Теорія Ймовір. та Матем. Статист.
Вип. 81, 2010

Theor. Probability and Math. Statist.
No. 81, 2010, Pages 159–175
S 0094-9000(2011)00817-4
Article electronically published on January 20, 2011

SINGULARITY OF THE DISTRIBUTION OF A RANDOM VARIABLE REPRESENTED BY AN A_2-CONTINUED FRACTION WITH INDEPENDENT ELEMENTS

UDC 519.21

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Abstract. We study the properties of the distribution of the random variable

$$\xi = \frac{1}{\eta_1 + \frac{1}{\eta_2 + \cdots}},$$

where η_k are independent random variables such that $\mathsf{P}\{\eta_k = \alpha_1\} = p_{\alpha_1 k} \geq 0$, $\mathsf{P}\{\eta_k = \alpha_2\} = p_{\alpha_2 k} \geq 0$, $0 < \alpha_1 < \alpha_2$, $\alpha_1\alpha_2 \geq \frac{1}{2}$, $p_{\alpha_1 k} + p_{\alpha_2 k} = 1$. It is proved that the distribution of ξ cannot be absolutely continuous. We find the criteria for the distribution of ξ to belong to one of the two types of singular distributions, Cantor and Salem types, depending on topological and metric properties of the topological support of the distribution.

1. Introduction

Recall [7] that the distribution of a random variable is called singular if it is continuous and is concentrated on a set of zero Lebesgue measure. The interest to singular distributions has been growing lately because of their close relations to fractals [7]. However, representation, investigation, and applications of such distributions are difficult, since there are no specific analytical techniques to study these objects. To solve the problem, various number systems are used, including the representation of real numbers by continued fractions (see [2, 9, 10]).

There are papers (see [2, 5, 6]) devoted to the studies of the structure of a distribution (in other words, studies of the Lebesgue discrete, absolutely continuous, and singularly continuous components of a distribution) as well as of the topological and metric properties of the topological support of the distribution of the random variable

$$\xi = \frac{1}{\eta_1 + \frac{1}{\eta_2 + \cdots}}, \tag{1}$$

where the η_k are independent random variables. The structure of the distribution of the random variable ξ is best studied in the case where the η_k assume only positive integer values. It is proved in [6] that the distribution of the random variable ξ represented by an elementary continued fraction with independent elements η_k cannot have an absolutely continuous component (thus, if the distribution is continuous, then it is singular).

2010 *Mathematics Subject Classification.* Primary 11K55; Secondary 11K50, 60E05, 26A30, 28A80.

Key words and phrases. Random continued fraction, A_2-continued fraction, Cantor type singular distribution, Salem type singular distribution.

The first author is supported by DFG 436 UKR Projects #113/80 and #113/97.

The second author is supported by DFG 436 UKR Project #113/80.

In this paper, we study the properties of the distribution of a random continued fraction ξ with independent elements η_k such that

$$\mathsf{P}\{\eta_k = \alpha_1\} = p_{\alpha_1 k} \geq 0, \qquad \mathsf{P}\{\eta_k = \alpha_2\} = p_{\alpha_2 k} \geq 0,$$
$$0 < \alpha_1 < \alpha_2, \qquad \alpha_1\alpha_2 \geq \frac{1}{2}, \qquad p_{\alpha_1 k} + p_{\alpha_2 k} = 1.$$

The main question we are concerned with is the Lebesgue structure of the distribution and its properties.

It is proved in [2] that the distribution of a random variable is pure if it is represented by a continued fraction with independent (not necessarily integer-valued) elements. A criterion for the discreteness of such a random variable is also proved in [2].

We give an independent proof that the distribution of the above random variable is pure, and we provide a criterion for the discreteness of the random variable ξ. Moreover, we prove that the distribution of the random variable ξ is either discrete or singularly continuous. We also show that the distribution of ξ belongs to a specific class of singular distributions depending on topological and metric properties of the topological support.

2. A random A_2-continued fraction with independent elements

Let

$$\xi = \frac{1}{\eta_1 + \frac{1}{\eta_2 + \cdots}} \equiv [\eta_1, \eta_2, \ldots],$$

where η_k are independent random variables with the distributions

$$\mathsf{P}\{\eta_k = \alpha_1\} = p_{\alpha_1 k} \geq 0, \qquad \mathsf{P}\{\eta_k = \alpha_2\} = p_{\alpha_2 k} \geq 0,$$
$$0 < \alpha_1 < \alpha_2, \qquad p_{\alpha_1 k} + p_{\alpha_2 k} = 1.$$

Recall [3] that an infinite continued fraction of the form

$$[a_1, a_2, \ldots, a_k, \ldots]$$

is called an A_2-continued fraction if $a_k \in A_2 = \{\alpha_1, \alpha_2\}$, $0 < \alpha_1 < \alpha_2$, $k = 1, 2, \ldots$. Put $\beta_1 = [\alpha_2, \alpha_1, \alpha_2, \alpha_1, \ldots] \equiv [(\alpha_2, \alpha_1)]$, $\beta_2 = [(\alpha_1, \alpha_2)]$.

It was proved in [3] that the condition $\alpha_1\alpha_2 = \frac{1}{2}$ implies that there is a countable set of points of the interval $[\beta_1, \beta_2]$ that have two different A_2-continued representations (the elements of this set are called the A_2-rational points), while the rest of the points of this interval have a unique representation in the form of an A_2-continued fraction (these are the so-called A_2-irrational points). In the first case, the two representations of a number x are given by

$$x = [\underbrace{c_1, c_2, \ldots}_{n}, \alpha_1, (\alpha_1, \alpha_2)] = [\underbrace{c_1, c_2, \ldots}_{n}, \alpha_2, (\alpha_2, \alpha_1],$$

where the first n elements coincide, $n = 0, 1, 2, \ldots$.

Let $L_{A_2} = \{x \colon x = [a_1, a_2, \ldots, a_k, \ldots], a_k \in A_2, k = 1, 2, \ldots\}$. It is easy to verify that, for $\alpha_1\alpha_2 > \frac{1}{2}$, every point $x \in [\beta_1, \beta_2]$ either does not have any A_2-continued representation ($x \notin L_{A_2}$) or such a representation exists and is unique.

Lemma 2.1. *If $\alpha_1\alpha_2 \geq \frac{1}{2}$ and the number $x \in [\beta_1, \beta_2]$ has a unique A_2-continued representation $[a_1, a_2, \ldots, a_k, \ldots]$, then*

$$\mathsf{P}\{\xi = x\} = \prod_{k=1}^{\infty} p_{a_k k}.$$

Proof. It is clear that the random events $\{\xi = x\}$ occur if and only if

$$\eta_k = a_k, \qquad k = 1, 2, \dots .$$

Then the independence of $\eta_1, \eta_2, \dots, \eta_k, \dots$ implies that

$$\mathsf{P}\{\xi = x\} = \mathsf{P}\{\eta_1 = a_1, \eta_2 = a_2, \dots, \eta_k = a_k, \dots\} = \prod_{k=1}^{\infty} \mathsf{P}\{\eta_k = a_k\} = \prod_{k=1}^{\infty} p_{a_k k}. \qquad \square$$

Lemma 2.2. *Let $\alpha_1\alpha_2 = \frac{1}{2}$. If $[a_1, \dots, a_k, \dots]$ and $[b_1, \dots, b_k, \dots]$ are two different A_2-continued representations of a real number x, then*

$$\mathsf{P}\{\xi = x\} = \prod_{k=1}^{\infty} p_{a_k k} + \prod_{k=1}^{\infty} p_{b_k k}. \tag{2}$$

Moreover, at least one of the infinite products on the right hand side of (2) *is equal to zero.*

Proof. Equality (2) is obvious. If x is not an atom of the distribution, then $\mathsf{P}\{\xi = x\} = 0$ and both products equal zero. If $\mathsf{P}\{\xi = x\} > 0$, then at least one of the products is positive. Suppose

$$\prod_{k=1}^{\infty} p_{a_k k} > 0.$$

According to the necessary condition for the convergence of an infinite product, $p_{a_k k} \to 1$ as $k \to \infty$. Since $b_k \neq a_k$ for all $k > n$, where n is equal to the number of coinciding initial elements in both representations, and since the set A_2 contains only two elements, we get $p_{b_k k} \to 0$ and the infinite product $\prod_{k=1}^{\infty} p_{b_k k}$ diverges to zero. $\square$

Let $c_1, \dots, c_n$ be a fixed family of elements of the set A_2. Recall [3] that the set

$$\Delta_{c_1 \dots c_n} = \big\{x \colon x = [c_1, \dots, c_n, a_{n+1}, a_{n+2}, \dots], a_{n+k} \in A_2, k = 1, 2, \dots\big\}$$

is called a cylindrical set (cylinder) of rank n with the base $c_1 \dots c_n$.

Define the set $\nabla_{c_1 \dots c_n}$ as follows:

$$\nabla_{c_1 \dots c_n} = \Delta_{c_1 \dots c_n} \setminus \big\{[c_1, \dots, c_n, (\alpha_1, \alpha_2)], [c_1, \dots, c_n, (\alpha_2, \alpha_1)]\big\}.$$

If $\alpha_1\alpha_2 = \frac{1}{2}$, then the sets $\Delta_{c_1 \dots c_n}$ ($\nabla_{c_1 \dots c_n}$) are the closed (open) intervals (see [3]). We call them cylindrical closed (open) intervals.

Lemma 2.3. *If $\alpha_1\alpha_2 \geq \frac{1}{2}$, then the probability of the event $\nabla_{c_1 c_2 \dots c_n}$ is given by*

$$\mathsf{P}\{\xi \in \nabla_{c_1 c_2 \dots c_n}\} = p_{c_1 1} p_{c_2 2} \cdots p_{c_n n}. \tag{3}$$

Proof. The random event $\{\xi \in \nabla_{c_1 c_2 \dots c_n}\}$ occurs if and only if all the events $\{\eta_1 = a_1\}$, $\{\eta_2 = a_2\}$, $\dots$, $\{\eta_n = a_n\}$ occur. Since the random variables η_k, $k = 1, \dots, n$, are independent,

$$\begin{aligned}\mathsf{P}\{\xi \in \nabla_{c_1 c_2 \dots c_n}\} &= \mathsf{P}\{\eta_1 = c_1, \eta_2 = c_2, \dots, \eta_n = c_n\} \\ &= \mathsf{P}\{\eta_1 = c_1\} \cdot \mathsf{P}\{\eta_2 = c_2\} \cdots \mathsf{P}\{\eta_n = c_n\} = p_{c_1 1} p_{c_2 2} \cdots p_{c_n n}. \qquad \square\end{aligned}$$

3. Criteria for the pure discreteness and pure continuity of the distribution of ξ

Theorem 3.1. *Let $\alpha_1\alpha_2 \geq \frac{1}{2}$. The distribution of the random variable ξ is discrete if and only if*

$$M \equiv \prod_{k=1}^{\infty} \max\{p_{\alpha_1 k}, p_{\alpha_2 k}\} > 0. \tag{4}$$

If the distribution is discrete, then the point topological support of the distribution of the random variable ξ consists of those numbers $x \in [\beta_1, \beta_2]$ whose A_2-continued representations have no more than a finite number of elements a_k such that $p_{a_k k} > 0$ and a_k differ from the corresponding elements of the representation of the number

$$x_0 = [b_1, b_2, \ldots, b_k, \ldots], \tag{5}$$

where $\mathsf{P}\{\eta_k = b_k\} = \max\{p_{\alpha_1 k}, p_{\alpha_2 k}\}$ for all $k \in \mathbb{N}$.

Proof. If the distribution of the random variable ξ is discrete, then there exists a point $x = [a_1, \ldots, a_k, \ldots]$ such that

$$\prod_{k=1}^{\infty} p_{a_k k} > 0.$$

Then

$$\prod_{k=1}^{\infty} \max\{p_{\alpha_1 k}, p_{\alpha_2 k}\} \geq \prod_{k=1}^{\infty} p_{a_k k} > 0.$$

Thus relation (4) holds if the distribution of the random variable ξ has atoms.

Now we assume that relation (4) holds. Consider an arbitrary point $x \in [\beta_1, \beta_2]$ for which the A_2-continued representation exists and differs from the representation of x_0 by at most a finite number of elements a_k with $p_{a_k k} > 0$. It is then obvious that x is an atom of the distribution of ξ.

Let $x_i^{(k)} \in [\beta_1, \beta_2]$ and let the elements of the A_2-continued representation of $x_i^{(k)}$ coincide, starting with the $(k+1)$th position, with those of the representation (5) for the number x_0. Then

$$\begin{aligned}
\mathsf{P}\left\{\xi \in \{x_i^{(k)}\}\right\} &= \sum_{\substack{a_1 \,:\, p_{a_1 1} > 0 \\ \cdots \\ a_k \,:\, p_{a_k k} > 0}} \left(p_{a_1 1} \cdots p_{a_k k} \prod_{j=k+1}^{\infty} p_{b_j j} \right) \\
&= \sum_{a_1 \,:\, p_{a_1 1} > 0} p_{a_1 1} \cdots \sum_{a_k \,:\, p_{a_k k} > 0} p_{a_k k} \prod_{j=k+1}^{\infty} p_{b_j j} = \frac{M}{p_{b_1 1} \cdots p_{b_k k}}.
\end{aligned}$$

It is clear that the set $D = \bigcup_{m=1}^{\infty} \{x_i^{(m)}\}$ is at most countable and

$$\mathsf{P}\{\xi \in D\} = \lim_{m\to\infty} \mathsf{P}\left\{\xi \in \left\{x_i^{(m)}\right\}\right\} = \lim_{m\to\infty} \frac{M}{p_{b_1 1} \cdots p_{b_m m}} = 1.$$

Therefore the random variable ξ is concentrated on at most countable set and thus ξ is discrete by definition. □

Corollary 3.1. *Let $\alpha_1\alpha_2 \geq \frac{1}{2}$. The distribution of the random variable ξ is continuous if and only if*

$$M \equiv \prod_{k=1}^{\infty} \max\{p_{\alpha_1 k}, p_{\alpha_2 k}\} = 0.$$

4. Properties of a continuously distributed random variable ξ

Recall [7] that the set

$$S_\xi = \big\{x\colon \mathsf{P}\{\xi \in (x-\varepsilon, x+\varepsilon)\} > 0 \text{ for all } \varepsilon > 0\big\} \tag{6}$$

is called the topological support of the distribution of the random variable ξ.

Lemma 4.1. *Let $\alpha_1\alpha_2 \geq \frac{1}{2}$. The topological support of the distribution of a continuously distributed random variable ξ coincides with the set*

$$A = \big\{x\colon x = [a_1, a_2, \ldots, a_k, \ldots], p_{a_k k} > 0, \text{ for all } k \in \mathbb{N}\big\}.$$

Proof. We prove that $S_\xi \subset A$. Let $x = [a_1, a_2, \ldots, a_k, \ldots] \in S_\xi$, where x is a point that has a unique A_2-continued representation. Then $\nabla_{a_1a_2\ldots a_k} \cap S_\xi \neq \varnothing$ and

$$\mathsf{P}\{\xi \in \nabla_{a_1a_2\ldots a_k}\} = \prod_{i=1}^{k} p_{a_i i} > 0$$

for an arbitrary positive integer k. This implies that $p_{a_k k} > 0$ for all $k \in \mathbb{N}$, that is, $x \in A$.

Let x be a point of S_ξ that has two different A_2-continued representations, say $[a_1, a_2, \ldots]$ and $[b_1, b_2, \ldots]$. Since

$$\mathsf{P}\{\xi \in (x-\varepsilon, x+\varepsilon)\} > 0$$

for all $\varepsilon > 0$, there exists at least one representation. Let $[a_1, a_2, \ldots]$ be such that

$$\mathsf{P}\{\xi \in \nabla_{a_1a_2\ldots a_k}\} > 0$$

for every positive integer number k. This implies that $p_{a_k k} > 0$ for all $k \in \mathbb{N}$, that is, $x \in A$.

Now we prove that $A \subset S_\xi$. Let $x = [a_1, a_2, \ldots, a_k, \ldots]$, where $p_{a_k k} > 0$ for all $k \in \mathbb{N}$, that is, $x \in A$. Then

$$\mathsf{P}\{\xi \in \nabla_{a_1a_2\ldots a_k}\} = \prod_{i=1}^{k} p_{a_i i} > 0$$

for all k. Since, for all $\varepsilon > 0$, there exists k such that $\nabla_{a_1a_2\ldots a_k} \subset (x-\varepsilon, x+\varepsilon)$, we get $x \in S_\xi$. □

Theorem 4.1. *Let $\alpha_1\alpha_2 > \frac{1}{2}$. If the distribution of the random variable ξ is continuous, then the distribution of ξ is a Cantor type singular distribution.*

Proof. Let $F_0 = [\beta_1, \beta_2]$ and $F_k = \bigcup_{c_1,\ldots,c_k \in A_2} F_{c_1,\ldots,c_k}$, where

$$F_{c_1,\ldots,c_k} = [\min \Delta_{c_1,\ldots,c_k}, \max \Delta_{c_1,\ldots,c_k}].$$

Then $S_\xi \subseteq \bigcap_{k=1}^{\infty} F_k$ and the Lebesgue measure of the topological support $\lambda(S_\xi)$ is such that

$$\begin{aligned}\lambda(S_\xi) &\leq \lim_{k\to\infty} \lambda(F_k) = (\beta_2 - \beta_1) \lim_{k\to\infty} \frac{\lambda(F_k)}{\lambda(F_{k-1})} \cdot \frac{\lambda(F_{k-1})}{\lambda(F_{k-2})} \cdots \frac{\lambda(F_1)}{\lambda(F_0)} \\ &= (\beta_2 - \beta_1) \prod_{k=1}^{\infty} \frac{\lambda(F_k)}{\lambda(F_{k-1})}.\end{aligned}$$

Let

$$D_{c_1,\ldots,c_k} = F_{c_1,\ldots,c_k} \setminus (F_{c_1,\ldots,c_k,\alpha_1} \cup F_{c_1,\ldots,c_k,\alpha_2}).$$

Then

$$D_{c_1\dots c_k} = \Big[\min\big\{[c_1,\dots,c_{k-1},\alpha_1,(\alpha_1,\alpha_2)],[c_1,\dots,c_{k-1},\alpha_2,(\alpha_2,\alpha_1)]\big\},$$
$$\max\big\{[c_1,\dots,c_{k-1},\alpha_1,(\alpha_1,\alpha_2)],[c_1,\dots,c_{k-1},\alpha_2,(\alpha_2,\alpha_1)]\big\}\Big].$$

Put $D_k = \bigcup_{c_1,\dots,c_k\in A_2} D_{c_1\dots c_k}$. Then $D_k = F_{k-1}\setminus F_k$, $F_{k-1} = F_k\cup D_k$, $D_k\cap S_\xi = \varnothing$ and thus

$$\lambda(F_{k-1}) = \lambda(F_k) + \lambda(D_k),$$
$$\frac{\lambda(F_{k-1})}{\lambda(F_{k-1})} = \frac{\lambda(F_k)}{\lambda(F_{k-1})} + \frac{\lambda(D_k)}{\lambda(F_{k-1})},$$
$$\frac{\lambda(F_k)}{\lambda(F_{k-1})} = 1 - \frac{\lambda(D_k)}{\lambda(F_{k-1})}.$$

Hence

$$\lambda(S_\xi) \le (\beta_2-\beta_1)\prod_{k=1}^{\infty}\left(1-\frac{\lambda(D_k)}{\lambda(F_{k-1})}\right). \tag{7}$$

Now we show that the series $\sum_{k=1}^{\infty}\lambda(D_k)/\lambda(F_{k-1})$ diverges. Applying the formula for the length of a cylinder (see [3]), we obtain

$$\lambda(F_{k-1}) = \sum_{c_1,\dots,c_{k-1}\in A_2}\frac{\beta_2-\beta_1}{(q_{k-1}+\beta_1 q_{q-2})(q_{k-1}+\beta_2 q_{k-2})},$$
$$\begin{aligned}\lambda(D_k) &= \sum_{c_1,\dots,c_{k-1}\in A_2}\big|[c_1,\dots,c_{k-1},\alpha_1+\beta_2]-[c_1,\dots,c_{k-1},\alpha_2+\beta_1]\big|\\ &= \sum_{c_1,\dots,c_{k-1}\in A_2}\left|\frac{(\alpha_1+\beta_2)p_{k-1}+p_{k-2}}{(\alpha_1+\beta_2)q_{k-1}+q_{k-2}}-\frac{(\alpha_2+\beta_1)p_{k-1}+p_{k-2}}{(\alpha_2+\beta_1)q_{k-1}+q_{k-2}}\right|\\ &= \sum_{c_1,\dots,c_{k-1}\in A_2}\frac{|(\alpha_2-\alpha_1)-(\beta_2-\beta_1)|}{((\alpha_1+\beta_2)q_{k-1}+q_{k-2})((\alpha_2+\beta_1)q_{k-1}+q_{k-2})}.\end{aligned}$$

Then

$$\begin{aligned}\frac{D_{c_1\dots c_{k-1}}}{F_{c_1\dots c_{k-1}}} &= \frac{|(\alpha_2-\alpha_1)-(\beta_2-\beta_1)|((\alpha_1+\beta_2)q_{k-1}+q_{k-2})((\alpha_2+\beta_1)q_{k-1}+q_{k-2})}{((\alpha_1+\beta_2)q_{k-1}+q_{k-2})((\alpha_2+\beta_1)q_{k-1}+q_{k-2})(\beta_2-\beta_1)}\\ &= \frac{|(\alpha_2-\alpha_1)-(\beta_2-\beta_1)|\left(1+\beta_1\frac{q_{k-2}}{q_{k-1}}\right)\left(1+\beta_2\frac{q_{k-2}}{q_{k-1}}\right)}{\left(\alpha_1+\beta_2+\frac{q_{k-2}}{q_{k-1}}\right)\left(\alpha_2+\beta_1+\frac{q_{k-2}}{q_{k-1}}\right)(\beta_2-\beta_1)}.\end{aligned}$$

Since $\alpha_1\alpha_2 > \frac{1}{2}$, we have $(\alpha_2-\alpha_1)-(\beta_2-\beta_1) = \theta > 0$ (see [3]). Since

$$\theta_1 \le \frac{q_{k-2}}{q_{k-1}} = [a_{k-1},a_{k-2},\dots,a_1] \le \theta_2,$$

where

$$\theta_1 = \frac{1}{\alpha_2+\frac{1}{\alpha_1}},\qquad \theta_2 = \frac{1}{\alpha_1},$$

we obtain

$$\frac{D_{c_1,\dots,c_{k-1}}}{F_{c_1,\dots,c_{k-1}}} > \frac{\theta(1+\beta_1\theta_1)(1+\beta_2\theta_1)}{(\alpha_1+\beta_2+\theta_2)(\alpha_2+\beta_1+\theta_2)(\beta_2-\beta_1)} = \delta > 0$$

for $k = 2,3,\dots$. Then

$$\frac{\lambda(D_k)}{\lambda(F_{k-1})} = \frac{\sum_{c_1,\dots,c_{k-1}\in A_2} D_{c_1\dots c_{k-1}}}{\sum_{c_1,\dots,c_{k-1}\in A_2} F_{c_1\dots c_{k-1}}} \ge \frac{\sum_{c_1,\dots,c_{k-1}\in A_2}\delta F_{c_1\dots c_{k-1}}}{\sum_{c_1,\dots,c_{k-1}\in A_2} F_{c_1\dots c_{k-1}}} = \delta > 0,$$

and the infinite product in (7) diverges to zero. Therefore $\lambda(S_\xi) = 0$ and the distribution of the random variable ξ is a Cantor type singular distribution. □

Lemma 4.2. *Let $\alpha_1\alpha_2 = \frac{1}{2}$. If the distribution of the random variable ξ is continuous, then its distribution function is represented as follows:*

$$F_\xi(x) = \begin{cases} 0 & \text{if } x \le \beta_1, \\ \beta_{a_1(x)} + \sum_{m=2}^{\infty} \left(\beta_{a_m(x)} \prod_{i=1}^{m-1} p_{a_i(x)i}\right) & \text{if } x \in L_{A_2}, \\ 1 & \text{if } x > \beta_2, \end{cases} \tag{8}$$

where $a_m(x)$ are the elements of the A_2-continued representation of a number x; the elements $a_m(x)$ are uniquely defined if x has a unique representation, while the $a_m(x)$ correspond to one of the representations of x if x has two representations; the $\beta_{a_m(x)}$ are given by

$$\beta_{a_m(x)} = \begin{cases} \sum_{\substack{j\in A_2 \\ j<a_m(x)}} p_{jm} & \text{if } m = 2k, \\ \sum_{\substack{j\in A_2 \\ j>a_m(x)}} p_{jm} & \text{if } m = 2k+1. \end{cases}$$

Proof. Let $x \in L_{A_2}$. By definition, $F_\xi(x) = \mathsf{P}\{\xi < x\}$. It is obvious that

$$\begin{aligned} \{\xi < x\} = \{\eta_1 > a_1(x)\} &\cup \{\eta_1 = a_1(x), \eta_2 < a_2(x)\} \cup \cdots \\ &\cup \{\eta_i = a_i(x), i = 1, \dots, 2k-1, \eta_{2k} < a_{2k}(x)\} \\ &\cup \{\eta_i = a_i(x), i = 1, \dots, 2k, \eta_{2k+1} > a_{2k+1}(x)\} \cup \cdots, \end{aligned}$$

where the random events on the right hand side are disjoint. Then

$$\begin{aligned} &\mathsf{P}\{\eta_i = a_i(x), i = 1, \dots, 2k-1, \eta_{2k} < a_{2k}(x)\} \\ &\quad = \left(\prod_{i=1}^{2k-1} \mathsf{P}\{\eta_i = a_i(x)\}\right) \sum_{\substack{j\in A_2 \\ j<a_{2k}(x)}} \mathsf{P}\{\eta_{2k} = j\} = \left(\prod_{i=1}^{2k-1} p_{a_i(x)i}\right) \beta_{a_{2k}(x)}, \\ &\mathsf{P}\{\eta_i = a_i(x), i = 1, \dots, 2k, \eta_{2k+1} > a_{2k+1}(x)\} \\ &\quad = \left(\prod_{i=1}^{2k} \mathsf{P}\{\eta_i = a_i(x)\}\right) \sum_{\substack{j\in A_2 \\ j>a_{2k+1}(x)}} \mathsf{P}\{\eta_{2k+1} = j\} = \left(\prod_{i=1}^{2k} p_{a_i(x)i}\right) \beta_{a_{2k+1}(x)}. \end{aligned}$$

Therefore

$$\begin{aligned} F_\xi(x) = \mathsf{P}\{\xi < x\} &= \sum_{k=1}^{\infty} \left(\beta_{a_{2k}(x)} \prod_{i=1}^{2k-1} p_{a_i(x)i} + \beta_{a_{2k+1}(x)} \prod_{i=1}^{2k} p_{a_i(x)i}\right) \\ &= \beta_{a_1(x)} + \sum_{m=2}^{\infty} \left(\beta_{a_m(x)} \prod_{i=1}^{m-1} p_{a_i(x)i}\right). \end{aligned}$$

□

Theorem 4.2. *Let $\alpha_1\alpha_2 = \frac{1}{2}$. Then the distribution of ξ is pure, that is, the decomposition of the distribution function $F_\xi(x)$ into the discrete, absolutely continuous, and continuously singular components,*

$$F_\xi(x) = aF_d(x) + bF_{ac}(x) + cF_{sc}(x), \qquad a + b + c = 1, \tag{9}$$

is such that one of the coefficients a, b, or c is equal to 1, while the other two are equal to 0.

Proof. According to Theorem 3.1, the distribution of the random variable ξ is either purely discrete or purely continuous. We show that if the distribution is continuous, then it is either absolutely continuous or continuously singular.

Let W_{A_2} be the set of all cylindrical intervals and closed intervals of an A_2-continued representation. The minimal σ-algebra $\mathfrak{F}$ containing W_{A_2} coincides with the σ-algebra of the Borel sets of the interval $[\beta_1, \beta_2]$.

Let $\delta_1, \dots, \delta_n$ be fixed numbers of the set A_2. By $T_{\delta_1 \dots \delta_n}(x)$ we denote the transformation of the point $x = [a_1, \dots, a_k, \dots]$ such that

$$T_{\delta_1 \dots \delta_n}(x) = [\delta_1, \dots, \delta_n, a_1, \dots, a_k, \dots].$$

For an arbitrary Borel set $E \subset [\beta_1, \beta_2]$, we put

$$T_{\delta_1 \dots \delta_n}(E) = \{y \colon y = T_{\delta_1 \dots \delta_n}(x), x \in E\}, \qquad T_0(E) = E,$$

$$T_n(E) = \bigcup_{\delta_1, \dots, \delta_n \in A_2} T_{\delta_1 \dots \delta_n}(E), \qquad T(E) = \bigcup_{n=0}^{\infty} T_n(E).$$

Consider the random event $A = \{\xi \in T(E)\}$. Since the random variables η_k are independent, A is a tail event with respect to all σ-algebras $\mathfrak{B}_k$ generated by the first k elements $\eta_1, \dots, \eta_k$ (k is an arbitrary finite number). Then Kolmogorov's 0-1 law implies that either $\mathsf{P}(A) = 0$ or $\mathsf{P}(A) = 1$.

It is easy to see that the Lebesgue measure of the sets $T_{\delta_1 \dots \delta_n}(E)$ and E is either zero or positive for both of them simultaneously, since the function $y = [\delta_1, \dots, \delta_n + x]$ is absolutely continuous and strictly monotone on the interval $[\beta_1, \beta_2]$. If $\lambda(T_{\delta_1 \dots \delta_n}(E)) = 0$, then $\lambda(T_n(E)) = 0$ and $\lambda(T(E)) = 0$. Since $E \subseteq T(E)$, the continuity of $\mathsf{P}\{\xi \in E\} > 0$ yields

$$\mathsf{P}\{\xi \in T(E)\} \geq \mathsf{P}\{\xi \in E\} > 0,$$

whence $\mathsf{P}\{\xi \in T(E)\} = 1$.

Since the random variable ξ is continuously distributed, only one of the following two cases is possible:

1) There is a set E such that $\lambda(E) = 0$ but $\mathsf{P}\{\xi \in E\} > 0$. Then

$$\mathsf{P}\{\xi \in T(E)\} = 1$$

and $\lambda(T(E)) = 0$, that is, the distribution of ξ is pure and continuously singular by definition.

2) For all sets E, if $\lambda(E) = 0$, then

$$\mathsf{P}\{\xi \in E\} = 0.$$

Then the distribution of ξ is absolutely continuous by definition.

Therefore the distribution of ξ is pure, indeed. □

5. The singularity of the distribution of the random variable ξ in the case of $\alpha_1\alpha_2 = \frac{1}{2}$

In what follows we assume that the random variable ξ is continuously distributed and that $\alpha_1\alpha_2 = \frac{1}{2}$. The denominator of the finite continued fraction of order k of a continued fraction $[a_1, \dots, a_k, \dots]$ is denoted by $q(a_1, \dots, a_k) \equiv q_k(x) \equiv q_k$.

Lemma 5.1. *Let $\alpha_1\alpha_2 = \frac{1}{2}$. If the distribution function $F_\xi(x)$ has the finite derivative $F'_\xi(x)$ at an A_2-irrational point $x = [a_1, a_2, \dots, a_k, \dots]$, then*

$$F'_\xi(x) = \frac{1}{\beta_2 - \beta_1} \lim_{k \to \infty} (q_k(x) + \beta_1 q_{k-1}(x))(q_k(x) + \beta_2 q_{k-1}(x)) \prod_{i=1}^{k} p_{a_i(x)i}. \tag{10}$$

Proof. It is clear that

$$F'_\xi(x) = \lim_{k\to\infty} \frac{\mathsf{P}\{\xi \in \Delta_{a_1\ldots a_k}\}}{|\Delta_{a_1\ldots a_k}|}.$$

Since the distribution of the random variable ξ is continuous,

$$\mathsf{P}\{\xi \in \Delta_{a_1\ldots a_k}\} = \mathsf{P}\{\xi \in \nabla_{a_1\ldots a_k}\}.$$

By relation (3) and by formulas for the probability of a cylinder and for the length of a cylinder (see [3]), we obtain

$$\begin{aligned} F'_\xi(x) &= \lim_{k\to\infty} \frac{\prod_{i=1}^k p_{a_i(x)i}}{|\Delta_{a_1\ldots a_k}|} \\ &= \frac{1}{\beta_2-\beta_1} \lim_{k\to\infty} (q_k(x)+\beta_1 q_{k-1}(x))(q_k(x)+\beta_2 q_{k-1}(x)) \prod_{i=1}^k p_{a_i(x)i}. \end{aligned}$$ □

Theorem 5.1. *For almost all (in the sense of Lebesgue measure) $x \in [\beta_1, \beta_2]$, we have*

$$\overline{\lim_{k\to\infty}} \sqrt[k]{q_k(x)} \le \sqrt{2}.$$

Proof. Let $p_{\alpha_1 k} = p_{\alpha_2 k} = \frac{1}{2}$ for all $k = 1, 2, \ldots$. It is known that every distribution function has a finite derivative almost everywhere (in the sense of Lebesgue measure), thus

$$F'_\xi(x) = \frac{1}{\beta_2-\beta_1} \lim_{k\to\infty} Q_k(x) \left(\frac{1}{2}\right)^k = c,$$

where $0 \le c < +\infty$ and $Q_k(x) = (q_k(x)+\beta_1 q_{k-1}(x))(q_k(x)+\beta_2 q_{k-1}(x))$. This implies that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{Q_k(x)\left(\frac{1}{2}\right)^k} \le 1$$

or

$$\overline{\lim_{k\to\infty}} \sqrt[k]{Q_k(x)} \le 2.$$

Hence

$$\overline{\lim_{k\to\infty}} \sqrt[k]{q_k^2(x)} \le \overline{\lim_{k\to\infty}} \sqrt[k]{Q_k(x)} \le 2$$

and $\overline{\lim}_{k\to\infty} \sqrt[k]{q_k(x)} \le \sqrt{2}$. □

Let $\widetilde{q}(\eta_1, \ldots, \eta_n) = \widetilde{q}_n$ be a random variable defined recursively as follows:

$$\begin{aligned} &\widetilde{q}_0 \equiv 1; \\ &\widetilde{q}_1 \equiv \eta_1; \\ &\widetilde{q}_n = \eta_n \widetilde{q}_{n-1} + \widetilde{q}_{n-2}, \qquad n \ge 2. \end{aligned}$$

Lemma 5.2. *If the random variables η_k are independent and identically distributed, then the expectation of the random variable $\widetilde{q}_n$ can be evaluated as follows:*

$$\mathsf{E}\,\widetilde{q}_n = \frac{1}{\sqrt{a^2+4}} \left(\left(\frac{a+\sqrt{a^2+4}}{2}\right)^{n+1} - \left(\frac{a-\sqrt{a^2+4}}{2}\right)^{n+1} \right), \tag{11}$$

where $a = \mathsf{E}\,\eta_k$.

Proof. It is clear that $\mathsf{E}\,\widetilde{q}_0 = 1$ and $\mathsf{E}\,\widetilde{q}_1 = \mathsf{E}\,\eta_1 = a$. Since $\widetilde{q}_n = \eta_n \widetilde{q}_{n-1} + \widetilde{q}_{n-2}$, where η_n and $\widetilde{q}_{n-1}$ are independent, we get

$$\mathsf{E}\,\widetilde{q}_n = \mathsf{E}\,\eta_n\,\mathsf{E}\,\widetilde{q}_{n-1} + \mathsf{E}\,\widetilde{q}_{n-2} = a\,\mathsf{E}\,\widetilde{q}_{n-1} + \mathsf{E}\,\widetilde{q}_{n-2}.$$

The expectation $\mathsf{E}\,\widetilde{q}_n$ is equal to the denominator $q(\underbrace{a,\dots,a}_{n})$, whence we obtain equality (11) by applying the corresponding formula for the evaluation of $q(\underbrace{a,\dots,a}_{n})$ obtained in [4]. □

Lemma 5.3. *For an arbitrary positive integer k, we have*

$$q(a_k, a_{k-1}, \dots, a_1) = q(a_1, a_2, \dots, a_k). \tag{12}$$

Proof. Applying the rule of formation of the denominators of convergents (see [3]), it is easy to check that equality (12) holds for $k = 1, 2, 3, 4$:

$$\begin{gathered} q(a_1) = a_1 = q(a_1), \\ q(a_1, a_2) = a_1 a_2 + 1 = q(a_2, a_1), \\ q(a_1, a_2, a_3) = a_1 a_2 a_3 + a_3 + a_1 = q(a_3, a_2, a_1), \\ q(a_1, a_2, a_3, a_4) = a_1 a_2 a_3 a_4 + a_3 a_4 + a_1 a_4 + a_1 a_2 + 1 = q(a_4, a_3, a_2, a_1). \end{gathered}$$

Assume that equality (12) holds for all $k < n$, where $n > 4$. Then, for $k = n$, we have

$$\begin{aligned} q(a_1, \dots, a_n) &= a_n q(a_1, \dots, a_{n-1}) + q(a_1, \dots, a_{n-2}) \\ &= a_1 a_n q(a_{n-1}, \dots, a_2) + a_n q(a_{n-1}, \dots, a_3) + q(a_1, \dots, a_{n-2}) \\ &= a_1 a_n q(a_{n-1}, \dots, a_2) + a_n q(a_{n-1}, \dots, a_3) \\ &\quad + a_1 q(a_2, \dots, a_{n-2}) + q(a_3, \dots, a_{n-2}), \\ q(a_n, \dots, a_1) &= a_1 q(a_n, \dots, a_2) + q(a_n, \dots, a_3) \\ &= a_1 a_n q(a_{n-1}, \dots, a_2) + a_1 q(a_{n-2}, \dots, a_2) + q(a_n, \dots, a_3) \\ &= a_1 a_n q(a_{n-1}, \dots, a_2) + a_1 q(a_{n-2}, \dots, a_2) \\ &\quad + a_n q(a_{n-1}, \dots, a_3) + q(a_3, \dots, a_{n-2}), \end{aligned}$$

whence we derive that $q(a_1, \dots, a_n) = q(a_n, \dots, a_1)$. Thus equality (12) holds for all positive integers k. □

By definition, let $q(\varnothing) = 1$.

Lemma 5.4. *For arbitrary positive integers k and m,*

$$\begin{aligned} &q(a_1, \dots, a_k, b_1, \dots, b_m) \\ &\qquad = q(a_1, \dots, a_k) q(b_1, \dots, b_m) + q(a_1, \dots, a_{k-1}) q(b_2, \dots, b_m). \end{aligned} \tag{13}$$

Proof. Using the equality $q(a_1) = a_1$ and Lemma 5.3 for the case of $k = 1$ and $m = 1$, we obtain

$$q(a_1, b_1) = a_1 b_1 + 1 = q(a_1) q(b_1) + q(\varnothing) q(\varnothing)$$

and equality (13) follows. If $k = 2$ and $m = 2$, then

$$\begin{aligned} q(a_1 a_2 b_1 b_2) &= a_1 a_2 b_1 b_2 + b_1 b_2 + a_1 b_2 + a_1 a_2 + 1 = b_1 b_2 (a_1 a_2 + 1) + a_1 b_2 + a_1 a_2 + 1 \\ &= (a_1 a_2 + 1)(b_1 b_2 + 1) + a_1 b_2 = q(a_1, a_2) q(b_1 b_2) + q(a_1) q(b_2) \end{aligned}$$

and equality (13) also follows.

Assume that equality (13) holds for the case of $k \le s$ and $m \le t$. Consider the case of $k = s$, $m = t+1$:

$$\begin{aligned}
&q(a_1,\dots,a_s,b_1,\dots,b_t,b_{t+1})\\
&\quad= b_{t+1}q(a_1,\dots,a_s,b_1,\dots,b_t) + q(a_1,\dots,a_s,b_1,\dots,b_{t-1})\\
&\quad= b_{t+1}\big(q(a_1,\dots,a_s)q(b_1,\dots,b_t) + q(a_1,\dots,a_{s-1})q(b_2,\dots,b_t)\big)\\
&\qquad+ q(a_1,\dots,a_s)q(b_1,\dots,b_{t-1}) + q(a_1,\dots,a_{s-1})q(b_2,\dots,b_{t-1})\\
&\quad= q(a_1,\dots,a_s)\big(b_{t+1}q(b_1,\dots,b_t) + q(b_1,\dots,b_{t-1})\big)\\
&\qquad+ q(a_1,\dots,a_{s-1})\big(b_{t+1}q(b_2,\dots,b_t) + q(b_2,\dots,b_{t-1})\big)\\
&\quad= q(a_1,\dots,a_s)q(b_1,\dots,b_{t+1}) + q(a_1,\dots,a_{s-1})q(b_2,\dots,b_{t+1}).
\end{aligned}$$

Now let $k = s+1$ and $m = t$. Since equality (13) holds for the case of $k \le s$ and $m \le t+1$, we conclude that

$$\begin{aligned}
q(a_1,\dots,a_{s+1}b_1,\dots,b_t) &= q(a_1,\dots,a_s)q(a_{s+1},b_1,\dots,b_t) + q(a_1,\dots,a_{s-1})q(b_1,\dots,b_t)\\
&= q(a_1,\dots,a_s)\big(a_{s+1}q(b_1,\dots,b_t) + q(b_t,\dots,b_2)\big)\\
&\quad+ q(a_1,\dots,a_{s-1})q(b_1,\dots,b_t)\\
&= a_{s+1}q(a_1,\dots,a_s) + q(a_1,\dots,a_{s-1})q(b_1,\dots,b_t)\\
&\quad+ q(a_1,\dots,a_s)q(b_t,\dots,b_2)\\
&= q(a_1,\dots,a_{s+1})q(b_1,\dots,b_t) + q(a_1,\dots,a_s)q(b_2,\dots,b_t).
\end{aligned}$$

Thus if equality (13) holds for $k \le s$ and $m \le t$, then it also holds for $k \le s$ and $m \le t+1$ or for $k \le s+1$ and $m \le t$. Therefore, equality (13) holds for all positive integers k and m. □

Lemma 5.5. *The following inequalities hold for all positive integers $n \ge 2$ and for all $k \ge 2$:*

$$\begin{aligned}
&\prod_{j=1}^{n} q\left(a_{(j-1)k+1},\dots,a_{(j-1)k+k}\right) < q_{nk}\\
&\qquad\qquad < \left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)^{n-1}\prod_{j=1}^{n} q\left(a_{(j-1)k+1},\dots,a_{(j-1)k+k}\right),
\end{aligned}\tag{14}$$

where $q_{nk} = q(a_1,\dots,a_{nk})$.

Proof. Since $q_n = a_n q_{n-1} + q_{n-2}$ (see [3]), we deduce that $q_n > a_n q_{n-1}$ and $q_{n-1} < q_n/\alpha_1$. Taking into account equalities (12) and (13), we obtain

$$\begin{aligned}
q(a_1,a_2,\dots,a_{nk}) &= q(a_1,\dots,a_k)q(a_{k+1},\dots,a_{nk}) + q(a_1,\dots,a_{k-1})q(a_{k+2},\dots,a_{nk})\\
&< q(a_1,\dots,a_k)q(a_{k+1},\dots,a_{nk}) + \frac{1}{\alpha_1}q(a_1,\dots,a_k)\frac{1}{\alpha_1}q(a_{k+1},\dots,a_{nk})\\
&= \left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)q(a_1,\dots,a_k)q(a_{k+1},\dots,a_{nk})\\
&< \dots < \left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)^{n-1}\prod_{j=1}^{n} q\left(a_{(j-1)k+1},a_{(j-1)k+2},\dots,a_{jk}\right).
\end{aligned}$$

Now we prove the left inequality in (14):

$$\begin{aligned} q(a_1, a_2, \dots, a_{nk}) &= q(a_1, \dots, a_k) q(a_{k+1}, \dots, a_{nk}) + q(a_1, \dots, a_{k-1}) q(a_{k+2}, \dots, a_{nk}) \\ &> q(a_1, \dots, a_k) q(a_{k+1}, \dots, a_{nk}) \\ &> \dots > \prod_{j=1}^{n} q\left(a_{(j-1)k+1}, a_{(j-1)k+2}, \dots, a_{jk}\right). \end{aligned}$$

□

Lemma 5.6. *If the random variables η_k are independent and identically distributed, then*

$$\mathsf{P}\left\{\xi \in \left\{x\colon \lim_{m\to\infty} \sqrt[m]{q_m(x)} = \frac{a + \sqrt{a^2+4}}{2}\right\}\right\} = 1,$$

where a is the mathematical expectation of the random variable η_k.

Proof. Since

$$\frac{1}{q_n(x)} = \prod_{k=1}^{n} [a_k, a_{k+1}, \dots, a_n]$$

for all $x \in [\beta_1, \beta_2]$ (see [1]), we derive from the multiplicative ergodic theorem that the limit $\lim_{m\to\infty} \sqrt[m]{q_m(x)}$ exists and is finite with probability one. Now we evaluate this limit. Lemma 5.5 implies that

$$q(a_1, a_2, \dots, a_{nk}) < \left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)^n q(a_1, \dots, a_k) q(a_{k+1}, \dots, a_{2k}) \cdots q(a_{(n-1)k+1}, \dots, a_{nk}),$$

where n and k are positive integers. Using the relation between arithmetical and geometrical means, we get

$$\begin{aligned} &q^{\frac{1}{nk}}(a_1, a_2, \dots, a_{nk}) \\ &\quad < \left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)^{\frac{1}{k}} \left(\sqrt[n]{q(a_1, \dots, a_k) q(a_{k+1}, \dots, a_{2k}) \cdots q\left(a_{(n-1)k+1}, \dots, a_{nk}\right)}\right)^{\frac{1}{k}} \\ &\quad \le \left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)^{\frac{1}{k}} \left(\frac{q(a_1, \dots, a_k) + q(a_{k+1}, \dots, a_{2k}) + \dots + q\left(a_{(n-1)k+1}, \dots, a_{nk}\right)}{n}\right)^{\frac{1}{k}}. \end{aligned}$$

Let

$$F = \left\{x\colon \lim_{n\to\infty} \frac{q(a_1, \dots, a_k) + q(a_{k+1}, \dots, a_{2k}) + \dots + q\left(a_{(n-1)k+1}, \dots, a_{nk}\right)}{n} = \mathsf{E}\,\widetilde{q}_k\right\}.$$

Since the random variables η_k are independent and identically distributed, the random variables

$$\widetilde{q}^{(i)} = \widetilde{q}\left(\eta_{(i-1)(k+1)}, \dots, a\eta_{ik}\right),$$

where $i = 1, \dots, n$, are also independent and identically distributed and, moreover, the strong law of large numbers holds:

$$\frac{\widetilde{q}^{(1)} + \dots + \widetilde{q}^{(n)}}{n} \to \mathsf{E}\,\widetilde{q}(\eta_1, \dots, \eta_k) = \mathsf{E}\,\widetilde{q}_k \quad \text{as} \quad n \to \infty$$

with probability one. Since $\xi = [\eta_1, \dots, \eta_k, \dots]$, we have $\mathsf{P}\{\xi \in F\} = 1$.

Applying equality (11) for the mathematical expectation $\mathsf{E}\,\widetilde{q}(\eta_1,\dots,\eta_k)$, we obtain for all $x\in F$ and positive integer k that

$$\lim_{n\to\infty} q^{\frac{1}{nk}}(a_1,\dots,a_{nk})$$
$$\le\left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)^{\frac{1}{k}}\left(\frac{1}{\sqrt{a^2+4}}\left(\left(\frac{a+\sqrt{a^2+4}}{2}\right)^{k+1}-\left(\frac{a-\sqrt{a^2+4}}{2}\right)^{k+1}\right)\right)^{\frac{1}{k}}.$$

The order of the denominator increases if the number is a multiple of k. Since

$$q^{\frac{1}{m}}(a_1,\dots,a_m)<\big(q(a_1,\dots,a_k)q(a_{k+1},\dots,a_{2k})\cdots q(a_{(n-1)k+1},\dots,a_{nk})\big)^{\frac{1}{k(n-1)}},$$

where $k(n-1)\le m<kn$, we have, for all positive integers k,

$$\lim_{\substack{k(n-1)\le m<kn\\ n\to\infty}} q^{\frac{1}{m}}(a_1,\dots,a_m)$$
$$\le\left(\frac{\alpha_1^2+1}{\alpha_1^2}\right)^{\frac{1}{k}}\left(\frac{1}{\sqrt{a^2+4}}\left(\left(\frac{a+\sqrt{a^2+4}}{2}\right)^{k+1}-\left(\frac{a-\sqrt{a^2+4}}{2}\right)^{k+1}\right)\right)^{\frac{1}{k}}.$$

Passing to the limit as $k\to\infty$ on the right hand side of the latter inequality, we prove that, for all $x\in F$,

$$\lim_{m\to\infty} q^{\frac{1}{m}}(a_1,\dots,a_m)\le\frac{a+\sqrt{a^2+4}}{2}. \tag{15}$$

Hence $\mathsf{P}\{\xi\in F\}=1$.

Let

$$G=\left\{x\colon \lim_{m\to\infty} q^{\frac{1}{m}}(a_1,\dots,a_m)<\frac{a+\sqrt{a^2+4}}{2}\right\}.$$

Then, for all $x\in G$,

$$\lim_{m\to\infty}\left(\frac{q_m}{\tau^m}\right)^{\frac{1}{m}}<1,$$

where $\tau=\frac{1}{2}(a+\sqrt{a^2+4})$. This implies that the series $\sum_{k=1}^{\infty} q_m/\tau^m$ converges. It is also clear that the series of mathematical expectations $\sum_{k=1}^{\infty}\mathsf{E}\,\widetilde{q}_m/\tau^m$ diverges with probability one. Relation (15) implies that there exists a sufficiently large number $c>0$ such that

$$\mathsf{P}\left\{\frac{\widetilde{q}_m}{\tau^m}\le c\right\}=1,\qquad m=1,2,\dots,$$

and the Kolmogorov two series theorem implies that the series $\sum_{k=1}^{\infty}\widetilde{q}_m/\tau^m$ diverges with probability one, whence $\mathsf{P}\{\xi\in G\}=0$.

Therefore

$$\mathsf{P}\left\{\xi\in\left\{x\colon \lim_{m\to\infty}\sqrt[m]{q_m(x)}=\frac{a+\sqrt{a^2+4}}{2}\right\}\right\}=1. \qquad \square$$

Corollary 5.1. *For almost all (in the sense of Lebesgue measure) $x\in[\beta_1,\beta_2]$, we have*

$$\lim_{k\to\infty}\frac{q_k}{(\sqrt{2})^k}=0.$$

Proof. Let $p_{\alpha_1 k}=p_{\alpha_2 k}=\frac{1}{2}$ for all $k=1,2,\dots$. Consider the set

$$E=\left\{x\colon \lim_{m\to\infty}\sqrt[m]{q_m(x)}=\frac{a+\sqrt{a^2+4}}{2}\right\},$$

where $a = \mathsf{E}\,\eta_k = \frac{3}{4}$. It is obvious that the distribution of ξ is continuous. Theorem 5.1 implies that the Lebesgue measure of the set E is equal to zero. On the other hand, Lemma 5.6 implies that $\mathsf{P}\{\xi \in E\} = 1$. Thus the distribution of ξ is purely singular and, for almost all $x \in [\beta_1, \beta_2]$,

$$\begin{aligned} F'_\xi(x) &= \frac{1}{\beta_2 - \beta_1} \lim_{k\to\infty} (q_k(x) + \beta_1 q_{k-1}(x))(q_k(x) + \beta_2 q_{k-1}(x)) \left(\frac{1}{2}\right)^k \\ &\geq \frac{1}{\beta_2 - \beta_1} \lim_{k\to\infty} q_k^2(x) \left(\frac{1}{2}\right)^k = 0, \end{aligned}$$

whence (5.1) follows. □

Theorem 5.2. *Suppose the random variable ξ has a continuous distribution, that is, $M = 0$, and $\alpha_1\alpha_2 = \frac{1}{2}$. If the matrix $\|p_{ik}\|$ contains only a finite number of zero entries, then ξ has a singular distribution of the Salem type (in other words, a singular distribution of the S-type).*

Proof. By assumption, the η_k are independent identically distributed random variables such that $\mathsf{P}\{\eta_k = \alpha_1\} = p_{\alpha_1 k}$ and $\mathsf{P}\{\eta_k = \alpha_2\} = p_{\alpha_2 k}$, where $p_{\alpha_1 k} + p_{\alpha_2 k} = 1$ for all $k = 1, 2, \dots,$

$$\prod_{k=1}^{\infty} \max\{p_{\alpha_1 k}, p_{\alpha_2 k}\} = 0.$$

Moreover, there exists a number k_0 such that $p_{\alpha_1 k} > 0$ and $p_{\alpha_2 k} > 0$ for all $k > k_0$.

Theorems 4.2 and 3.1 imply that ξ has a pure absolutely continuous or pure singular distribution. Assume that the distribution of ξ is absolutely continuous. Then the distribution function $F_\xi(x)$ has a nonzero finite derivative on a set of a positive Lebesgue measure.

Let $\ln \widetilde{p}_{a_i}$, $i = 1, 2, \dots,$ be independent random variables, where $\widetilde{p}_{a_i} = p_{\alpha_1 i}$ if $\eta_i = \alpha_1$, and $\widetilde{p}_{a_i} = p_{\alpha_2 i}$ if $\eta_i = \alpha_2$.

We evaluate the mathematical expectation and variance of the random variable $\ln \widetilde{p}_{a_i}$:

$$\begin{aligned} \mathsf{E} \ln \widetilde{p}_{a_i} &= \ln p_{\alpha_1 i}^{p_{\alpha_1 i}} (1 - p_{\alpha_1 i})^{1 - p_{\alpha_1 i}}, \\ \operatorname{Var} \ln \widetilde{p}_{a_i} &= \ln^2 p_{\alpha_1 i}^{p_{\alpha_1 i}} + \ln^2 (1 - p_{\alpha_1 i})^{1 - p_{\alpha_1 i}} - \ln^2 \left(p_{\alpha_1 i}^{p_{\alpha_1 i}} (1 - p_{\alpha_1 i})^{1 - p_{\alpha_1 i}}\right). \end{aligned}$$

Since the function $y = \ln^2 x^x + \ln^2 (1 - x)^{1-x} - \ln^2 \left(x^x (1 - x)^{1-x}\right)$ is bounded on the interval $(0, 1)$, the series

$$\sum_{k=1}^{\infty} \frac{\operatorname{Var} \ln \widetilde{p}_{a_k}}{k^2}$$

converges and the strong law of large numbers holds, whence $\mathsf{P}\{\xi \in W\} = 1$, where the set W is defined by

$$W = \left\{ x \colon \lim_{k\to\infty} \frac{1}{k - k_0 + 1} \sum_{i=k_0}^{k>k_0} \ln p_{a_i(x) i} = \lim_{k\to\infty} \frac{1}{k - k_0 + 1} \sum_{i=k_0}^{k>k_0} \mathsf{E} \ln \widetilde{p}_{a_i} \right\}.$$

By assumption, the distribution of ξ is absolutely continuous, whence

$$\lambda(W) = \lambda(S_\xi) > 0$$

(here λ stands for Lebesgue measure). We introduce the set

$$L = \left\{ x \colon \lim_{k\to\infty} \frac{q_k(x)}{\left(\sqrt{2}\right)^k} = 0 \right\}.$$

By Corollary 5.1, $\lambda(L) = \beta_2 - \beta_1$, whence $\lambda(W \cap L) = \lambda(S_\xi) > 0$.

For all $x \in W \cap L$, we have

$$\begin{aligned}
F'_\xi(x) &= \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty}(q_k(x)+\beta_1 q_{k-1}(x))(q_k(x)+\beta_2 q_{k-1}(x))\prod_{i=1}^{k} p_{a_i(x)i} \\
&\le \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty}(q_k(x)+\beta_1 q_{k-1}(x))(q_k(x)+\beta_2 q_{k-1}(x))\prod_{i=k_0}^{k>k_0} p_{a_i(x)i} \\
&\le \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty} q_k^2 \exp\left\{\ln\prod_{i=k_0}^{k>k_0} p_{a_i(x)i}\right\} \\
&= \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty} q_k^2 \exp\left\{\sum_{i=k_0}^{k>k_0} \ln p_{a_i(x)i}\right\} \\
&= \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty} q_k^2 \exp\left\{\sum_{i=k_0}^{k>k_0} \mathsf{E}\ln \widetilde{p}_{a_i}\right\} \\
&= \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty} q_k^2 \exp\left\{\sum_{i=k_0}^{k>k_0} \ln p_{\alpha_1 i}^{p_{\alpha_1 i}}(1-p_{\alpha_1 i})^{1-p_{\alpha_1 i}}\right\} \\
&= \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty} q_k^2 \exp\left\{\ln\prod_{i=k_0}^{k>k_0} p_{\alpha_1 i}^{p_{\alpha_1 i}}(1-p_{\alpha_1 i})^{1-p_{\alpha_1 i}}\right\} \\
&= \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty} q_k^2 \prod_{i=k_0}^{k>k_0} p_{\alpha_1 i}^{p_{\alpha_1 i}}(1-p_{\alpha_1 i})^{1-p_{\alpha_1 i}}.
\end{aligned}$$

Since the function $y = x^x(1-x)^{1-x}$ attains its minimum on $(0,1)$ and this minimum is equal to $\frac{1}{2}$, we have

$$F'_\xi(x) \le \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty} q_k^2\left(\frac{1}{2}\right)^{k-k_0+1} = 0$$

for all $x \in W \cap L$, which means that the latter relation holds almost surely on S_ξ, and this contradicts the assumption that the distribution of ξ is absolutely continuous.

Therefore the random variable ξ has a pure singular distribution. Since the number of zeros in the matrix $\|p_{ik}\|$ is finite, it is easy to check that the topological support is a union of intervals and thus ξ has a pure singular distribution of the Salem type (S-type). □

Theorem 5.3. *Suppose the random variable ξ has a continuous distribution, that is, $M = 0$ and let $\alpha_1\alpha_2 = \frac{1}{2}$. The random variable ξ has a singular distribution of the Cantor type (C-type) if and only if the matrix $\|p_{ik}\|$ has an infinite number of zeros.*

Proof. The necessity is obvious. We prove the sufficiency.

Let $F_0 = [\beta_1, \beta_2]$. Note that F_k is a union of cylinders of rank k and that the points of the topological support S_ξ are interior points of F_k. Then $S_\xi = \bigcap_{k=1}^{\infty} F_k$ and the Lebesgue measure $\lambda(S_\xi)$ of the topological support satisfies

$$\begin{aligned}
\lambda(S_\xi) &= \lim_{k\to\infty}\lambda(F_k) = \frac{1}{\beta_2-\beta_1}\lim_{k\to\infty}\frac{\lambda(F_k)}{\lambda(F_{k-1})}\cdot\frac{\lambda(F_{k-1})}{\lambda(F_{k-2})}\cdots\frac{\lambda(F_1)}{\lambda(F_0)} \\
&= \frac{1}{\beta_2-\beta_1}\prod_{k=1}^{\infty}\frac{\lambda(F_k)}{\lambda(F_{k-1})}.
\end{aligned}$$

Let $D_k = F_{k-1} \setminus F_k$. Then $F_{k-1} = F_k \cup D_k$, $D_k \cap S_\xi = \varnothing$, and thus

$$\lambda(F_{k-1}) = \lambda(F_k) + \lambda(D_k),$$
$$\frac{\lambda(F_{k-1})}{\lambda(F_{k-1})} = \frac{\lambda(F_k)}{\lambda(F_{k-1})} + \frac{\lambda(D_k)}{\lambda(F_{k-1})},$$
$$\frac{\lambda(F_k)}{\lambda(F_{k-1})} = 1 - \frac{\lambda(D_k)}{\lambda(F_{k-1})}.$$

Hence

$$\lambda(S_\xi) = \frac{1}{\beta_2 - \beta_1} \prod_{k=1}^{\infty} \left(1 - \frac{\lambda(D_k)}{\lambda(F_{k-1})}\right).$$

We show that the series

$$\sum_{k=1}^{\infty} \frac{\lambda(D_{k+1})}{\lambda(F_k)} \tag{16}$$

diverges. It is obvious that there are numbers k such that $p_{a_{k+1}k+1} = 0$. For such numbers k, we have (see also [3])

$$\frac{|\Delta_{a_1\dots a_k c}|}{|\Delta_{a_1\dots a_k}|} = \frac{\left(1 + c\frac{q_{n-1}}{q_n}\right)}{2c^2 + 1 + 2c\frac{q_{n-1}}{q_n}} > \frac{1 + c\theta_1}{2c^2 + 1 + 2c\theta_2} = \gamma > 0,$$

where

$$\theta_1 = \frac{1}{\alpha_2 + \frac{1}{\alpha_1}}, \qquad \theta_2 = \frac{1}{\alpha_1},$$

and thus

$$\frac{\lambda(D_{k+1})}{\lambda(F_k)} = \frac{\sum_{\substack{p_{a_i i} > 0,\, i=1,\dots,k \\ p_{a_{k+1}k+1}=0}} |\Delta_{a_1\dots a_{k+1}}|}{\sum_{p_{a_i i} > 0,\, i=1,\dots,k} |\Delta_{a_1\dots a_k}|} \geq \gamma > 0.$$

Since the matrix $\|p_{ik}\|$ contains an infinite number of zeros, the ratio $\lambda(D_{k+1})/\lambda(F_k)$ is bounded away from zero for infinitely many values of k, whence we conclude that the series (16) diverges.

Therefore the Lebesgue measure $\lambda(S_\xi)$ of the topological support equals zero and the distribution of ξ is Cantor type singular. □

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Received 14/JUL/2009

Translated by N. SEMENOV