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GENERALIZED SOLUTIONS OF A HYPERBOLIC EQUATION WITH A φ-SUB-GAUSSIAN RIGHT HAND SIDE

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B. V. DOVGAĬ

Abstract. A hyperbolic equation with a φ-sub-Gaussian right hand side is considered. Conditions for the existence of a generalized solution of such an equation are found.

1. Introduction

This paper is devoted to studying generalized solutions of the wave equation with zero initial and boundary conditions and with a φ-sub-Gaussian right hand side. We find sufficient conditions for the existence of a generalized solution expressed in terms of convergence of a certain deterministic series and in terms of the covariance function of the random field presented on the right hand side of the equation.

2. Generalized solution of a boundary problem for the deterministic case

Let $T > 0$ be a constant, let $p(x)$ be a continuously differentiable function on the interval $[a, b]$ such that $p(x) > 0$, let $\rho(x)$ and $q(x)$ be continuous functions on the interval $[a, b]$ such that $\rho(x) > 0$ and $q(x) \geq 0$, and let $f(x, t)$, $x \in [a, b]$, $t \in [0, T]$, be a continuous function.

Consider the boundary problem for the nonhomogeneous hyperbolic equation on the interval $[a, b]$:

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)u - \rho(x)\frac{\partial^2 u}{\partial t^2} = -\rho(x)f(x,t), \qquad x \in [a,b], \qquad t \in [0,T], \tag{1}$$

$$u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{2}$$

$$u|_{x=a} = 0, \qquad u|_{x=b} = 0. \tag{3}$$

The corresponding Sturm–Liouville problem is written as follows:

$$\frac{d}{dx}\left(p\frac{dX}{dx}\right) - qX + \lambda\rho X = 0,$$
$$X(a) = X(b) = 0.$$

Let $X_n(x)$ be the orthonormal eigenfunctions with respect to the weight ρ and let λ_n be the corresponding eigenvalues for this problem. We assume that λ_n are taken in

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ascending order. In view of the restrictions imposed on the functions p, q, and ρ, all eigenvalues are positive, and zero is not an eigenvalue.

Put $\mu_n = \sqrt{\lambda_n}$.

Denote by $u_n(x,t)$ a solution of the problem (1)–(3) where a partial Fourier sum of the function f substitutes the function f itself on the right hand side of equation (1), namely

$$f_n(x,t) = \sum_{k=1}^{n} f_k(t) X_k(x)$$

replaces f. Here $f_k(t) = \int_a^b f(x,t) X_k(x) \rho(x)\, dx$. Thus

$$u_n(x,t) = \sum_{k=1}^{n} X_k(x) \frac{1}{\mu_k} \int_0^t \sin \mu_k (t-u) f_k(u)\, du.$$

Definition 1. Following [11], a function $u(x,t)$ is called a generalized solution of the problem (1)–(3) if there is a subsequence $u_{n_k}(x,t)$, $k \geq 1$, such that $u_{n_k}(x,t) \to u(x,t)$ as $k \to \infty$ in the norm of the space $C([a,b] \times [0,T])$.

3. Sufficient conditions for the existence of a generalized solution

Let $T > 0$ be a constant; let $p(x)$ and $\rho(x)$, $x \in [0,\pi]$, be twice continuously differentiable functions such that $p(x) > 0$ and $\rho(x) > 0$; let $q(x)$, $x \in [0,\pi]$, be a continuously differentiable function such that $q(x) \geq 0$; and let $\xi(x,t)$, $x \in [0,\pi]$, $t \in [0,T]$, be an almost surely continuous random field.

Consider the following problem:

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)u - \rho(x)\frac{\partial^2 u}{\partial t^2} = -\rho(x)\xi(x,t), \qquad x \in [0,\pi],\ t \in [0,T], \tag{4}$$

$$u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{5}$$

$$u|_{x=0} = 0, \qquad u|_{x=\pi} = 0. \tag{6}$$

The corresponding Sturm–Liouville problem is as follows:

$$\frac{d}{dx}\left(p\frac{dX}{dx}\right) - qX + \lambda\rho X = 0,$$
$$X(0) = X(\pi) = 0.$$

Let $X_n(x)$ be the orthonormal eigenfunctions with respect to the weight ρ and let λ_n be the corresponding eigenvalues of the problem. We assume that λ_n are taken in ascending order. In view of the assumptions imposed on p, q, and ρ, all eigenvalues are positive, and zero is not an eigenvalue.

Put $\mu_n = \sqrt{\lambda_n}$.

Below is the definition of a generalized condition compatible with Definition 1.

Definition 2. A random field $u(x,t)$, $x \in [0,\pi]$, $t \in [0,T]$, is called a generalized solution of the problem (4)–(6) if

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) \frac{1}{\mu_n} \int_0^t \sin \mu_n (t-u) \zeta_n(u)\, du, \tag{7}$$

where

$$\zeta_n(t) = \int_0^\pi \xi(x,t) X_n(x) \rho(x)\, dx$$

and the series on the right hand side of (7) converges in probability in the norm of the space $C([0,\pi]\times[0,T])$.

Remark 1. If a series of random variables converges in probability, then there exists a subsequence of partial sums that converges almost surely. Thus Definition 2 is indeed compatible with Definition 1.

Theorem 1. *Let $\xi(x,t)$ be a strictly φ-sub-Gaussian random field that is continuous with probability one. Assume that $p>1$ and $\varphi(x)=|x|^p$ for $|x|>1$. If the following double series:*

$$\sum_{k,m=1}^{\infty}\frac{\left(\ln^{\alpha}k\ln^{1-\alpha}m\right)^{2\delta}}{km}C_{k,m}$$

converges for some $\alpha\in[0,1]$ and $\delta>1-1/p$, where

$$C_{k,m}=\sup_{0\le s\le T,\,0\le t\le T}\left|\mathsf{E}\,\zeta_k(s)\zeta_m(t)\right|,$$

then $u(x,t)$ defined by (7) is a generalized solution of the problem (4)–(6).

Proof. It is known [11] that there exists a constant $C_X>0$ such that

$$|X_n(x)|\le C_X$$

for all $x\in[0,\pi]$. Moreover, in the case under consideration,

$$\mu_n=\frac{n\pi}{l}+O\left(\frac{1}{n}\right),$$

$$X_n(x)=\frac{1}{\sqrt[4]{p(x)\rho(x)}}\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi}{l}\int_0^x\sqrt{\frac{\rho(u)}{p(u)}}\,du\right)+\frac{K_n(x)}{n}$$

(see [11]), where

$$l=\int_0^{\pi}\sqrt{\rho(u)/p(u)}\,du,$$

$|K_n(x)|\le C'$ for all $n\ge1$ and $x\in[0,\pi]$, and

$$u(x,t)=\frac{1}{\sqrt[4]{p(x)\rho(x)}}\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\frac{1}{\mu_n}\sin\left(\frac{n\pi}{l}\int_0^x\sqrt{\frac{\rho(u)}{p(u)}}\,du\right)\int_0^t\sin\mu_n(t-u)\zeta_n(u)\,du$$
$$+\sum_{n=1}^{\infty}\frac{1}{n\mu_n}K_n(x)\int_0^t\sin\mu_n(t-u)\zeta_n(u)\,du,$$

$$\sum_{n=1}^{\infty}\frac{1}{n\mu_n}K_n(x)\int_0^t\sin\mu_n(t-u)\zeta_n(u)\,du\le C'C_XT\pi\max_{\substack{x\in[0,\pi]\\ t\in[0,T]}}\{|\xi(x,t)\rho(x)|\}\sum_{n=1}^{\infty}\frac{1}{n\mu_n}<\infty$$

with probability 1. Thus the series

$$\sum_{n=1}^{\infty}\frac{1}{n\mu_n}K_n(x)\int_0^t\sin\mu_n(t-u)\zeta_n(u)\,du$$

converges with probability 1 in the norm of the space $C([0,\pi]\times[0,T])$, and therefore it converges in probability.

Further,

$$\sum_{k,n=1}^{\infty} \frac{1}{\mu_n\mu_k} \frac{1}{\sqrt{p(x)\rho(x)}} \frac{2}{l} \sin\left(\frac{n\pi}{l}\int_0^x \sqrt{\frac{\rho(u)}{p(u)}}\,du\right) \sin\left(\frac{k\pi}{l}\int_0^x \sqrt{\frac{\rho(u)}{p(u)}}\,du\right)$$
$$\times \int_0^t\int_0^t \sin\mu_n(t-u)\sin\mu_k(t-v)\,\mathsf{E}\,\zeta_n(u)\zeta_k(v)\,du\,dv$$
$$\le C'' \sum_{k,n=1}^{\infty} \frac{1}{\mu_n\mu_k} C_{n,k} < \infty.$$

Let

$$R_{k,m}(t,s) = \int_0^t\int_0^s \sin\mu_k(t-u)\sin\mu_m(s-v)\,\mathsf{E}\,\zeta_k(u)\zeta_m(v)\,dv\,du.$$

Similarly to the proof of Lemma 5 of the paper [5], we obtain

$$\mathsf{E}\Bigg|\sum_{k=1}^{n} \frac{1}{\mu_k} \sin\left(\frac{k\pi}{l}\int_0^x \sqrt{\frac{\rho}{p}}\,du\right) \int_0^t \sin\mu_k(t-u)\zeta_k(u)\,du$$
$$- \sum_{k=1}^{n} \frac{1}{\mu_k} \sin\left(\frac{k\pi}{l}\int_0^y \sqrt{\frac{\rho}{p}}\,du\right) \int_0^s \sin\mu_k(s-u)\zeta_k(u)\,du\Bigg|^2$$
$$= \sum_{k,m=1}^{n} \frac{1}{\mu_k\mu_m} \Bigg[\sin\left(\frac{k\pi}{l}\int_0^x \sqrt{\frac{\rho}{p}}\,du\right) \sin\left(\frac{m\pi}{l}\int_0^x \sqrt{\frac{\rho}{p}}\,du\right) R_{k,m}(t,t)$$
$$- \sin\left(\frac{k\pi}{l}\int_0^x \sqrt{\frac{\rho}{p}}\,du\right) \sin\left(\frac{m\pi}{l}\int_0^y \sqrt{\frac{\rho}{p}}\,du\right) R_{k,m}(t,s)$$
$$- \sin\left(\frac{k\pi}{l}\int_0^y \sqrt{\frac{\rho}{p}}\,du\right) \sin\left(\frac{m\pi}{l}\int_0^x \sqrt{\frac{\rho}{p}}\,du\right) R_{k,m}(s,t)$$
$$+ \sin\left(\frac{k\pi}{l}\int_0^y \sqrt{\frac{\rho}{p}}\,du\right) \sin\left(\frac{m\pi}{l}\int_0^y \sqrt{\frac{\rho}{p}}\,du\right) R_{k,m}(s,s)\Bigg]$$
$$\le 4T^2 \sum_{k,m=1}^{\infty} \frac{C_{k,m}}{\mu_k\mu_m} \left(\ln^{2\delta}\left(e^{2\delta} + \frac{m\pi C_1}{2l}\right) + \ln^{2\delta}\left(e^{2\delta} + \frac{\mu_m}{2}\right) + A\right)^{\alpha}$$
$$\times \left(\ln^{2\delta}\left(e^{2\delta} + \frac{k\pi C_1}{2l}\right) + \ln^{2\delta}\left(e^{2\delta} + \frac{\mu_k}{2}\right) + A\right)^{1-\alpha} \cdot |\ln h|^{-2\delta},$$

where $h = \max\{|t-s|, |x-y|\}$ and where $C_1 > 0$ and $A > 0$ are some constants such that $\left|\sqrt{\rho(x)/p(x)}\right| \ge C_1$ for all $x \in [0,\pi]$ and

$$\frac{|t-s|}{2T} \le \frac{A}{\left|\ln|t-s|\right|^{2\delta}}$$

for sufficiently small $|t-s|$. □

4. Conditions for the existence of a generalized solution expressed in terms of the covariance function

Let $B(x,y,t,s) = \mathsf{E}\,\xi(x,t)\xi(y,s)$ for $(x,y,t,s) \in [0,\pi]^2 \times [0,T]^2$. Assume that

$$B(0,y,t,s) = B(\pi,y,t,s) = 0, \qquad y \in [0,\pi],\ t \in [0,T],\ s \in [0,T],$$
$$B(x,0,t,s) = B(x,\pi,t,s) = 0, \qquad x \in [0,\pi],\ t \in [0,T],\ s \in [0,T].$$

Fix a pair $(t,s) \in [0,T]^2$ and extend $B(x,y,t,s)$ as a function of x and y to the whole plane $\mathbb{R}^2$ in such a way that the extension is a periodic function of period 2π in x and in y and

$$B(-x,y,t,s) = -B(x,y,t,s) = B(x,-y,t,s).$$

Such an extension exists in view of the assumptions imposed above on the function B.

Consider a particular case of the problem (4)–(6), namely let $\rho(x) = p(x)$ in (4).

We extend the functions $p(x)$ and $q(x)$ to the whole axis such that the extension is periodic with period 2π and

$$p(x) = p(-x), \qquad q(x) = q(-x).$$

We also assume that the extension of $p(x)$ is twice continuously differentiable, while that of $q(x)$ is continuously differentiable.

Theorem 2. *Let $\xi(x,t)$ be a strictly φ-sub-Gaussian almost surely continuous random field whose covariance function $B(x,y,t,s)$ is continuous. Assume that $p > 1$ and $\varphi(x) = |x|^p$ for $x > 1$. If*

$$\sup_{\substack{y\in[0,\pi],\ t\in[0,T]\\ s\in[0,T]}} \int_{-\pi}^{\pi} \left| B(x,y,t,s)\sqrt{p(x)} - B(x+\beta,y,t,s)\sqrt{p(x+\beta)} \right| dx \le \frac{C}{|\ln\beta|^{2\gamma}}$$

for a sufficiently small constant $\beta > 0$ and some $C > 0$ and $\gamma > 2 - 1/p$, then $u(x,t)$ defined by (7) is a generalized solution of the problem (4)–(6).

Remark 2. The above condition holds if, for example, the derivatives $\partial B/\partial x$ and $p'(x)$ are continuous.

Proof. We show that the assumptions of the theorem imply that the series

$$\sum_{k,m=1}^{\infty} \frac{\left((\ln k)^{\alpha}(\ln m)^{(1-\alpha)}\right)^{2\delta}}{km} C_{k,m}$$

converges for some constants $\alpha \in [0,1]$ and $\delta > 1 - 1/p$, where

$$C_{k,m} = \sup_{0\le t,s\le T} \left|\mathsf{E}\,\zeta_k(t)\zeta_m(s)\right|.$$

Let $\alpha = \frac{1}{2}$. It is sufficient to show that

$$C_{k,m} \le \frac{C}{(\ln k \ln m)^{\gamma}}, \qquad C > 0,\ \gamma > 2 - \frac{1}{p}$$

for all sufficiently large k and m. In the case under consideration,

$$X_n(x) = \frac{1}{\sqrt{p(x)}}\sqrt{\frac{2}{\pi}}\sin nx + \frac{K_n(x)}{n}, \qquad |K_n(x)| \le L$$

(see [11]). Put $B^*(x,y,t,s) = p(x)p(y)B(x,y,t,s)$.

Similarly to the proof of Theorem 2 of the paper [6] we obtain

$$
\begin{aligned}
|\mathsf{E}\,\zeta_k(t)\zeta_m(s)| &= \left|\int_0^\pi\int_0^\pi X_k(x)X_m(y)B^*(x,y,t,s)\,dx\,dy\right| \\
&\le \left|\frac{2}{\pi}\int_0^\pi\int_0^\pi B^*(x,y,t,s)\frac{1}{\sqrt{p(x)p(y)}}\sin kx\sin my\,dx\,dy\right| \\
&\quad+\left|\frac{\sqrt{2}}{m\sqrt{\pi}}\int_0^\pi\int_0^\pi B^*(x,y,t,s)\frac{1}{\sqrt{p(x)}}\sin kxK_m(y)\,dx\,dy\right| \\
&\quad+\left|\frac{\sqrt{2}}{k\sqrt{\pi}}\int_0^\pi\int_0^\pi B^*(x,y,t,s)\frac{1}{\sqrt{p(y)}}\sin myK_k(x)\,dx\,dy\right| \\
&\quad+\left|\frac{1}{km}\int_0^\pi\int_0^\pi B^*(x,y,t,s)K_k(x)K_m(y)\,dx\,dy\right| \\
&\le \frac{1}{2\pi}\left(\int_0^\pi\int_{-\pi}^\pi\left|B^*(x,y,t,s)\frac{1}{\sqrt{p(x)}}-B^*\left(x+\frac{\pi}{k},y,t,s\right)\frac{1}{\sqrt{p(x+\frac{\pi}{k})}}\right|\right. \\
&\qquad\qquad\left.\times\frac{1}{\sqrt{p(y)}}\left|\sin kx\sin my\right|\,dx\,dy\right)^{1/2} \\
&\quad\times\left(\int_0^\pi\int_{-\pi}^\pi\left|B^*(y,x,s,t)\frac{1}{\sqrt{p(y)}}-B^*\left(y+\frac{\pi}{m},x,s,t\right)\frac{1}{\sqrt{p(y+\frac{\pi}{m})}}\right|\right. \\
&\qquad\qquad\left.\times\frac{1}{\sqrt{p(x)}}\left|\sin kx\sin my\right|\,dy\,dx\right)^{1/2} \\
&\quad+\frac{\sqrt{2}}{4m\sqrt{\pi}}\int_0^\pi\int_{-\pi}^\pi\left|B^*(x,y,t,s)\frac{1}{\sqrt{p(x)}}-B^*\left(x+\frac{\pi}{k},y,t,s\right)\frac{1}{\sqrt{p(x+\frac{\pi}{k})}}\right| \\
&\qquad\qquad\times\left|\sin kxK_m(y)\right|\,dx\,dy \\
&\quad+\frac{\sqrt{2}}{4k\sqrt{\pi}}\int_0^\pi\int_{-\pi}^\pi\left|B^*(y,x,s,t)\frac{1}{\sqrt{p(y)}}-B^*\left(y+\frac{\pi}{m},x,s,t\right)\frac{1}{\sqrt{p(y+\frac{\pi}{m})}}\right| \\
&\qquad\qquad\times\left|\sin myK_k(x)\right|\,dy\,dx \\
&\quad+\frac{1}{km}\int_0^\pi\int_0^\pi\left|B^*(x,y,t,s)K_k(x)K_m(y)\right|\,dx\,dy.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|B^*(x,y,t,s)\frac{1}{\sqrt{p(x)}}-B^*\left(x+\frac{\pi}{k},y,t,s\right)\frac{1}{\sqrt{p\big(x+\frac{\pi}{k}\big)}}\right| \\
&\quad=\left|B(x,y,t,s)\frac{\sqrt{p(x)}}{p(y)}-B\left(x+\frac{\pi}{k},y,t,s\right)\frac{\sqrt{p\big(x+\frac{\pi}{k}\big)}}{p(y)}\right| \\
&\quad\le\left|B(x,y,t,s)\sqrt{p(x)}-B\left(x+\frac{\pi}{k},y,t,s\right)\sqrt{p\left(x+\frac{\pi}{k}\right)}\right|\cdot\frac{1}{\min_{0\le y\le\pi}p(y)},
\end{aligned}
$$

there exist constants $C_i > 0$, $i = 0, \dots, 4$, such that

$$|\mathsf{E}\,\zeta_k(t)\zeta_m(s)| \le \frac{C_1}{\left|\ln\frac{\pi}{k}\ln\frac{\pi}{m}\right|^{\gamma}} + \frac{C_2}{m\left|\ln\frac{\pi}{k}\right|^{2\gamma}} + \frac{C_3}{k\left|\ln\frac{\pi}{m}\right|^{2\gamma}} + \frac{C_4}{km} \le \frac{C_0}{(\ln k\ln m)^{\gamma}}$$

for sufficiently large k and m. □

5. Concluding remarks

We consider generalized solutions of a hyperbolic equation with zero initial and boundary conditions and find sufficient conditions for the existence of a generalized solution of the problem (4)–(6).

Bibliography

1. V. V. Buldygin and Yu. V. Kozachenko, *On a question of the applicability of the Fourier method for solving problems with random boundary conditions*, Random Processes in Problems of Mathematical Physics, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1979, pp. 4–35. (Russian) MR587658 (83i:35080)
2. E. Beisenbaev and Yu. V. Kozachenko, *Uniform convergence in probability of random series, and solution of boundary value problems with random initial conditions*, Teor. Veroyatnost. i Mat. Statist. **21** (1979), 9–23; English transl. in Theory Probab. Math. Statist. **21** (1980), 9–23. MR550238 (81g:60036)
3. E. Barrasa de la Krus and Yu. V. Kozachenko, *Boundary-value problems for equations of mathematical physics with strictly Orlicz random initial conditions*, Random Oper. Stochastic Equations **3** (1995), no. 3, 201–220. MR1354813 (96h:60009)
4. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, TViMS, Kiev, 1998; English transl., American Mathematical Society, Providence, RI, 2000. MR1743716 (2001g:60089)
5. B. V. Dovgay and Yu. V. Kozachenko, *The condition for application of Fourier method to the solution of nonhomogeneous string oscillation equation with φ-sub-Gaussian right hand side*, Random Oper. Stochastic Equations **13** (2005), no. 3, 281–296. MR2165325 (2006h:60042)
6. B. V. Dovgay and Yu. V. Kozachenko, *The properties of solution of nonhomogeneous string oscillation equation with φ-sub-Gaussian right side*, Random Oper. Stochastic Equations **17** (2009), no. 3, 221–241. MR2574102 (2010k:35281)
7. M. A. Krasnosel'skiĭ and J. B. Rutitskiĭ, *Convex Functions and Orlicz Spaces*, Fizmatgiz, Moscow, 1958; English transl., P. Noordhoff Ltd., Groningen, 1961. MR0126722 (23:A4016)
8. Yu. V. Kozachenko and G. I. Slivka, *Justification of the Fourier method for a hyperbolic equation with random initial conditions*, Teor. Imovir. Mat. Stat. **69** (2003), 63–78; English transl. in Theory Probab. Math. Statist. **69** (2004), 67–83. MR2110906 (2005k:60127)
9. Yu. V. Kozachenko and Yu. A. Kovalchuk, *Boundary value problems with random initial conditions, and functional series from* $\mathrm{Sub}_{\varphi}(\Omega)$, Ukrain. Mat. Zh. **50** (1998), no. 4, 504–515; English transl. in Ukrainian Math. J. **50** (1999), no. 4, 572–585. MR1698149 (2000f:60029)
10. R. Giuliano Antonini, Yu. Kozachenko, and T. Nikitina, *Spaces of φ-sub-Gaussian random variables*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **XXVII** (2003), no. 1, 95–124. MR2056414 (2005f:60036)
11. G. N. Polozhiĭ, *Equations of Mathematical Physics*, Vysshaya shkola, Moscow, 1964. (Russian)

Department of Operations Research, Faculty for Cybernetics, National Taras Shevchenko University, Glushkov Avenue 6, Kyiv 03127, Ukraine

E-mail address: bogdov@gmail.com

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