GENERALIZED SOLUTIONS OF A HYPERBOLIC EQUATION WITH A $\varphi$-SUB-GAUSSIAN RIGHT HAND SIDE

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ABSTRACT. A hyperbolic equation with a $\varphi$-sub-Gaussian right hand side is considered. Conditions for the existence of a generalized solution of such an equation are found.

1. INTRODUCTION

This paper is devoted to studying generalized solutions of the wave equation with zero initial and boundary conditions and with a $\varphi$-sub-Gaussian right hand side. We find sufficient conditions for the existence of a generalized solution expressed in terms of convergence of a certain deterministic series and in terms of the covariance function of the random field presented on the right hand side of the equation.

2. GENERALIZED SOLUTION OF A BOUNDARY PROBLEM FOR THE DETERMINISTIC CASE

Let $T > 0$ be a constant, let $p(x)$ be a continuously differentiable function on the interval $[a, b]$ such that $p(x) > 0$, let $\rho(x)$ and $q(x)$ be continuous functions on the interval $[a, b]$ such that $\rho(x) > 0$ and $q(x) \geq 0$, and let $f(x, t)$, $x \in [a, b]$, $t \in [0, T]$, be a continuous function.

Consider the boundary problem for the nonhomogeneous hyperbolic equation on the interval $[a, b]$:  
\[
\begin{align*}
\frac{\partial}{\partial x} \left( p(x) \frac{\partial u}{\partial x} \right) - q(x) u - \rho(x) \frac{\partial^2 u}{\partial t^2} &= -\rho(x) f(x, t), \\
&\quad x \in [a, b], \quad t \in [0, T], \\
|u|_{t=0} &= 0, \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = 0, \\
|u|_{x=a} &= 0, \quad |u|_{x=b} = 0.
\end{align*}
\]

The corresponding Sturm–Liouville problem is written as follows:  
\[
\begin{align*}
\frac{d}{dx} \left( p \frac{dX}{dx} \right) - qX + \lambda \rho X &= 0, \\
X(a) &= X(b) = 0.
\end{align*}
\]

Let $X_n(x)$ be the orthonormal eigenfunctions with respect to the weight $\rho$ and let $\lambda_n$ be the corresponding eigenvalues for this problem. We assume that $\lambda_n$ are taken in...
ascending order. In view of the restrictions imposed on the functions $p$, $q$, and $\rho$, all eigenvalues are positive, and zero is not an eigenvalue.

Put $\mu_n = \sqrt{\lambda_n}$.

Denote by $u_n(x,t)$ a solution of the problem (1)-(3) where a partial Fourier sum of the function $f$ substitutes the function $f$ itself on the right hand side of equation (1), namely

$$f_n(x,t) = \sum_{k=1}^{n} f_k(t)X_k(x)$$

replaces $f$. Here $f_k(t) = \int_{a}^{b} f(x,t)X_k(x)\rho(x)\,dx$. Thus

$$u_n(x,t) = \sum_{k=1}^{n} X_k(x)\frac{1}{\mu_k}\int_{0}^{t} \sin \mu_k(t-u)f_k(u)\,du.$$  

**Definition 1.** Following [11], a function $u(x,t)$ is called a generalized solution of the problem (1)-(3) if there is a subsequence $u_{n_k}(x,t)$, $k \geq 1$, such that $u_{n_k}(x,t) \to u(x,t)$ as $k \to \infty$ in the norm of the space $C([a,b] \times [0,T])$.

3. **Sufficient conditions for the existence of a generalized solution**

Let $T > 0$ be a constant; let $p(x)$ and $\rho(x)$, $x \in [0,\pi]$, be twice continuously differentiable functions such that $p(x) > 0$ and $\rho(x) > 0$; let $q(x)$, $x \in [0,\pi]$, be a continuously differentiable function such that $q(x) \geq 0$; and let $\xi(x,t)$, $x \in [0,\pi]$, $t \in [0,T]$, be an almost surely continuous random field.

Consider the following problem:

\begin{align}
(4) \quad \frac{\partial}{\partial x} \left( p(x) \frac{\partial u}{\partial x} \right) - q(x)u - \rho(x)\frac{\partial^2 u}{\partial t^2} &= -\rho(x)\xi(x,t), \quad x \in [0,\pi], \ t \in [0,T], \\
(5) \quad u|_{t=0} &= 0, \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = 0, \\
(6) \quad u|_{x=0} &= 0, \quad u|_{x=\pi} = 0.
\end{align}

The corresponding Sturm–Liouville problem is as follows:

\begin{align}
\frac{d}{dx} \left( p \frac{dX}{dx} \right) - qX + \lambda \rho X &= 0, \\
X(0) = X(\pi) &= 0.
\end{align}

Let $X_n(x)$ be the orthonormal eigenfunctions with respect to the weight $\rho$ and let $\lambda_n$ be the corresponding eigenvalues of the problem. We assume that $\lambda_n$ are taken in ascending order. In view of the assumptions imposed on $p$, $q$, and $\rho$, all eigenvalues are positive, and zero is not an eigenvalue.

Put $\mu_n = \sqrt{\lambda_n}$.

Below is the definition of a generalized condition compatible with Definition [11]

**Definition 2.** A random field $u(x,t)$, $x \in [0,\pi]$, $t \in [0,T]$, is called a generalized solution of the problem (1)-(3) if

\begin{align}
(7) \quad u(x,t) &= \sum_{n=1}^{\infty} X_n(x) \frac{1}{\mu_n} \int_{0}^{t} \sin \mu_n(t-u)\zeta_n(u)\,du, \\
\text{where} \quad \zeta_n(t) &= \int_{0}^{\pi} \xi(x,t)X_n(x)\rho(x)\,dx.
\end{align}
and the series on the right hand side of (7) converges in probability in the norm of the space \( C([0, \pi] \times [0, T]) \).

Remark 1. If a series of random variables converges in probability, then there exists a subsequence of partial sums that converges almost surely. Thus Definition 2 is indeed compatible with Definition 1.

Theorem 1. Let \( \xi(x, t) \) be a strictly \( \varphi \)-sub-Gaussian random field that is continuous with probability one. Assume that \( p > 1 \) and \( \varphi(x) = |x|^p \) for \( |x| > 1 \). If the following double series:

\[
\sum_{k, m=1}^{\infty} \frac{\left( \ln^\alpha k \ln^{1-\alpha} m \right)^{2\delta}}{km} C_{k,m}
\]

converges for some \( \alpha \in [0, 1] \) and \( \delta > 1 - 1/p \), where

\[
C_{k,m} = \sup_{0 \leq s \leq T, 0 \leq t \leq T} |E \zeta_k(s) \zeta_m(t)|,
\]

then \( u(x, t) \) defined by (7) is a generalized solution of the problem (4)–(6).

Proof. It is known [11] that there exists a constant \( C_X > 0 \) such that

\[
|X_n(x)| \leq C_X
\]

for all \( x \in [0, \pi] \). Moreover, in the case under consideration,

\[
\mu_n = \frac{n\pi}{l} + O \left( \frac{1}{n} \right),
\]

\[
X_n(x) = \frac{1}{\sqrt{p(x)p(x)}} \sqrt{\frac{2}{l}} \left( \frac{n\pi}{l} \int_0^x \sqrt{\frac{\rho(u)}{p(u)}} \, du \right) + \frac{K_n(x)}{n}
\]

(see [11]), where

\[
l = \int_0^\pi \sqrt{\rho(u)/p(u)} \, du,
\]

\[
|K_n(x)| \leq C' \quad \text{for all } n \geq 1 \text{ and } x \in [0, \pi],
\]

and

\[
u(x, t) = \frac{1}{\sqrt{p(x)p(x)}} \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \frac{1}{\mu_n} \sin \left( \frac{n\pi}{l} \int_0^x \sqrt{\frac{\rho(u)}{p(u)}} \, du \right) \int_0^t \sin \mu_n(t-u) \zeta_n(u) \, du
\]

\[
+ \sum_{n=1}^{\infty} \frac{1}{n\mu_n} K_n(x) \int_0^t \sin \mu_n(t-u) \zeta_n(u) \, du,
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n\mu_n} K_n(x) \int_0^t \sin \mu_n(t-u) \zeta_n(u) \, du \leq C'C_X T \pi \max_{x \in [0, \pi]} \{ |\xi(x, t)\rho(x)| \} \sum_{n=1}^{\infty} \frac{1}{n\mu_n} < \infty
\]

with probability 1. Thus the series

\[
\sum_{n=1}^{\infty} \frac{1}{n\mu_n} K_n(x) \int_0^t \sin \mu_n(t-u) \zeta_n(u) \, du
\]

converges with probability 1 in the norm of the space \( C([0, \pi] \times [0, T]) \), and therefore it converges in probability.
Further,
\[\sum_{k,n=1}^{\infty} \frac{1}{\mu_k \mu_n} \int_{0}^{\frac{x}{l}} \int_{0}^{\frac{y}{p}} \sin \left(\frac{n\pi}{l} \int_{0}^{x} \sqrt{\rho(u)} \, du\right) \sin \left(\frac{k\pi}{l} \int_{0}^{x} \sqrt{\rho(u)} \, du\right) \times \int_{0}^{t} \int_{0}^{t} \sin \mu_n(t-u) \sin \mu_k(t-v) \mathbb{E} \zeta_n(u) \zeta_k(v) \, du \, dv \leq C^{''} \sum_{k,n=1}^{\infty} \frac{1}{\mu_k \mu_n} C_{n,k} < \infty.\]

Let
\[R_{k,m}(t,s) = \int_{0}^{t} \int_{0}^{s} \sin \mu_k(t-u) \sin \mu_m(s-v) \mathbb{E} \zeta_k(u) \zeta_m(v) \, du \, dv.\]

Similarly to the proof of Lemma 5 of the paper [5], we obtain
\[\mathbb{E} \left[ \sum_{k=1}^{n} \frac{1}{\mu_k} \sin \left(\frac{k\pi}{l} \int_{0}^{x} \sqrt{\rho \phi} \, du\right) \int_{0}^{t} \sin \mu_k(t-u) \zeta_k(u) \, du \right]^2 \leq \sum_{k,m=1}^{n} \frac{1}{\mu_k \mu_m} \left[ \sin \left(\frac{k\pi}{l} \int_{0}^{x} \sqrt{\rho \phi} \, du\right) \sin \left(\frac{m\pi}{l} \int_{0}^{x} \sqrt{\rho \phi} \, du\right) \right] R_{k,m}(t,t) - \sin \left(\frac{k\pi}{l} \int_{0}^{x} \sqrt{\rho \phi} \, du\right) \sin \left(\frac{m\pi}{l} \int_{0}^{x} \sqrt{\rho \phi} \, du\right) R_{k,m}(t,s) - \sin \left(\frac{k\pi}{l} \int_{0}^{y} \sqrt{\rho \phi} \, du\right) \sin \left(\frac{m\pi}{l} \int_{0}^{y} \sqrt{\rho \phi} \, du\right) R_{k,m}(s,t) + \sin \left(\frac{k\pi}{l} \int_{0}^{y} \sqrt{\rho \phi} \, du\right) \sin \left(\frac{m\pi}{l} \int_{0}^{y} \sqrt{\rho \phi} \, du\right) R_{k,m}(s,s) \]
\[\leq 4T^2 \sum_{k,m=1}^{\infty} \frac{C_{k,m}}{\mu_k \mu_m} \left( \ln^{2\delta} \left( e^{2\delta} + \frac{m\pi C_1}{2l} \right) + \ln^{2\delta} \left( e^{2\delta} + \frac{\mu_m}{2} \right) + A \right)^{\alpha} \times \left( \ln^{2\delta} \left( e^{2\delta} + \frac{k\pi C_1}{2l} \right) + \ln^{2\delta} \left( e^{2\delta} + \frac{\mu_k}{2} \right) + A \right)^{1-\alpha} \cdot |\ln h|^{-2\delta},\]

where \( h = \max\{|t-s|, |x-y|\} \) and where \( C_1 > 0 \) and \( A > 0 \) are some constants such that \( |\sqrt{\rho(x)/p(x)}| \geq C_1 \) for all \( x \in [0, \pi] \) and
\[\frac{|t-s|}{2T} \leq \frac{A}{|\ln |t-s||^{2\delta}}\]
for sufficiently small \( |t-s| \). \( \square \)

4. Conditions for the existence of a generalized solution expressed in terms of the covariance function

Let \( B(x, y, t, s) = \mathbb{E} \xi(x, t) \xi(y, s) \) for \((x, y, t, s) \in [0, \pi]^2 \times [0, T]^2\). Assume that
\(B(0, y, t, s) = B(y, y, t, s) = 0, \quad y \in [0, \pi], \quad t \in [0, T], \quad s \in [0, T],\)
\(B(x, 0, t, s) = B(x, \pi, t, s) = 0, \quad x \in [0, \pi], \quad t \in [0, T], \quad s \in [0, T].\)
Fix a pair \((t, s) \in [0, T]^2\) and extend \(B(x, y, t, s)\) as a function of \(x\) and \(y\) to the whole plane \(\mathbb{R}^2\) in such a way that the extension is a periodic function of period \(2\pi\) in \(x\) and in \(y\) and

\[ B(-x, y, t, s) = -B(x, y, t, s) = B(x, -y, t, s). \]

Such an extension exists in view of the assumptions imposed above on the function \(B\).

Consider a particular case of the problem (4)–(6), namely let \(\rho(x) = p(x)\) in (4).

We extend the functions \(p(x)\) and \(q(x)\) to the whole axis such that the extension is periodic with period \(2\pi\) and

\[ p(x) = p(-x), \quad q(x) = q(-x). \]

We also assume that the extension of \(p(x)\) is twice continuously differentiable, while that of \(q(x)\) is continuously differentiable.

**Theorem 2.** Let \(\xi(x, t)\) be a strictly \(\varphi\)-sub-Gaussian almost surely continuous random field whose covariance function \(B(x, y, t, s)\) is continuous. Assume that \(p > 1\) and \(\varphi(x) = |x|^p\) for \(x > 1\). If

\[
\sup_{y \in [0, \pi], t \in [0, T]} \int_{-\pi}^{\pi} \left| B(x, y, t, s) \sqrt{p(x)} - B(x + \beta, y, t, s) \sqrt{p(x + \beta)} \right| dx \leq \frac{C}{|\ln \beta|^\gamma}
\]

for a sufficiently small constant \(\beta > 0\) and some \(C > 0\) and \(\gamma > 2 - 1/p\), then \(u(x, t)\) defined by (7) is a generalized solution of the problem (4)–(6).

**Remark 2.** The above condition holds if, for example, the derivatives \(\partial B/\partial x\) and \(p'(x)\) are continuous.

**Proof.** We show that the assumptions of the theorem imply that the series

\[
\sum_{k, m=1}^{\infty} \frac{(\ln k)^\alpha (\ln m)^{(1-\alpha)}}{km} C_{k,m} \delta (25)
\]

converges for some constants \(\alpha \in [0, 1]\) and \(\delta > 1 - 1/p\), where

\[ C_{k,m} = \sup_{0 \leq t, s \leq T} |E \zeta_k(t) \zeta_m(s)|. \]

Let \(\alpha = \frac{1}{2}\). It is sufficient to show that

\[ C_{k,m} \leq \frac{C}{(\ln k \ln m)^\gamma}, \quad C > 0, \quad \gamma > 2 - \frac{1}{p} \]

for all sufficiently large \(k\) and \(m\). In the case under consideration,

\[ X_n(x) = \frac{1}{\sqrt{p(x)}} \sqrt{\frac{2}{\pi}} \sin nx + \frac{K_n(x)}{n}, \quad |K_n(x)| \leq L \]

(see [11]). Put \(B^*(x, y, t, s) = p(x)p(y)B(x, y, t, s)\).
Similarly to the proof of Theorem 2 of the paper [6] we obtain

\[
|E \zeta_k(t) \zeta_m(s)| = \left| \int_0^\pi \int_0^\pi X_k(x)X_m(y)B^*(x, y, t, s) \, dx \, dy \right|
\]

\[
\leq \frac{2}{\pi} \int_0^\pi \int_0^\pi B^*(x, y, t, s) \frac{1}{\sqrt{p(x)p(y)}} \sin kx \sin my \, dx \, dy
\]

\[
+ \frac{\sqrt{2}}{m\sqrt{\pi}} \int_0^\pi \int_0^\pi B^*(x, y, t, s) \frac{1}{\sqrt{p(x)}} \sin kxK_m(y) \, dx \, dy
\]

\[
+ \frac{\sqrt{2}}{k\sqrt{\pi}} \int_0^\pi \int_0^\pi B^*(x, y, t, s) \frac{1}{\sqrt{p(y)}} \sin myK_k(x) \, dx \, dy
\]

\[
+ \frac{1}{km} \int_0^\pi \int_0^\pi B^*(x, y, t, s)K_k(x)K_m(y) \, dx \, dy
\]

\[
\leq \frac{1}{2\pi} \left( \int_0^\pi \int_\pi^{-\pi} \left| B^*(x, y, t, s) \frac{1}{\sqrt{p(x)}} - B^*(x + \frac{\pi}{k}, y, t, s) \frac{1}{\sqrt{p(x + \frac{\pi}{k})}} \right| \right.
\]

\[
\times \frac{1}{\sqrt{p(y)}} \sin kx \sin my \, dx \, dy \biggr)^{1/2}
\]

\[
\times \left( \int_0^\pi \int_\pi^{-\pi} \left| B^*(y, x, s, t) \frac{1}{\sqrt{p(y)}} - B^*(y + \frac{\pi}{m}, x, s, t) \frac{1}{\sqrt{p(y + \frac{\pi}{m})}} \right| \right.
\]

\[
\times \frac{1}{\sqrt{p(x)}} \sin kx \sin my \, dy \, dx \biggr)^{1/2}
\]

\[
+ \frac{\sqrt{2}}{4m\sqrt{\pi}} \int_0^\pi \int_\pi^{-\pi} \left| B^*(x, y, t, s) \frac{1}{\sqrt{p(x)}} - B^*(x + \frac{\pi}{k}, y, t, s) \frac{1}{\sqrt{p(x + \frac{\pi}{k})}} \right| \sin kxK_m(y) \, dx \, dy
\]

\[
+ \frac{\sqrt{2}}{4k\sqrt{\pi}} \int_0^\pi \int_\pi^{-\pi} \left| B^*(y, x, s, t) \frac{1}{\sqrt{p(y)}} - B^*(y + \frac{\pi}{m}, x, s, t) \frac{1}{\sqrt{p(y + \frac{\pi}{m})}} \right| \sin myK_k(x) \, dy \, dx
\]

\[
+ \frac{1}{km} \int_0^\pi \int_0^\pi \left| B^*(x, y, t, s)K_k(x)K_m(y) \right| \, dx \, dy.
\]

Since

\[
\left| B^*(x, y, t, s) \frac{1}{\sqrt{p(x)}} - B^*(x + \frac{\pi}{k}, y, t, s) \frac{1}{\sqrt{p(x + \frac{\pi}{k})}} \right|
\]

\[
= \left| B(x, y, t, s)\sqrt{\frac{p(x)}{p(y)}} - B \left( x + \frac{\pi}{k}, y, t, s \right) \sqrt{\frac{p(x + \frac{\pi}{k})}{p(y)}} \right|
\]

\[
\leq \left| B(x, y, t, s)\sqrt{\frac{p(x)}{p(y)}} - B \left( x + \frac{\pi}{k}, y, t, s \right) \sqrt{\frac{p \left( x + \frac{\pi}{k} \right)}{p(y)}} \right| \min_{0 \leq y \leq \pi} p(y),
\]
there exist constants $C_i > 0$, $i = 0, \ldots, 4$, such that

$$|E\zeta_k(t)\zeta_m(s)| \leq \frac{C_1}{\ln^\alpha \ln k} + \frac{C_2}{m \ln^\alpha \ln m} + \frac{C_3}{k \ln^\alpha \ln k} + \frac{C_4}{km \ln^\alpha \ln m} \leq C_0 \left(\frac{\ln k \ln m}{\ln^\alpha \ln k}\right)$$

for sufficiently large $k$ and $m$. \hfill \Box

5. CONCLUDING REMARKS

We consider generalized solutions of a hyperbolic equation with zero initial and boundary conditions and find sufficient conditions for the existence of a generalized solution of the problem (4)–(6).

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