. *Let the response function of a shot noise process $X \equiv (X(t), t \in \mathbb{R})$ be of the form $g(t,s) = g(s-t)$ and let*

$$|g(x) - g(y)| \leq \hat{\varrho}(x-y), \tag{34}$$

where $\hat{\varrho}(x)$, $x > 0$, $\hat{\varrho}(0) = 0$, is a continuous increasing function such that the integral

$$\int_0^\varepsilon \left[\ln\left(\frac{T}{2\sigma^{(-1)}(u)} + 1 \right) \right]^{1/\delta} du \tag{35}$$

converges for $\delta = \min(a,2)$ and for all $\varepsilon > 0$, where

$$\sigma(h) = \begin{cases} C_1 \hat{\varrho}^{\beta/a}(h), & \hat{\varrho}(h) < 1, \\ C_1 \hat{\varrho}(h), & \hat{\varrho}(h) \geq 1, \end{cases} \tag{36}$$

and

$$C_1 \equiv \frac{1}{1-c} \sup_{k\geq 2} \left[\frac{2}{k!} 2^{(1-\beta)k} |\pi_k| \int_{\mathbb{R}} |g(\tau)|^{(1-\beta)k} \, d\tau \right]^{\frac{1}{ak-2(a-1)}} \tag{37}$$

for $\beta \in [0,1]$.

If, for $\beta \in [0,1]$,

$$\sup_{k\geq 2} \left[\frac{2|\pi_k| \int_{\mathbb{R}} |g(x)|^{(1-\beta)k} \, dx}{k!} \right]^{\frac{1}{ak-2(a-1)}} < +\infty, \tag{38}$$

then inequalities (24)–(26) hold for all $x > 0$.

Proof. We derive Theorem 4.3 from Theorem 4.2. Consider the integral

$$\int_{\mathbb{R}} |g(\tau - t) - g(\tau - s)|^k \, d\tau = \int_{\mathbb{R}} |g(x + s - t) - g(x)|^k \, dx.$$

Condition (34) with $\beta \in [0, 1]$ implies that

$$\begin{aligned} \int_{\mathbb{R}} |g(x + s - t) - g(x)|^k \, dx &= \int_{\mathbb{R}} |g(x + s - t) - g(x)|^{\beta k} |g(x + s - t) - g(x)|^{(1-\beta)k} \, dx \\ &\le (\hat{\varrho}(s - t))^{\beta k} \int_{\mathbb{R}} |g(x + s - t) - g(x)|^{(1-\beta)k} \, dx. \end{aligned}$$

Further

$$\begin{aligned} &\sqrt[(1-\beta)k]{\int_{\mathbb{R}} |g(x + s - t) - g(x)|^{(1-\beta)k} \, dx} \\ &\quad \le \sqrt[(1-\beta)k]{\int_{\mathbb{R}} |g(x + s - t)|^{(1-\beta)k} \, dx} + \sqrt[(1-\beta)k]{\int_{\mathbb{R}} |g(x)|^{(1-\beta)k} \, dx} \\ &\quad = 2 \sqrt[(1-\beta)k]{\int_{\mathbb{R}} |g(x)|^{(1-\beta)k} \, dx}; \end{aligned}$$

that is,

$$\int_{\mathbb{R}} |g(x + s - t) - g(x)|^{(1-\beta)k} \, dx \le 2^{(1-\beta)k} \int_{\mathbb{R}} |g(x)|^{(1-\beta)k} \, dx.$$

Thus

$$\begin{aligned} \sup_{\{t,s \in T \colon \rho(t,s) \le h\}} \theta(t, s) &= \frac{1}{1 - c} \sup_{\{t,s \in T \colon \rho(t,s) \le h\}} \beta_X(t, s) \\ &\le \frac{1}{1 - c} \sup_{\{\rho(t,s) \le h\}} \Bigg(\sup_{k \ge 2} \bigg[\frac{2}{k!} \pi_k 2^{(1-\beta)k} \hat{\varrho}^{\beta k}(t - s) \\ &\qquad \times \int_{\mathbb{R}} |g(\tau)|^{(1-\beta)k} \, d\tau \bigg]^{\frac{1}{ak - 2(a-1)}} \Bigg). \end{aligned} \tag{39}$$

Hence conditions (17) and (18) of Theorem 4.2 follow from (38). Condition (28) follows from (34). Theorem 4.3 is proved. ☐

Consider some examples of functions g satisfying the assumptions of Theorem 4.3.

Example 4.3. Let $g(x) = \exp\{-|x|^\alpha\}$ and

$$I(\alpha) \equiv \int_{\mathbb{R}} g(x) \, dx = \int_{\mathbb{R}} \exp\{-|x|^\alpha\} \, dx.$$

First we prove that condition (18) holds for this function. If $\alpha > 1$ and $k \ge 2$, then

$$\begin{aligned} \int_{\mathbb{R}} |g(x)|^k \, dx &= \int_{\mathbb{R}} \left(\exp\left\{-|x|^\alpha\right\}\right)^k \, dx = \int_{\mathbb{R}} \exp\left\{-\left|k^{1/\alpha} x\right|^\alpha\right\} \, dx \\ &= k^{-1/\alpha} \int_{\mathbb{R}} \exp\left\{-|t|^\alpha\right\} \, dt = k^{-1/\alpha} I(\alpha) \end{aligned}$$

and

$$\sup_{k \ge 2} \left[\left(\frac{2}{k!} \right)^u \frac{I(\alpha)}{k^{1/\alpha}} \right]^{\frac{1}{ak - 2(a-1)}} \le \sqrt{\frac{I(\alpha)}{2^{1/\alpha}}}. \tag{40}$$

In particular, Example 4.1 for $\alpha = 2$ shows that

$$\int_{\mathbb{R}} |g(x)|^k \, dx = \frac{\sqrt{\pi}}{\sqrt{k}},$$

while (40) implies that

$$\sup_{k\geq 2} \left[\left(\frac{2}{k!}\right)^u \frac{\sqrt{\pi}}{\sqrt{k}} \right]^{\frac{1}{ak-2(a-1)}} = \sqrt[4]{\frac{\pi}{2}} \approx 1.119515. \tag{41}$$

The integrals $I(\alpha)$ can be evaluated explicitly for various values of α. For example, $I(1) = 2$, $I(2) = \sqrt{\pi}$, $\dots$. Taking into account the inequality $1 - e^x \leq x$, $x > 0$, we obtain for $t > s$ and $\alpha \leq 1$ that

$$\left| e^{-t^\alpha} - e^{-s^\alpha} \right| = \left| e^{-s^\alpha} \left(e^{-(t^\alpha - s^\alpha)} - 1 \right) \right| \leq \left| 1 - e^{-(t^\alpha - s^\alpha)} \right| \leq |t^\alpha - s^\alpha| \leq |t - s|^\alpha.$$

Further

$$\left| e^{-t^\alpha} - e^{-s^\alpha} \right| \leq \sup_t \left[\alpha |t|^{\alpha-1} e^{-|t|^\alpha} \right] \cdot |t - s|$$

for $t > s$ and $\alpha > 1$. Maximizing the function $f(t) = \alpha t^{\alpha-1} e^{-t^\alpha}$ with respect to t, we get

$$\left| e^{-t^\alpha} - e^{-s^\alpha} \right| \leq C_\alpha \cdot |t - s|, \qquad C_\alpha \equiv \alpha \left(\frac{\alpha - 1}{\alpha} \right)^{\frac{\alpha-1}{\alpha}}.$$

Note that the constant C_α increases with α.

Example 4.4. Consider a function g such that $|g(x)| < A$, $A > 1$. Put $g_1(x) = g(x)/A$ and assume that

$$\int_{\mathbb{R}} |g_1(x)|^2 \, dx < +\infty.$$

Then

$$\int_{\mathbb{R}} |g(x)|^k \, dx = A^k \int_{\mathbb{R}} \left| \frac{g(x)}{A} \right|^k \, dx = A^k \int_{\mathbb{R}} |g_1(x)|^k \, dx.$$

Since $k \geq 2$ and

$$\int_{\mathbb{R}} |g_1(x)|^k \, dx = \int_{\mathbb{R}} |g_1(x)|^2 \cdot |g_1(x)|^{k-2} \, dx \leq \int_{\mathbb{R}} |g_1(x)|^2 \, dx,$$

we have

$$\begin{aligned} &\sup_{k\geq 2} \left[\left(\frac{2}{k!}\right)^u \int_{\mathbb{R}} |g(x)|^k \, dx \right]^{\frac{1}{ak-2(a-1)}} \\ &= \sup_{k\geq 2} \left\{ \left[\left(\frac{2}{k!}\right)^u A^k \right]^{\frac{1}{ak-2(a-1)}} \left[\int_{\mathbb{R}} |g_1(x)|^k \, dx \right]^{\frac{1}{ak-2(a-1)}} \right\} \\ &\leq Z \sup_{k\geq 2} \left[\left(\frac{2}{k!}\right)^u A^k \right]^{\frac{1}{ak-2(a-1)}}, \end{aligned}$$

where

$$Z \equiv \sup_{k\geq 2} \left[\int_{\mathbb{R}} |g_1(x)|^k \, dx \right]^{\frac{1}{ak-2(a-1)}}.$$

Then we evaluate

$$\sup_{k\geq 2} \left[\left(\frac{2}{k!}\right)^u A^k \right]^{\frac{1}{ak-2(a-1)}},$$

where $A > 1$. To evaluate the latter expression we consider the following function of the argument k that also depends on the three parameters u, a, and A:

$$f(k; u, a, A) \equiv \left[\left(\frac{2}{k!} \right)^u A^k \right]^{\frac{1}{ak-2(a-1)}}, \qquad u \in (0,1],\ a \geq 1,\ A \geq 1.$$

The function $f(k; u, a, A)$ defined on the right hand side of the latter equality is decreasing for all values of the parameters (except for the case of $u = 0$). This means that f attains its maximal value at $k = 2$. Thus

$$\sup_{k \geq 2} \left[\left(\frac{2}{k!} \right)^u A^k \right]^{\frac{1}{ak-2(a-1)}} = A,$$

and therefore

$$\sup_{k \geq 2} \left[\left(\frac{2}{k!} \right)^u \int_{\mathbb{R}} |g(x)|^k \, dx \right]^{\frac{1}{ak-2(a-1)}} \leq A \cdot Z.$$

5. Estimates of the distribution of the supremum of a normalized pre-Gaussian process on $\mathbb{R}$

Let

$$T \equiv \mathbb{R} = \bigcup_{k=-\infty}^{+\infty} B_k$$

be a partition of $\mathbb{R}$ by disjoint intervals $B_k \equiv [a_k, a_{k+1})$, where $a_k < a_{k+1}$, $a_k \to +\infty$ as $k \to +\infty$ and $a_k \to -\infty$ as $k \to -\infty$. Further let $\rho(t, s) = |t - s|$.

Theorem 3.8 of the paper [2] contains estimates of the distribution of the supremum of a $\Theta(a)$-pre-Gaussian stochastic process. We state this result below for the sake of completeness.

Theorem 5.1. *Let $X \equiv (X(t), t \in \mathbb{R})$ be a $\Theta(a)$-pre-Gaussian stochastic process. Assume that there exist*

1) *continuous increasing functions $\sigma_k(\cdot)$ such that*

$$\sup_{\{t,s \in B_k : \rho(t,s) \leq h\}} \theta(t, s) \leq \sigma_k(h) \tag{42}$$

for $h \in (0, a_{k+1} - a_k)$,

2) *the numbers $a \geq 1$, $\delta = \min(a, 2)$, $w_k \equiv \sigma_k(\beta_k)$, $\beta_k \leq \inf_{t \in B_k} \sup_{s \in B_k} \rho(t, s)$,*

$$0 < A_k < \min\left(1, w_k^{2(2-a)}, \beta_{1k}^{2(2-a)}\right),$$

$\{p_k, \varkappa\} \subset (0, 1)$, $\beta_{1k} \equiv \sup_{t \in B_k} \beta_X(t)$, and a positive even function $c(t)$, $t \in \mathbb{R}$, such that c is increasing for $t \geq 0$,

a) *the entropy integral*

$$\int_0^{\varepsilon} H^{1/\delta}\left(\sigma_k^{(-1)}(u)\right) du \tag{43}$$

is finite for all $\varepsilon > 0$,

b) *the series*

$$\sum_{k \in \mathbb{Z}} \frac{\beta_{1k}}{c^{\varkappa}(a_k)}, \qquad \sum_{k \in \mathbb{Z}} \frac{w_k}{c^{\varkappa}(a_k)(1 - p_k)}, \qquad \sum_{k \in \mathbb{Z}} \frac{\psi_{1k}(p_k)}{c(a_k)(1 - p_k)} \tag{44}$$

converge, where

$$\psi_{1k}(p_k) \equiv \frac{1}{p_k} \max\left(1; \frac{\sqrt{2}}{\sqrt{A_k}\gamma}\left(\frac{1}{p_k}\int_0^{w_k p_k} H^{1/\delta}\left(\sigma^{(-1)}(u)\right) du\right)^{\delta-1}\right)$$

$$\times \int_0^{w_k p_k^2} H^{1/\delta}\left(\sigma^{(-1)}(u)\right) du$$

$$+ \frac{1}{p_k}\int_0^{w_k p_k^2} H^{1-1/\delta}\left(\sigma^{(-1)}(u)\right) du \tag{45}$$

$$+ \beta_{1k}(1-p_k)\left(\min\left(H^{1-1/\delta}\left(\sigma^{(-1)}(w_k p_k)\right); \sqrt{A_k}\frac{\gamma}{\sqrt{2}}\beta_{1k}^{1-\delta}\right)\right.$$

$$\left. + \frac{H\left(\sigma^{(-1)}(w_k p_k)\right)}{\min\left(H^{1-1/\delta}\left(\sigma^{(-1)}(w_k p_k)\right); \sqrt{A_k}\frac{\gamma}{\sqrt{2}}\beta_{1k}^{1-\delta}\right)}\right)$$

and

$$R_{1k} \equiv \frac{1}{1-p_k}\left(\beta_{1k}^2 + \frac{w_k^2 p_k}{1-p_k}\right).$$

Then

$$\mathsf{P}\left\{\sup_{t\in\mathbb{R}} \frac{X(t)}{c(t)} > x\right\} \le 2G(x) \tag{46}$$

for all $x > 0$, where

$$G(x) \equiv \begin{cases} 1, & 0 < x < \Delta_1, \\ \exp\left\{-\frac{(x-\Delta_1)^2}{2R_1^2}\right\}, & \Delta_1 \le x \le \Delta_1 + R_1^2\Delta_2, \\ \exp\left\{-\Delta_2\left(x - \frac{R_1^2\Delta_2}{2} - \Delta_1\right)\right\}, & x > \Delta_1 + R_1^2\Delta_2, \end{cases}$$

$$R_1^2 \equiv \sum_{k\in\mathbb{Z}} \frac{R_{1k}}{c^{\varkappa}(a_k)} \cdot \sum_{k\in\mathbb{Z}} \frac{R_{1k}}{c^{2-\varkappa}(a_k)}, \qquad \Delta_1 \equiv \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \frac{\psi_k(p_k)}{c(a_k)(1-p_k)},$$

and

$$\Delta_2 \equiv \begin{cases} \gamma\left[\sum_{j\in\mathbb{Z}} \frac{R_j}{c^{\varkappa}(a_j)}\right]^{-1} & \\ \quad\times \min_{k\in\mathbb{Z}}\left\{c^{1-\varkappa}(a_k)\min\left(w_{0k}^{1-a}\sqrt{1-A_k}; \beta_1^{1-a}(t_{0k})\right)\right\}, & 1 \le a \le 2, \\ \gamma\left[\sum_{j\in\mathbb{Z}} \frac{R_j}{c^{\varkappa}(a_j)}\right]^{-1} \min_{k\in\mathbb{Z}}\left\{c^{1-\varkappa}(a_k)M(k,a)\right\}, & a > 2, \end{cases} \tag{47}$$

and where

$$M(k,a) = \min\left(w_{0k}^{1-a}\sqrt{1 - A_k w_{0k}^{2(a-2)}}; \beta_1^{1-a}(t_{0k})\right).$$

Theorem 5.2. *Let $X \equiv (X(t), t\in\mathbb{R})$ be a shot noise stochastic process. Assume that conditions* (17) *and* (18) *hold for $0 \le u \le 1$. Assume also that the response function $g(t,s)$, $t,s\in\mathbb{R}$, is such that*

$$|g(t,\tau) - g(s,\tau)| \le \varrho_k(|t-s|)\hat{c}(\tau) \tag{48}$$

for $\tau \in (0, a_{k+1} - a_k)$, where the function $\hat{c}(\tau) \ge 0$ satisfies

$$\int_{\mathbb{R}} |\hat{c}(\tau)|^k \, d\tau < +\infty$$

for all $k \geq 2$, and

$$\sup_{k\geq 2}\left[\left(\frac{2}{k!}\right)^u\int_{\mathbb{R}}|\hat{c}(\tau)|^k\,d\tau\right]^{\frac{1}{ak-2(a-1)}}<+\infty, \tag{49}$$

while $\varrho_k(h)$, $h>0$, are continuous increasing functions such that $\varrho_k(0)=0$ and the integrals

$$\int_0^\varepsilon\left[\ln\left(\frac{a_{k+1}-a_k}{2\sigma_k^{(-1)}(u)}+1\right)\right]^{1/\delta}du \tag{50}$$

converge with $\delta=\min(a,2)$ and for all $\varepsilon>0$, where

$$\sigma_k(h)=\begin{cases}C_1\varrho_k^{\frac{1}{a}}(h), & \varrho_k(h)<1,\\ C_1\varrho_k(h), & \varrho_k(h)\geq 1,\end{cases}$$
$$C_1\equiv\frac{1}{1-c}\sup_{k\geq 2}\left[\frac{2}{k!}|\pi_k|\int_{\mathbb{R}}|\hat{c}(\tau)|^k\,d\tau\right]^{\frac{1}{ak-2(a-1)}}. \tag{51}$$

Finally we assume that the following three series

$$\sum_{k\in\mathbb{Z}}\frac{\beta_{1k}}{c^{\varkappa}(a_k)},\qquad \sum_{k\in\mathbb{Z}}\frac{w_k}{c^{\varkappa}(a_k)(1-p_k)},\qquad \sum_{k\in\mathbb{Z}}\frac{\psi_{1k}(p_k)}{c(a_k)(1-p_k)} \tag{52}$$

converge for some positive, increasing for $t\geq 0$, even function $c(t)$, $t\in\mathbb{R}$, where

$$w_k\equiv\sigma_k(\beta_k),\qquad \beta_k\leq\inf_{t\in B_k}\sup_{s\in B_k}\rho(t,s),\qquad p_k\in(0,1),$$

and where ψ_{1k} is defined by (45).

Then inequality (46) holds for all $x>0$.

Proof. Similarly to the proof of Theorem 4.2, one can show that condition (42) follows from (48) and (17), that is,

$$\sup_{\{t,s\in B_k:\rho(t,s)\leq h\}}\theta(t,s)=\frac{1}{1-c}\sup_{\{t,s\in B_k:\rho(t,s)\leq h\}}\beta_X(t,s)\leq\sigma_k(h).$$

Conditions (43) and (44) of Theorem 5.1 follow from (50) and (52). Thus inequality (46) holds for all $x>0$, and this is what had to be proved. □

6. Concluding remarks

We obtained conditions that a shot noise process $X(t)$ belongs to the class of Θ-pre-Gaussian stochastic processes and found estimates for the distribution of the supremum of shot noise processes for a finite interval and for the whole space $\mathbb{R}$. In the latter case we use normalizing functions $C(t)$, $t\in\mathbb{R}$.

These results can be applied when studying conditions for the uniform convergence with probability one of wavelet expansions for shot noise stochastic processes.

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